

UE Math Review: Grad, etc.

Pressure field

Figure 1 shows a Control Volume or circuit placed in a pressure field $p(x, y)$. To evaluate the pressure force on the CV, we need to examine the pressures on the CV faces.

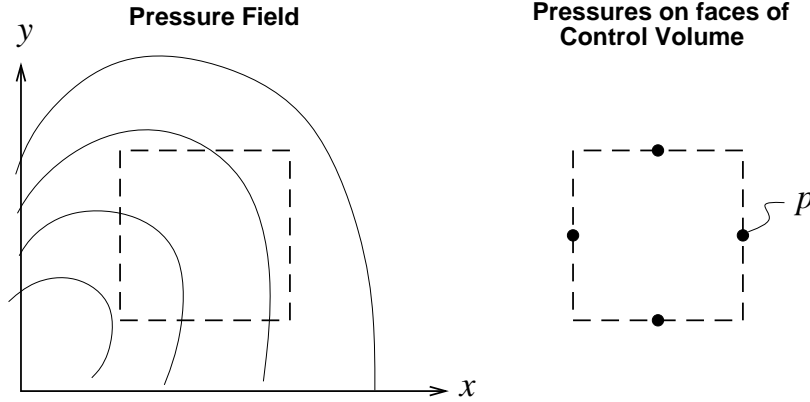


Figure 1: Pressure field and Control Volume.

Pressure force – infinitesimal rectangular CV

Let us now assume the CV is an infinitesimal rectangle, with dimensions dx and dy . We wish to compute the net *pressure force* (per unit z depth) on this CV,

$$-d\vec{F}' = \oint (p \hat{n}) ds$$

where ds is the CV side arc length, either dx or dy depending on the side in question. As shown in Figure 2, the pressures across the opposing faces 1,2 and 3,4 are related by using the local pressure gradients $\partial p / \partial x$ and $\partial p / \partial y$.

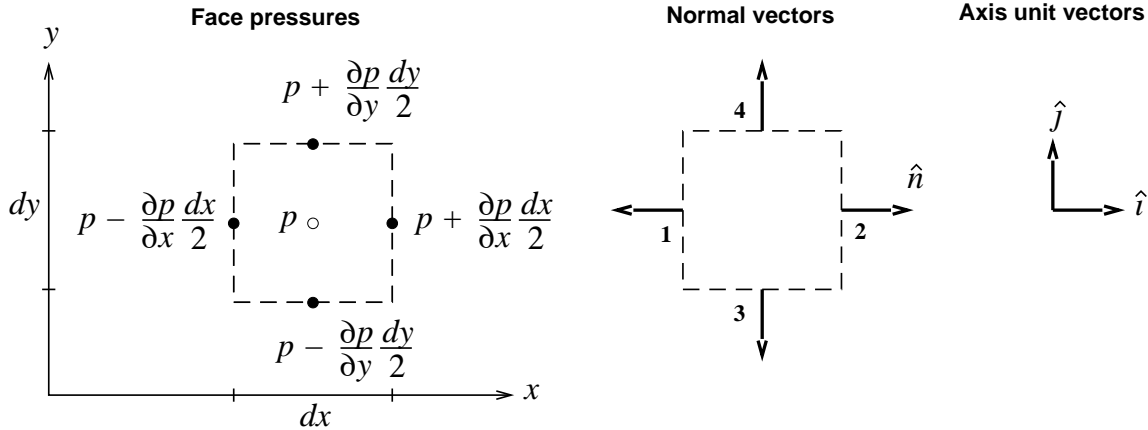


Figure 2: Infinitesimal CV surface pressures and normal vectors.

The pressure force is then computed by evaluating the integral as a sum over the four faces.

$$\begin{aligned} -d\vec{F}' &= \left[(p \hat{n}) ds \right]_1 + \left[(p \hat{n}) ds \right]_2 + \left[(p \hat{n}) ds \right]_3 + \left[(p \hat{n}) ds \right]_4 \\ &= \left(-p + \frac{\partial p}{\partial x} \frac{dx}{2} \right) \hat{i} dy + \left(p + \frac{\partial p}{\partial x} \frac{dx}{2} \right) \hat{i} dy + \left(-p + \frac{\partial p}{\partial y} \frac{dy}{2} \right) \hat{j} dx + \left(p + \frac{\partial p}{\partial y} \frac{dy}{2} \right) \hat{j} dx \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} \right) dx dy \\
-d\vec{F}' &= (\nabla p) dx dy
\end{aligned} \tag{1}$$

where ∇p is a convenient shorthand for the gradient expression in the parentheses. This final result for any 2-D infinitesimal CV can be stated as follows:

$$\text{2-D : } \boxed{-d(\text{force/depth}) = (\text{pressure gradient}) \times d(\text{area})} \tag{2}$$

For a 3-D infinitesimal “box” CV, a slightly more involved analysis gives

$$-d\vec{F} = \left(\frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} + \frac{\partial p}{\partial z} \hat{k} \right) dx dy dz = (\nabla p) dx dy dz \tag{3}$$

$$\text{3-D : } \boxed{-d(\text{force}) = (\text{pressure gradient}) \times d(\text{volume})} \tag{4}$$

