UE Math Review: Div, Curl, etc.

Velocity field

Figure 1 shows a Control Volume or circuit placed in a velocity field $\vec{V}(x,y)$. To evaluate the volume flow through the CV, we need to examine the velocities on the CV faces. It is convenient to work directly with the x,y velocity components u, v.
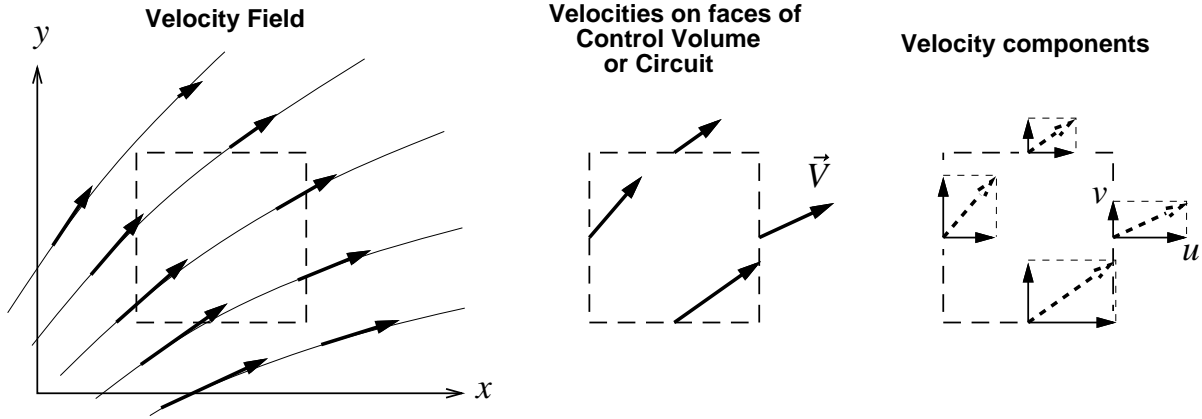

Figure 1: Velocity field and Control Volume.

Volume outflow – infinitesimal rectangular CV

Let us now assume the CV is an infinitesimal rectangle, with dimensions dx and dy. We wish to compute the net *volume outflow* (per unit z depth) out of this CV,

$$d\dot{\mathcal{V}}' \;=\; \oint \left(\vec{V} \cdot \hat{n} \right) ds \;=\; \oint \left(u\, n_x \,+\, v\, n_y \right) ds$$

where ds is the CV side arc length, either dx or dy depending on the side in question. Figure 2 shows the normal velocity components which are required. The velocities across the opposing faces 1,2 and 3,4 are related by using the local velocity gradients $\partial u/\partial x$ and $\partial v/\partial y$.
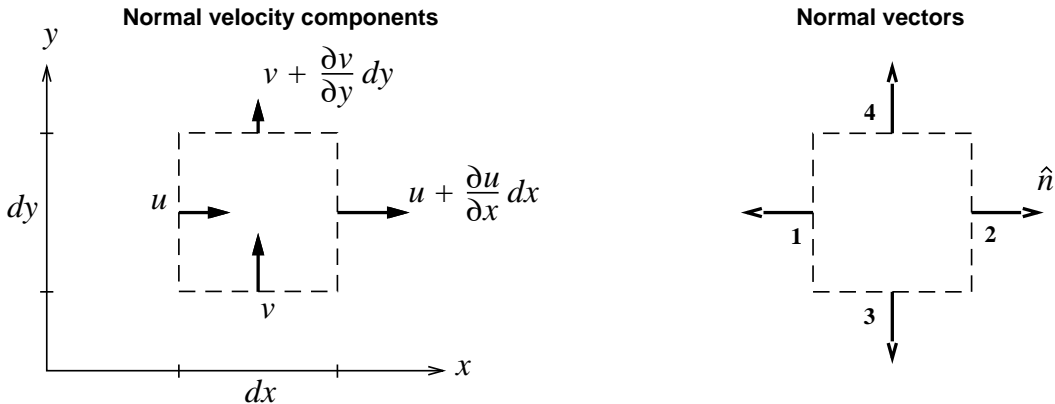

Figure 2: Infinitesimal CV with normal velocity components.

