Signals and systems
Lecture (S3)

Square Wave Example, Signal Power and Properties of Fourier Series

March 18, 2008

Today's Topics

1. Derivation of a Fourier series representation of a square wave signal
2. Power in signals
3. Properties of Fourier series

Take Away

Fourier series can represent a wide class of functions including discontinuous functions. The Fourier coefficients indicate the power in the signal at the frequencies associated with those coefficients

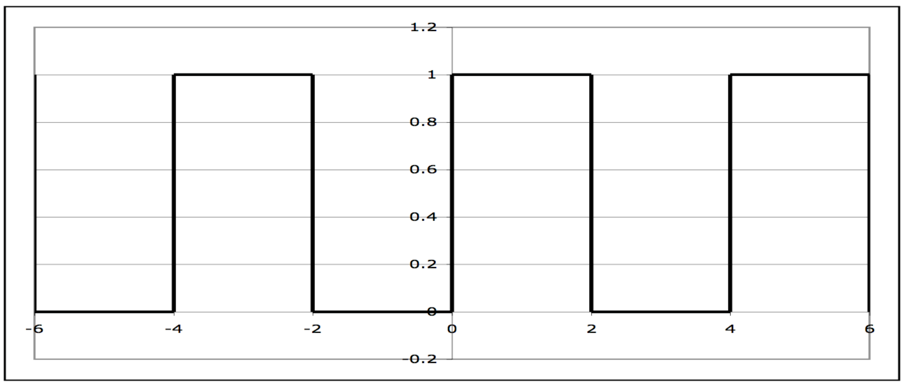
Required Reading

O&W-3.3.2 (Example 3.5), 3.4, 3.5

Last time we defined Fourier series and showed how to determine the Fourier coefficients. We begin today by deriving the Fourier series representation of the square wave. This is an important and illustrative example because of the discontinuities inherent in the square wave.

Example 1- Fourier Series Representation of a Periodic Square Wave

The signal *x(t)* is a square wave



As shown in the figure, the signal is periodic with period *T=4*. Each pulse has value *+1* and is of duration *T/2,* followed by a zero signal of duration *T/2.* Over one cycle the signal is defined as-

$$x(t) = \begin{cases} 1 & 0 \leq t < 2 \\ 0 & 2 \leq t < 4 \end{cases}$$

From our earlier work we know that the Fourier coefficients can be determined from

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

The *k=0* coefficient is the average value of the signal

$$a_0 = \frac{1}{T} \int_0^T x(t) dt = \frac{1}{T} \int_0^{\frac{T}{2}} dt = \frac{1}{2}$$

and the other coefficients for *k>0* are

$$a_k = \frac{1}{T} \int_0^{\frac{T}{2}} e^{-jk\omega_0 t} dt = -\frac{1}{Tjk\omega_0} \left[e^{-jk\omega_0 t} \right]_0^{\frac{T}{2}}$$

$$= \frac{1}{Tjk\omega_0} \left[1 - e^{-jk\omega_0 \frac{T}{2}} \right] = \frac{1}{Tjk\omega_0} \left[1 - e^{-jk\pi} \right]$$

$$= \begin{cases} \frac{1}{\pi j k} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

Hence, except for *k=0,* all of the coefficients for even values of *k* are zero and the coefficients for odd values of *k* are

$$a_1 = \frac{1}{\pi j} \qquad a_{-1} = -\frac{1}{\pi j}$$

$$a_3 = \frac{1}{3\pi j} \qquad a_{-1} = -\frac{1}{3\pi j}$$

$$a_5 = \frac{1}{5\pi j} \qquad a_{-5} = -\frac{1}{5\pi j}$$

$$\vdots \qquad \vdots$$

Substituting these coefficients yields

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} = a_0 + \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{1}{\pi j k} e^{jk\left(\frac{2\pi}{T}\right)t}$$

