Fluids – Lecture 13 Notes

1. Bernoulli Equation
2. Uses of Bernoulli Equation

Reading: Anderson 3.2, 3.3

Bernoulli Equation

Derivation – 1-D case
The 1-D momentum equation, which is Newton’s Second Law applied to fluid flow, is written as follows.

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} + \rho g_x + (F_x)_{\text{viscous}}$$

We now make the following assumptions about the flow.

- Steady flow: $\partial/\partial t = 0$
- Negligible gravity: $\rho g_x \approx 0$
- Negligible viscous forces: $(F_x)_{\text{viscous}} \approx 0$
- Low-speed flow: $\rho$ is constant

These reduce the momentum equation to the following simpler form, which can be immediately integrated.

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = 0$$

$$\frac{1}{2} \rho \frac{d(u^2)}{dx} + \frac{dp}{dx} = 0$$

$$\frac{1}{2} \rho u^2 + p = \text{constant} \equiv p_o$$

The final result is the one-dimensional Bernoulli Equation, which uniquely relates velocity and pressure if the simplifying assumptions listed above are valid. The constant of integration $p_o$ is called the stagnation pressure, or equivalently the total pressure, and is typically set by known upstream conditions.

Derivation – 2-D case
The 2-D momentum equations are

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \rho g_x + (F_x)_{\text{viscous}}$$

$$\rho \frac{\partial v}{\partial t} + \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \rho g_y + (F_y)_{\text{viscous}}$$

Making the same assumptions as before, these simplify to the following.

$$\rho u \frac{du}{dx} + \rho v \frac{du}{dy} + \frac{dp}{dx} = 0 \quad (1)$$

$$\rho u \frac{dv}{dx} + \rho v \frac{dv}{dy} + \frac{dp}{dy} = 0 \quad (2)$$
Before these can be integrated, we must first restrict ourselves only to flowfield variations along a streamline. Consider an incremental distance $ds$ along the streamline, with projections $dx$ and $dy$ in the two axis directions. The speed $V$ likewise has projections $u$ and $v$.

Along the streamline, we have

$$\frac{dy}{dx} = \frac{v}{u}$$

or

$$u \, dy = v \, dx \quad (3)$$

We multiply the $x$-momentum equation (1) by $dx$, use relation (3) to replace $v \, dx$ by $u \, dy$, and combine the $u$-derivative terms into a $du$ differential.

$$\rho u \, \frac{\partial u}{\partial x} \, dx + \rho v \, \frac{\partial u}{\partial y} \, dy + \frac{\partial p}{\partial x} \, dx = 0$$

$$\rho u \left( \frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy \right) + \frac{\partial p}{\partial x} \, dx = 0$$

$$\rho u \, du + \frac{\partial p}{\partial x} \, dx = 0$$

$$\frac{1}{2} \rho \, d\left( u^2 \right) + \frac{\partial p}{\partial x} \, dx = 0 \quad (4)$$

We multiply the $y$-momentum equation (2) by $dy$, and performing a similar manipulation, we get

$$\rho u \, \frac{\partial v}{\partial x} \, dy + \rho v \, \frac{\partial v}{\partial y} \, dy + \frac{\partial p}{\partial y} \, dy = 0$$

$$\rho v \left( \frac{\partial v}{\partial x} \, dx + \frac{\partial v}{\partial y} \, dy \right) + \frac{\partial p}{\partial y} \, dy = 0$$

$$\rho v \, dv + \frac{\partial p}{\partial y} \, dy = 0$$

$$\frac{1}{2} \rho \, d\left( v^2 \right) + \frac{\partial p}{\partial y} \, dy = 0 \quad (5)$$

Finally, we add equations (4) and (5), giving

$$\frac{1}{2} \rho \, d\left( u^2 + v^2 \right) + \frac{\partial p}{\partial x} \, dx + \frac{\partial p}{\partial y} \, dy = 0$$

$$\frac{1}{2} \rho \, d\left( u^2 + v^2 \right) + dp = 0$$
which integrates into the general Bernoulli equation
\[ \frac{1}{2} \rho V^2 + p = \text{constant} \equiv p_o \] (along a streamline) \hspace{1cm} (6)

where \( V^2 = u^2 + v^2 \) is the square of the speed. For the 3-D case the final result is exactly the same as equation (6), but now the \( w \) velocity component is nonzero, and hence \( V^2 = u^2 + v^2 + w^2 \).