The volume outflow is then computed by evaluating the integral as a sum over the four faces.

$$d\dot{\mathcal{V}}' \;=\; \left[\left(\vec{V} \cdot \hat{n} \right) ds \right]_1 \,+\, \left[\left(\vec{V} \cdot \hat{n} \right) ds \right]_2 \,+\, \left[\left(\vec{V} \cdot \hat{n} \right) ds \right]_3 \,+\, \left[\left(\vec{V} \cdot \hat{n} \right) ds \right]_4$$

1

$$= \left(-u\right) dy + \left(u + \frac{\partial u}{\partial x} dx\right) dy + \left(-v\right) dx + \left(v + \frac{\partial v}{\partial y} dy\right) dx$$

$$= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\, dy$$

$$\text{or} \qquad d\dot{\mathcal{V}}' = \left(\nabla \cdot \vec{V}\right) dx\, dy \tag{1}$$

where $\nabla \cdot \vec{V}$ is a convenient shorthand for the velocity divergence in the parentheses. This final result for any 2-D infinitesimal CV can be stated as follows:

2-D : $\boxed{\textbf{\textit{d(volume outflow/depth) = (velocity divergence)}} \times \textbf{\textit{d(area)}}}$ $\tag{2}$

For a 3-D infinitesimal "box" CV, a slightly more involved analysis gives

$$d\dot{\mathcal{V}} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dx\, dy\, dz = \left(\nabla \cdot \vec{V}\right) dx\, dy\, dz \tag{3}$$

3-D : $\boxed{\textbf{\textit{d(volume outflow) = (velocity divergence)}} \times \textbf{\textit{d(volume)}}}$ $\tag{4}$

Volume outflow – infinitesimal triangular CV
Although statements (2) and (4) were derived for a rectangular infinitesimal CV, they are in fact correct for any infinitesimal shape. For illustration, consider the volume outflow from a right-triangular CV, pictured in Figure 3. One complication is that the unit normal vector $\hat{n}$ on the hypotenuse is not parallel to one of the axes. However, its n_x, n_y components are easily obtained from the geometry as shown in the figure.
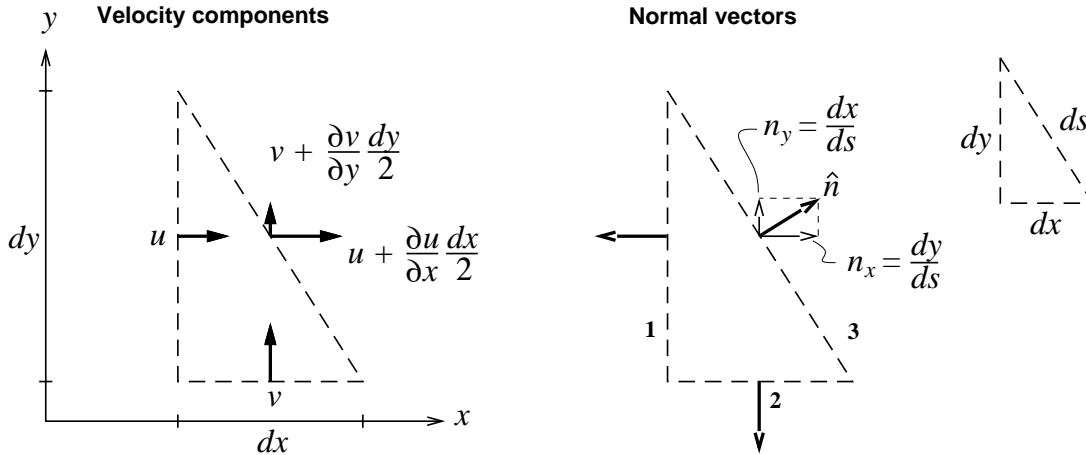

Figure 3: Infinitesimal triangular Control Volume.