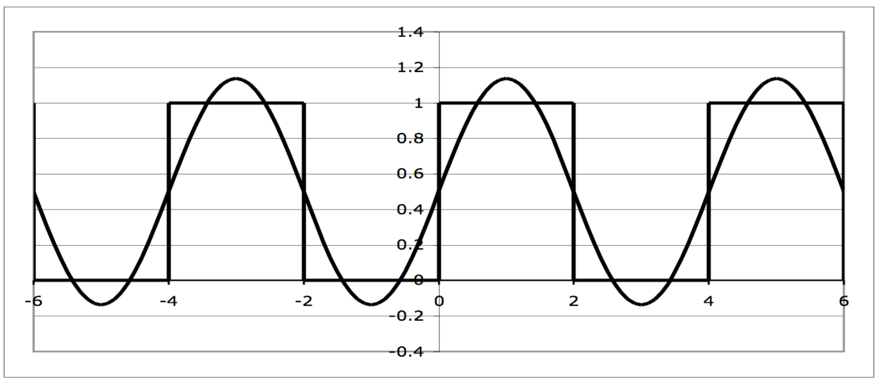
$$= \frac{1}{2} + \frac{1}{\pi j}\left[e^{j\left(\frac{2\pi}{T}\right)t} - e^{-j\left(\frac{2\pi}{T}\right)t}\right] + \frac{1}{\pi j 3}\left[e^{j3\left(\frac{2\pi}{T}\right)t} - e^{-j3\left(\frac{2\pi}{T}\right)t}\right]$$

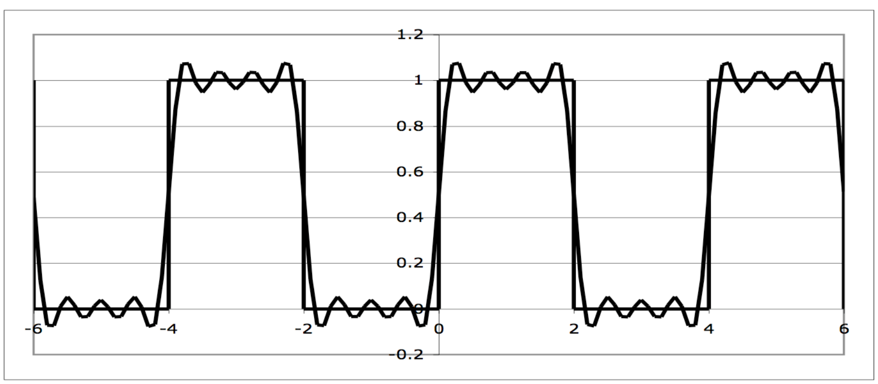
$$+ \cdots$$

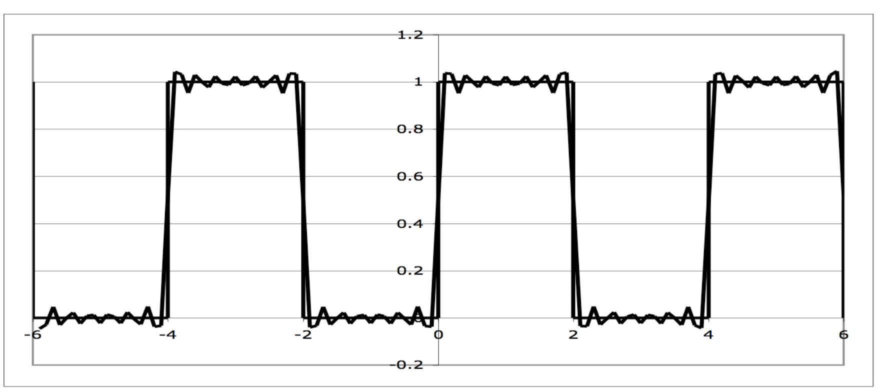
which is a Fourier sine series

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\sin\left(\frac{2\pi t}{T}\right) + \frac{2}{3\pi}\sin\left(\frac{6\pi t}{T}\right) + \frac{2}{5\pi}\sin\left(\frac{10\pi t}{T}\right) + \cdots$$

and is consistent with the fact that $x(t)$ is an odd function of time. In any real world situation only a finite number of terms would be used to characterize a square wave using a Fourier series. Approximations to a square wave of period 4, using a truncated Fourier series with maximum values of $k=1$, 7 and 13 are illustrated in the following plots.