**Irrotational Flow**

Because of the assumptions used in the derivations above, in particular the streamline relation (3), the Bernoulli Equation (6) relates \( p \) and \( V \) only along any given streamline. Different streamlines will in general have different \( p_o \) constants, so \( p \) and \( V \) cannot be directly related between streamlines. For example, the simple shear flow on the left of the figure has parallel flow with a linear \( u(y) \), and a uniform pressure \( p \). Its \( p_o \) distribution is therefore parabolic as shown. Hence, there is no unique correspondence between velocity and pressure in such a flow.

![Parallel Rotational flow](image1)

![Nonparallel Irrotational flow](image2)

However, if the flow is irrotational, i.e. if \( \vec{V} = \nabla \phi \) and \( V^2 = |\nabla \phi|^2 \), then \( p_o \) takes on the same value for all streamlines, and the Bernoulli Equation (6) becomes usable to relate \( p \) and \( V \) in the entire irrotational flowfield. Fortunately, a flowfield is irrotational if the upstream flow is irrotational (e.g. uniform), which is a very common occurrence in aerodynamics. From the uniform far upstream flow we can evaluate

\[ p_o = p_\infty + \frac{1}{2} \rho V_\infty^2 \equiv p_{o\infty} \]

and the Bernoulli equation (6) then takes the more general form.

\[ \frac{1}{2} \rho V^2 + p = p_{o\infty} \] (everywhere in an irrotational flow) \hspace{1cm} (7)

**Uses of Bernoulli Equation**

**Solving potential flows**

Having the Bernoulli Equation (7) in hand allows us to devise a relatively simple two-step solution strategy for potential flows.

1. Determine the potential field \( \phi(x, y, z) \) and resulting velocity field \( \vec{V} = \nabla \phi \) using the

\[ \frac{1}{2} \rho V^2 + p = p_{o\infty} \]
governing equations.

2. Once the velocity field is known, insert it into the Bernoulli Equation to compute the pressure field \( p(x, y, z) \).

This two-step process is simple enough to permit very economical aerodynamic solution methods which give a great deal of physical insight into aerodynamic behavior. The alternative approaches which do not rely on Bernoulli Equation must solve for \( \vec{V}(x, y, z) \) and \( p(x, y, z) \) simultaneously, which is a tremendously more difficult problem which can be approached only through brute force numerical computation.

**Venturi flow**

Another common application of the Bernoulli Equation is in a *venturi*, which is a flow tube with a minimum cross-sectional area somewhere in the middle.

Assuming incompressible flow, with \( \rho \) constant, the mass conservation equation gives

\[
A_1 V_1 = A_2 V_2
\]  

(8)

This relates \( V_1 \) and \( V_2 \) in terms of the geometric cross-sectional areas.

\[
V_2 = V_1 \frac{A_1}{A_2}
\]

Knowing the velocity relationship, the Bernoulli Equation then gives the pressure relationship.

\[
p_1 + \frac{1}{2} \rho V_1^2 = p_o = p_2 + \frac{1}{2} \rho V_2^2
\]

(9)

Equations (8) and (9) together can be used to determine the inlet velocity \( V_1 \), knowing only the pressure difference \( p_1 - p_2 \) and the geometric areas. By direct substitution we have

\[
V_1 = \sqrt{\frac{2(p_1 - p_2)}{\rho \left[ (A_1/A_2)^2 - 1 \right]}}
\]

A venturi can therefore be used as an *airspeed indicator*, if some means of measuring the pressure difference \( p_1 - p_2 \) is provided.