The volume outflow per unit depth is now

$$d\dot{\mathcal{V}}' = \left[\left(\vec{V} \cdot \hat{n}\right) ds\right]_1 + \left[\left(\vec{V} \cdot \hat{n}\right) ds\right]_2 + \left[\left(\vec{V} \cdot \hat{n}\right) ds\right]_3$$

$$= \left(-u\right) dy + \left(-v\right) dx + \left[\left(u + \frac{\partial u}{\partial x} \frac{dx}{2}\right) \frac{dy}{ds} + \left(v + \frac{\partial v}{\partial y} \frac{dy}{2}\right) \frac{dx}{ds}\right] ds$$

2

$$= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \frac{dx\, dy}{2}$$

or $\quad d\dot{\mathcal{V}}' = \left(\nabla \cdot \vec{V}\right) \dfrac{dx\, dy}{2}$

The area of the triangular CV is $dx\, dy/2$, so statement (2) is still correct for this case.

Circulation – infinitesimal rectangular circuit

We now wish to compute the clockwise circulation $d\Gamma$ on the perimeter of the infinitesimal rectangular CV, called a *circuit* by convention.

$$-d\Gamma = \oint \vec{V} \cdot d\vec{s} = \oint \left(\vec{V} \cdot \hat{s}\right) ds$$

This now requires knowing the tangential velocity components, shown in Figure 4.

Tangential velocity components
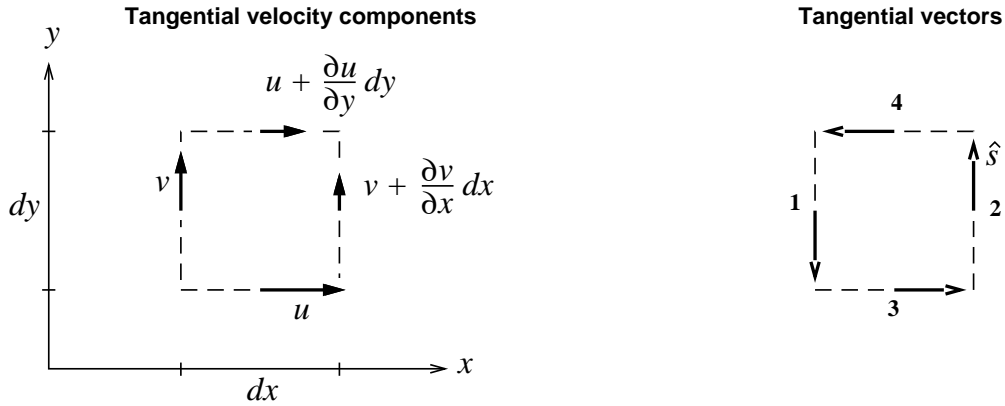
Tangential vectors

Figure 4: Infinitesimal CV with tangential velocity components.

The circulation is then computed by evaluating the integral as a sum over the four faces.

$$-d\Gamma = \left[\left(\vec{V} \cdot \hat{s}\right) ds\right]_1 + \left[\left(\vec{V} \cdot \hat{s}\right) ds\right]_2 + \left[\left(\vec{V} \cdot \hat{s}\right) ds\right]_3 + \left[\left(\vec{V} \cdot \hat{s}\right) ds\right]_4$$

$$= \left(-v\right) dy + \left(v + \frac{\partial v}{\partial x} dx\right) dy + \left(u\right) dx + \left(-u - \frac{\partial u}{\partial y} dy\right) dx$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy$$

or $\quad -d\Gamma = \left(\nabla \times \vec{V}\right) \cdot \hat{k}\, dx\, dy \qquad (5)$

where $\left(\nabla \times \vec{V}\right) \cdot \hat{k}$ is a convenient (?) shorthand for the z-component of the velocity curl in the parentheses. For a triangular circuit, equation (5) would have $dx\, dy/2$ as expected. The general result for any infinitesimal circuit can be stated as follows:

$$\boxed{\textbf{\textit{-d(circulation) = (normal component of velocity curl)}} \times \textbf{\textit{d(area)}}} \qquad (6)$$

Volume outflow – finite CV

We now wish to integrate the velocity divergence over a *finite* CV.

$$\iint \left(\nabla \cdot \vec{V}\right) dx\, dy = \iint \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dx\, dy$$