Note that for most values of time the approximations become increasingly better as more terms are added to the series. However, for regions near the discontinuities the series approximations tend not to improve as more terms are added. Although the approximations always pass through the half-way-point at all discontinuities, there are also overshoots to the left and right of each discontinuity. More terms in the series do not reduce the size of these overshoots, but more terms tend to confine the overshoots to smaller and smaller regions near the discontinuities. Hence, there are narrower and narrower spikes on each sided of the discontinuities. Fig. 3.9 in the text shows the series approximation for 79 terms in such a series.

Convergence of Fourier Series

Mathematicians have determined three conditions that are necessary to assure that a periodic signal *x(t)* will have a Fourier series representation. These are known as the Dirichlet conditions, which must hold over the interval T-

<u>Condition 1</u>-the signal *x(t)* must be absolutely integrable over any period

$$\int_T |x(t)| < \infty$$

This condition assures that all of the coefficients of the series will be finite.

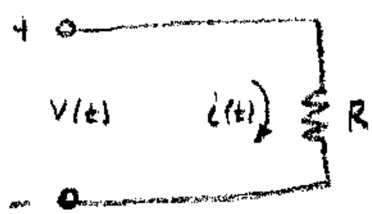
Condition 2-the signal must be of bounded variation, which requires that there be only a finite number of maxima and minima in the interval T

Condition 3-there can be only a finite number of discontinuities in the interval T

Page 199 of the text illustrates some functions that do not satisfy Dirichlet conditions. Virtually all signals that are of practical interest do satisfy these conditions

Power in Signals

Often it is useful to compare signals in terms of some measure of their intensity. We will do this by defining a generalized power measure for signals. To motivate this definition we will examine two physical systems. The first is the power dissipated as heat in an electrical resistor. Suppose we have the simple circuit



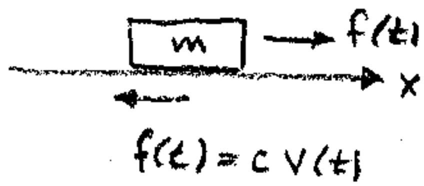
which is described by the constitutive relationship for the resistor.

$$v(t) = i(t) R$$

The power dissipated as heat in the resistor is

$$P(t) = i(t)v(t) = i^2(t)R = \frac{v^2(t)}{R}$$

Similarly, in a mechanical system, heat can be dissipated by frication that is proportional to velocity. In steady state, a body moving at constant velocity against friction requires a force to keep it in constant motion



and the power dissipated as heat of friction is equal to the rate of change of the work done, so

$$P(t) = \frac{dw(t)}{dt} = f(t)\frac{dx(t)}{dt} = f(t)v(t) = cv^2(t)$$

Note that in both cases the power is proportional to the square of a function (signal) that characterized what was happening (e.g., voltage or current in the first case and velocity in the second).

We will generalize this interpretation for signals by defining the average power in the signal, over an interval of time, as

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt$$

where $|x(t)|$ implies the magnitude of the (possibly complex) variable $x(t)$. Hence

$$|x(t)|^2 = x(t)\, x^*(t)$$

A very important result, called Parsival's relation, can be readily obtained for periodic signals that can be represented as Fourier series.

Parsival's Relationship for Periodic Signals

If $x(t)$ can be represented by a Fourier series so

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k \phi_k(t)$$

then

$$|x(t)|^2 = x(t)x^*(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) \cdot \sum_{l=-\infty}^{\infty} a_l^* \phi_l^*(t) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} a_k a_l^* \phi_k(t)\phi_l^*(t)$$

and applying the equation for average power over one period obtains

$$P = \frac{1}{T}\int_T \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} a_k a_l^* \phi_k(t)\phi_l^*(t)\,dt = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} a_k a_l^* \frac{1}{T}\int_T \phi_k(t)\phi_l^*(t)\,dt$$

Now the integral on the right is the inner product of orthogonal functions. Hence, all terms for which $l \neq k$ are zero and each of the remaining terms, for which $l=k$, is the power in the signal component associated with that value of k.

$$P = \sum_{k=-\infty}^{\infty} a_k a_k^* = \sum_{k=-\infty}^{\infty} |a_k|^2$$

This expression has a very important interpretation. Each term in the sum is associated with a value of k, which represents an individual harmonic frequency. Hence, each term is the power in that frequency component of the signal $x(t)$. Whence, the Fourier coefficients indicate the distribution of power over the frequency spectrum.