Pressure force – finite CV

We now wish to integrate the pressure gradient over a *finite* CV.

$$\iint (\nabla p) dx dy = \iint \left(\frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} \right) dx dy$$

As shown in Figure 3, this is equivalent to summing the pressure forces on all the infinitesimal CVs in the finite CV's interior. We then note that the contributions of all the interior faces

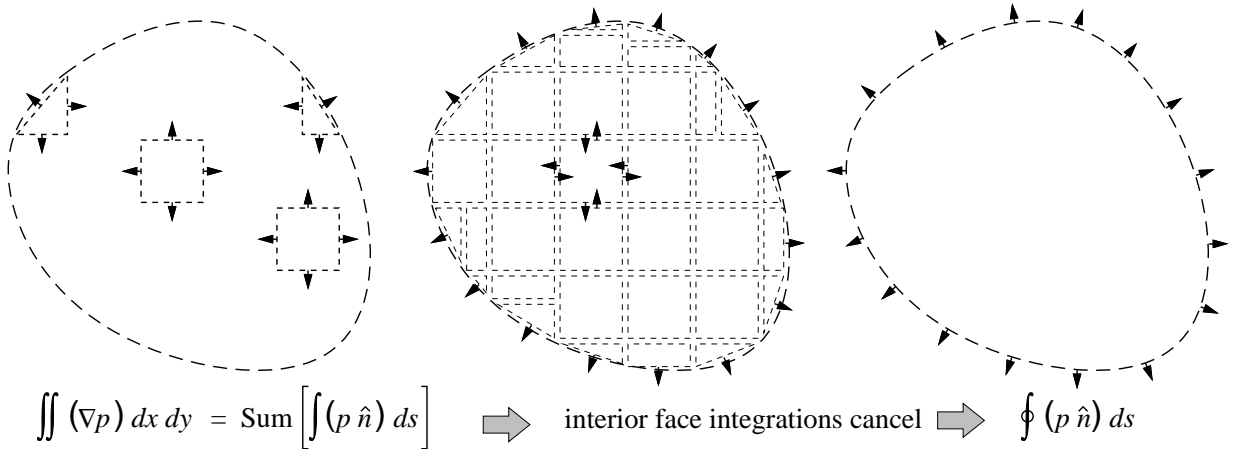


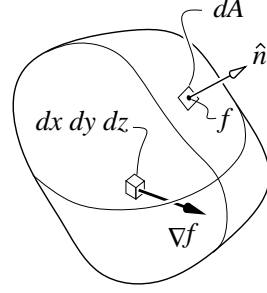
Figure 3: Summation of pressure gradient over interior of a finite CV

cancel, since these have directly opposing \hat{n} normal vectors, leaving only the boundary faces in the overall summation which give the net pressure force on the CV.

$$\iint (\nabla p) dx dy = \oint (p \hat{n}) ds$$

This general result is known as the *Gradient Theorem*, and it applies to any scalar field f , not just fluid pressure fields. The general Gradient Theorem in 3-D is

$$\iiint \nabla f \, dx \, dy \, dz = \oint f \, \hat{n} \, dA$$



where the volume integral is over the interior of the CV, and the area integral is over the surface of the CV.

Constant pressure field contribution

It is useful to note that adding a constant pressure p_c to the pressure field $p(x, y)$ has no net effect on the calculation of \vec{F}' . Because the grad operation involves derivatives, any constant component in p will disappear, e.g.

$$\nabla (p + p_c) = \nabla p + \nabla p_c = \nabla p \quad (5)$$

Similarly, the normal vector integral of a constant around a closed CV is zero.

$$\oint p_c \, \hat{n} \, ds = 0$$

These properties can be used to see the important feature of a pressure field. We can take the four face pressures shown in Figure 1 and define their average

$$p_{\text{avg}} = \frac{1}{4} (p_1 + p_2 + p_3 + p_4)$$

and subtract this constant p_{avg} from each pressure to give the local deviations Δp , as shown in Figure 4.

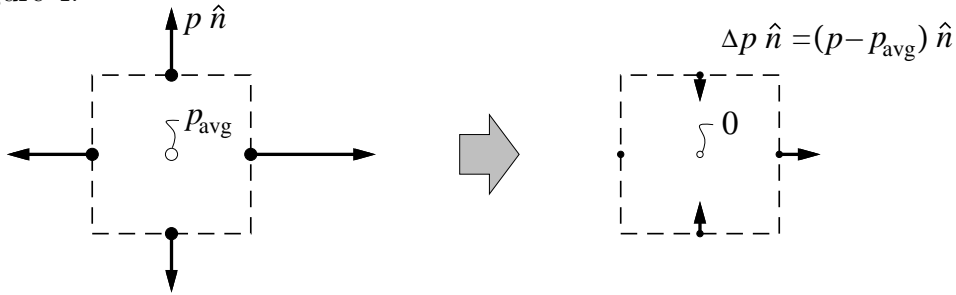


Figure 4: Average pressure subtracted out to give local pressure force deviations

From (5) we see that

$$\nabla (\Delta p) = \nabla p$$

so that these pressure deviations have exactly the same \vec{F}' as the full pressure. Removal of the large constant p_{avg} makes it easier to visualize the net magnitude and direction of the pressure force acting on the CV.