As shown in Figure 5, this is equivalent to summing the volume outflows from all the infinitesimal CVs in the finite CV's interior, either rectangular or triangular in shape, as needed to conform to the boundary. We then note that the contributions of all the interior faces

$$\iint (\nabla \cdot \vec{V})\, dx\, dy = \text{Sum}\left[\int (\vec{V} \cdot \hat{n})\, ds\right] \implies \text{interior face integrations cancel} \implies \oint (\vec{V} \cdot \hat{n})\, ds$$
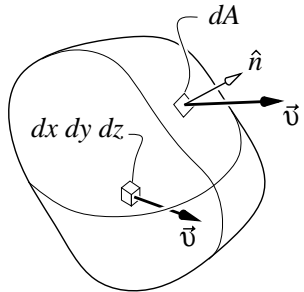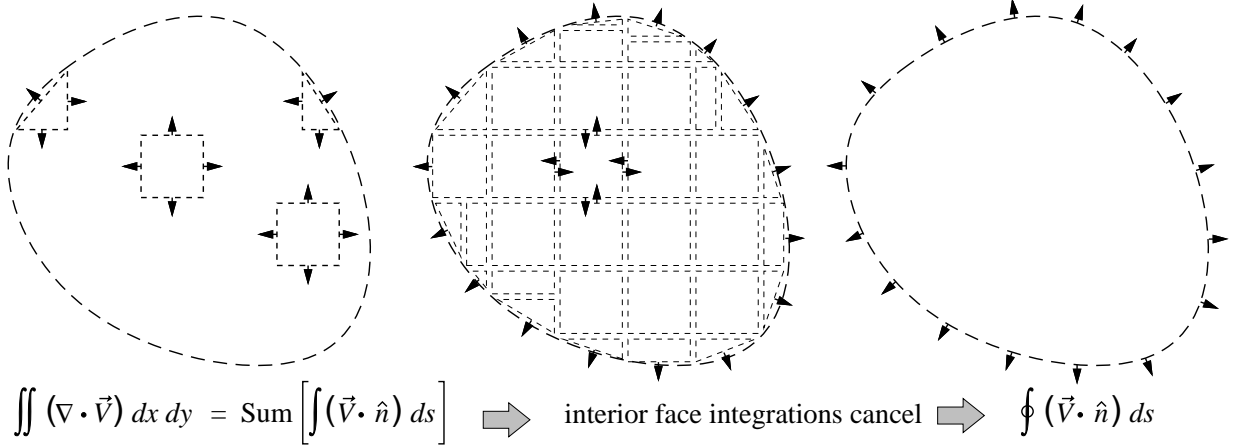
Figure 5: Summation of divergence over interior of a finite CV

cancel, since these have directly opposing $\hat{n}$ normal vectors, leaving only the boundary faces in the overall summation which the give the net outflow out of the CV.

$$\iint \left(\nabla \cdot \vec{V}\right) dx\, dy = \oint \left(\vec{V} \cdot \hat{n}\right) ds$$

This general result is known as *Gauss's Theorem*, and it applies to any vector field $\vec{v}$, not just fluid velocity fields. The general Gauss's Theorem in 3-D is

$$\boxed{\iiint \nabla \cdot \vec{v}\ dx\, dy\, dz = \oiint \vec{v} \cdot \hat{n}\ dA}$$

where the volume integral is over the interior of the CV, and the area integral is over the surface of the CV.

Circulation – finite circuit

We now wish to integrate the normal component of the curl over the interior of a *finite* circuit.

$$\iint \left(\nabla \times \vec{V}\right) \cdot \hat{k}\, dx\, dy = \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy$$

As shown in Figure 6, this is equivalent to summing the circulations from all the infinitesimal circuits in the circuit's interior, either rectangular or triangular in shape, as needed to