A Fourier series that is truncated, at some maximum value of *k,* is the closest approximation to that function, in the sense that the power in the difference, between the signal and the approximation, is minimized.

Properties of Fourier Series

A number of significant properties of Fourier series can be readily obtained from our results so far. In what follows we will use the notation in the text to indicate the relationship between a signal *x(t)* and its Fourier coefficients

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

Linearity

If *x(t)* and *y(t)* are two periodic signals with the same period T , so

$$x(t) \overset{FS}{\longleftrightarrow} a_k$$

$$y(t) \overset{FS}{\longleftrightarrow} b_k$$

Then if the signal *z(t)* is a linear combination of *x(t)* and *y(t),* its Fourier coefficients c_k are the same linear combination of coefficients of *x(t)* and *y(t)*

$$z(t) = Ax(t) + By(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k$$

<u>Time Shifting</u>

If the periodic signal *x(t)* is shifted in time so that $y(t) = x(t - t_0)$, then the Fourier coefficients of *y(t)* are

$$b_k = \frac{1}{T}\int_T x(t - t_0)e^{-jk\omega_0 t}dt$$

and defining a new variable of integration as

$$\tau = t - t_0 \qquad d\tau = dt$$

obtains

$$b_k = \frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau + t_0)}d\tau = \frac{1}{T}e^{-jk\omega_0 t_0}\int_T e^{-jk\omega_0 \tau}d\tau$$

$$= e^{-jk\omega_0 t_0}a_k$$

so we can write that if

$$x(t) \xleftrightarrow{FS} a_k$$

then

$$x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} a_k \equiv e^{-jk\left(\frac{2\pi}{T}\right)t_0} a_k$$

Time Reversal

If *y(t)* is the time reversal of the signal *x(t)* so that

$$y(t) = x(-t)$$

then the Fourier series for *y(t)* is

$$y(t) = x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk\left(\frac{2\pi}{T}\right)t}$$

and substituting *k=-m*

$$y(t) = x(-t) = \sum_{m=\infty}^{-\infty} a_{-m} e^{jm\left(\frac{2\pi}{T}\right)t} = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\left(\frac{2\pi}{T}\right)t}$$

we infer that if b_k are the Fourier coefficients for *y(t)* then

$$b_k = a_{-k}$$

so if

$$x(t) \xleftrightarrow{FS} a_k$$

then

$$x(-t) \xleftrightarrow{FS} a_{-k}$$

Time Scaling

Time scaling will extend or shorten the period of a signal. In particular, if *x(t)* has period *T,* with Fourier representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Then if *a* is a real, positive, number the signal *x(at)* will have period *T/a* and fundamental frequency $a\omega_0$. So the Fourier series representation of *x(at)* is

$$x(at) = \sum_{k=-\infty}^{\infty} a_k e^{jk(a\omega_0)t}$$

Multiplication

If $x(t)$ and $y(t)$ are both periodic with period T and Fourier series representations

$$x(t) \xleftrightarrow{FS} a_k$$
$$y(t) \xleftrightarrow{FS} b_k$$

then their product must also be periodic with period T. The product will have a Fourier representation with coefficients

$$x(t)y(t) \xleftrightarrow{FS} h_k = \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$$

Conjugation

Complex conjugation of the signal $x(t)$ has the effect of both conjugation and time reversal for the Fourier coefficients of $x(t)$. In particular if $x(t)$ has the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

then its complex conjugate is

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}$$

or, if we substitute *k*=-*m*

$$x^*(t) = \sum_{m=\infty}^{-\infty} a^*_{-m} e^{jm\omega_0 t} = \sum_{m=-\infty}^{\infty} a^*_{-m} e^{jm\omega_0 t}$$

Hence if

$$x(t) \stackrel{FS}{\longleftrightarrow} a_k$$

then

$$x^*(t) \stackrel{FS}{\longleftrightarrow} a^*_{-k}$$

Recall that we showed earlier that if *x(t)* is a real valued function of time, so that it must equal its complex conjugate, then the Fourier coefficients will be conjugate symmetric.

$$a^*_k = a_{-k}$$