$$\iint (\nabla \times \vec{v}) \cdot \hat{k} \ dx \ dy \ = \ \mathrm{Sum}\left[\int \vec{V} \cdot d\vec{s}\right] \quad \Longrightarrow \quad \text{interior face integrations cancel} \quad \Longrightarrow \quad \oint \vec{V} \cdot d\vec{s}$$
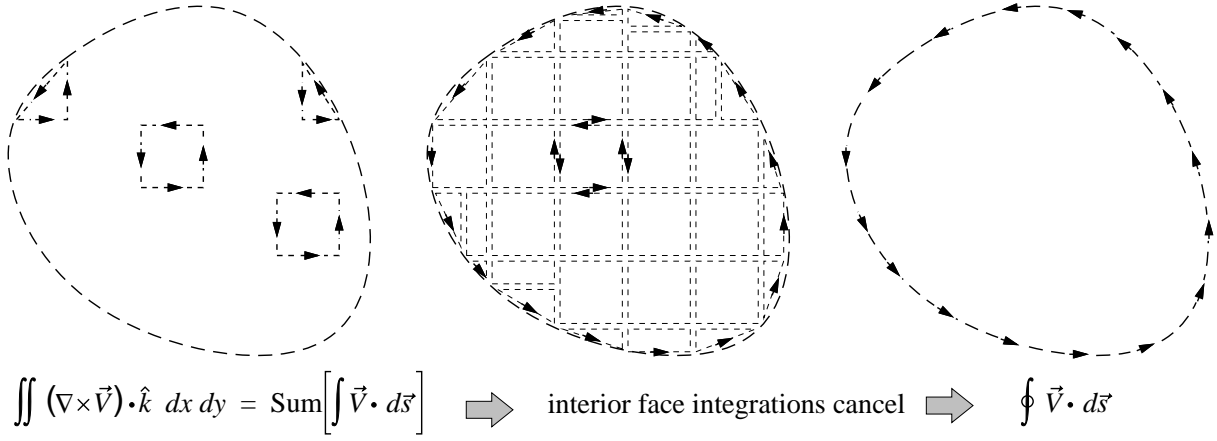
Figure 6: Summation of curl over interior of a finite circuit

conform to the boundary. We then note that the contributions of all the interior faces cancel, since these have directly opposing $\hat{s}$ tangential vectors, leaving only the boundary faces in the overall summation.

$$\iint \left(\nabla \times \vec{V}\right) \cdot \hat{k} \ dx \ dy \ = \ \oint \left(\vec{V} \cdot \hat{s}\right) \ ds$$

This general result is known as *Stokes's Theorem*, and it applies to any vector field $\vec{v}$, not just fluid velocity fields. The general Stokes's Theorem in 3-D is

$$\boxed{\iint (\nabla \times \vec{v}) \cdot \hat{n} \ dA \ = \ \oint \vec{v} \cdot d\vec{s}}$$
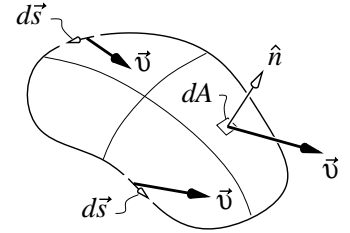
where the area integral is over any "potato chip" surface spanned by the circuit, and the line integral is over the circuit itself. The circuit and spanned surface do not need to be flat.

Constant velocity field contribution

It is useful at this point to observe that adding a constant vector $\vec{V}_{\mathrm{C}}$ to the velocity field $\vec{V}(x,y)$ has no net effect on the calculation of $\dot{\mathcal{V}}'$ or Γ. Because the div and curl operations involve derivatives, any constant component in $\vec{V}$ will disappear, e.g.

$$\nabla \cdot \left(\vec{V} + \vec{V}_{\mathrm{C}}\right) \ = \ \nabla \cdot \vec{V} \ + \ \nabla \cdot \vec{V}_{\mathrm{C}} \ = \ \nabla \cdot \vec{V} \tag{7}$$

$$\nabla \times \left(\vec{V} + \vec{V}_{\mathrm{C}}\right) \ = \ \nabla \times \vec{V} \ + \ \nabla \times \vec{V}_{\mathrm{C}} \ = \ \nabla \times \vec{V} \tag{8}$$

Similarly, the normal or tangential integral of a constant around a closed CV or circuit is zero.

$$\oint \vec{V}_{\mathrm{C}} \cdot \hat{n} \ ds \ = \ 0$$

$$\oint \vec{V}_{\mathrm{C}} \cdot \hat{s} \ ds \ = \ 0$$

These properties can be used to help "visualize" the divergence or curl of the velocity field. We can take the four face velocities shown in Figure 1 and define their average

$$\vec{V}_{\mathrm{avg}} \ = \ \frac{1}{4}\left(\vec{V}_1 \ + \ \vec{V}_2 \ + \ \vec{V}_3 \ + \ \vec{V}_4\right)$$

and subtract this constant $\vec{V}_{\mathrm{avg}}$ from each velocity to give the local deviations $\Delta\vec{V}$, as shown in Figure 7.
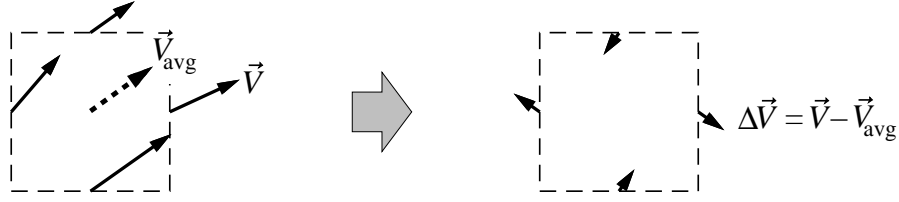

Figure 7: Average velocity subtracted out to give local velocity deviations

From (7) and (8) we see that

$$
\begin{aligned}
\nabla \cdot \left(\Delta\vec{V}\right) &= \nabla \cdot \vec{V} \\
\nabla \times \left(\Delta\vec{V}\right) &= \nabla \times \vec{V}
\end{aligned}
$$

so that these velocity deviations have exactly the same $\dot{\mathcal{V}}'$ and Γ as the full velocities. Removal of the large constant $\vec{V}_{\mathrm{avg}}$ makes it easier to visualize a net outflow or circulation around the perimeter. Figure 8 shows four $\vec{V}$ distributions, along with their corresponding $\Delta\vec{V}$ distributions. The latter readily indicate zero or nonzero $\dot{\mathcal{V}}'$ or Γ, or equivalently, zero or nonzero divergence or curl.
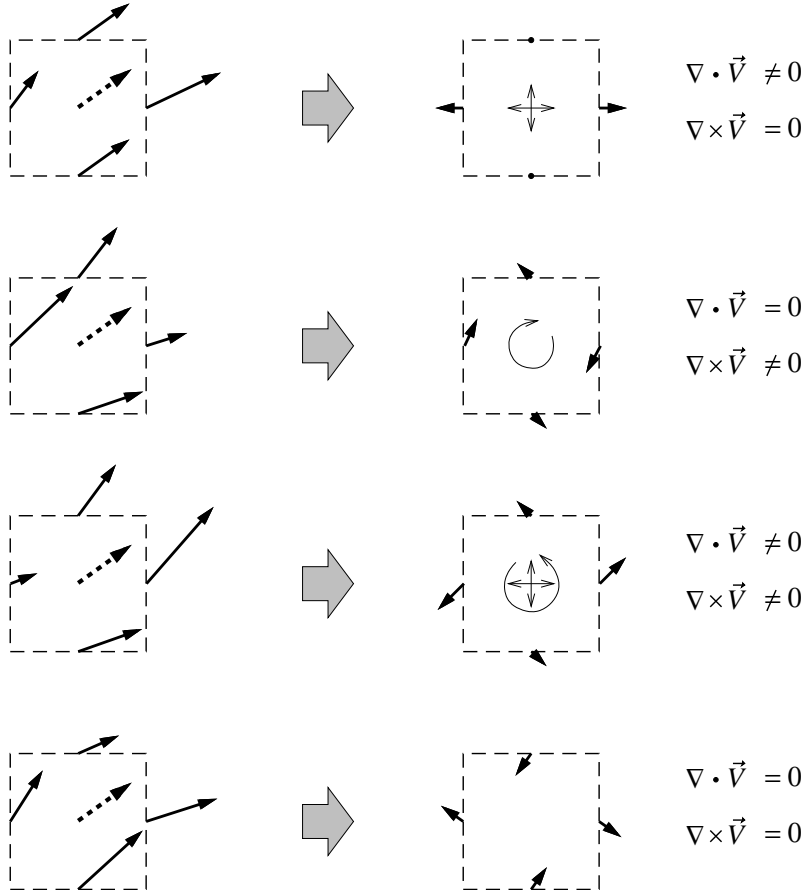

Figure 8: Example velocity distributions around a CV or circuit.