# Block 2:

# Stress and Strain

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# Unit M2.1
# (More) Language for Stress and Strain

Readings:

CDL 4.1

16.001/002 -- *"Unified Engineering"*
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# LEARNING OBJECTIVES FOR UNIT **M2.1**

*Through participation in the lectures, recitations, and work associated with Unit M2.1, it is intended that you will be able to………*

- ….**employ** the tensor/indicial notation to express equations and relations
- ….**recognize, explain,** and **apply** two special parameters (Kronecker delta, permutation tensor)

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Many times in engineering, a number of equations of similar form need to be written.

This will be particularly true as we look at stress and strain in this section/block.

In the earlier "U" lectures, we looked at such cases (as for vectors):

$$\underline{R} = R_1 \, \underline{i}_1 + R_2 \, \underline{i}_2 + R_3 \, \underline{i}_3$$

And we have seen this written as:

$$\underline{R} = \sum_{m=1}^{3} R_m \, \underline{i}_m$$

This suggests a "shorthand" often used in engineering known as:

# Tensor (/Summation/Indicial) Notation

- "Easy" to write complicated formulae
- "Easy" to mathematically manipulate
- "Elegant", rigorous
- Use for derivations or to succinctly express a set of equations or a long equation

Example: $x_i = f_{ij} y_j$

--> Rules for subscripts

NOTE: index ≡ subscript

- Latin subscripts (m, n, p, q, …) take on the values 1, 2, 3 (3-D)
- Greek subscripts ($\alpha$, $\beta$, $\gamma$ …) take on the values 1, 2 (2-D)
- When subscripts are repeated on one side of the equation within one term, they are called *dummy indices* and are to be summed on

Thus:

$$f_{ij}\, y_j = \sum_{j=1}^{3} f_{ij}\, y_j$$

*But* $f_{ij}\, y_j + g_i$ ... **do not sum on i !**

- Subscripts which appear only once on the left side of the equation within one term are called free indices and represent a separate equation

Thus:

$x_i$ = .....

$\Rightarrow$ $x_1$ = .....

$x_2$ = .....

$x_3$ = .....

- No subscript can appear more than twice in a single term

Thus: $x_i = f_{ij} \, y_j$

i = free index

j = dummy index

represents:

$$x_1 = f_{11} y_1 + f_{12} y_2 + f_{13} y_3$$
$$x_2 = f_{21} y_1 + f_{22} y_2 + f_{23} y_3$$
$$x_3 = f_{31} y_1 + f_{32} y_2 + f_{33} y_3$$

--> To go along with tensor notation, we introduce two useful parameters

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1. Kronecker delta

$$\delta_{mn} = \begin{cases} 1 & \text{when m = n} \\ 0 & \text{when m} \neq \text{n} \end{cases}$$

Where does this come from?

Consider dot products of unit vectors:

$$\underline{i}_m \cdot \underline{i}_m = 1 \quad \text{(parallel)}$$

$$\underline{i}_m \cdot \underline{i}_n = 0 \quad \text{(perpendicular)}$$

So we see: $\delta_{mn} = \underline{i}_m \cdot \underline{i}_n$

So dot product of two vectors becomes:

$$\begin{aligned} \underline{F} \cdot \underline{G} &= F_m \underline{i}_m \cdot G_n \underline{i}_n \\ &= F_m G_n \left(\underline{i}_m \cdot \underline{i}_n\right) \\ &= F_m G_n \delta_{mn} \end{aligned}$$

--> useful elsewhere as well

## 2. Permutation tensor

$$e_{mnp} = \begin{cases} 0 & \text{when any two indices are equal} \\ 1 & \text{when } mnp \text{ is even permutation of 1, 2, 3} \\ -1 & \text{when } mnp \text{ is odd permutation of 1, 2, 3} \end{cases}$$

(even permutation: find 1 and can progress to 2, 3)

(odd permutation: can't!)

| | |
|---|---|
| 123 | 132 |
| 312 | 213 |
| 231 | 321 |

So where does this one come from?

Consider cross products of unit vectors:

$$\underline{i}_1 \times \underline{i}_1 = 0 \qquad \underline{i}_1 \times \underline{i}_2 = \underline{i}_3 \qquad \underline{i}_2 \times \underline{i}_1 = -\underline{i}_3$$

$$\underline{i}_2 \times \underline{i}_2 = 0 \qquad \underline{i}_2 \times \underline{i}_3 = \underline{i}_1 \qquad \underline{i}_3 \times \underline{i}_2 = -\underline{i}_1$$

$$\underline{i}_3 \times \underline{i}_3 = 0 \qquad \underline{i}_3 \times \underline{i}_1 = \underline{i}_2 \qquad \underline{i}_1 \times \underline{i}_3 = -\underline{i}_2$$

So: $\underline{i}_m \times \underline{i}_n = e_{mnp}\, \underline{i}_p$

Example:

$$\underline{i}_1 \times \underline{i}_2 = \underbrace{e_{121}}_{0} \underline{i}_1 + \underbrace{e_{122}}_{0} \underline{i}_2 + \underbrace{e_{123}}_{1} \underline{i}_3$$

$$\Rightarrow \underline{i}_1 \times \underline{i}_2 = \underline{i}_3$$

So a general vector cross-product can be written as:

$$\underline{H} = \underline{F} \times \underline{G} = F_m \, \underline{i}_m \times G_n \, \underline{i}_n$$

$$= F_m \, G_n \left( \underline{i}_m \times \underline{i}_n \right)$$

$$= F_m \, G_n \, e_{mnp} \, \underline{i}_p$$

So this represents a shorthand we will find quite useful.

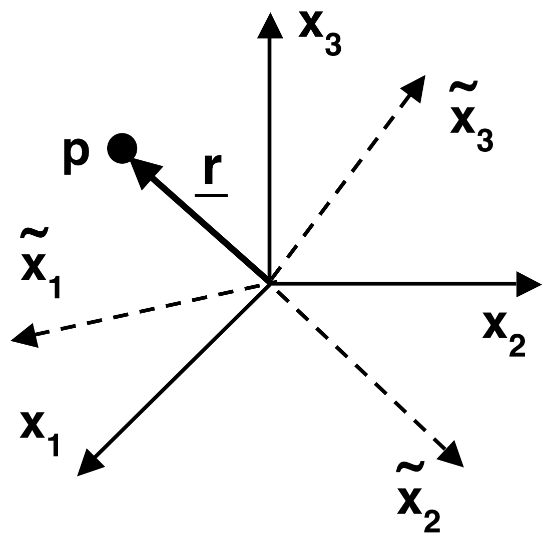
To illustrate this via an example, let's

# Revisit Transformation of Coordinates

Issue is describing the same "thing" in 2 different coordinate systems.

--> Consider this formally via the mathematics:

***Figure M2.1-1*** **Two rectangular cartesian coordinate systems with the same origin:**




~ = "tilde" ⇒ rotated coordinate system

point p is located by the vector r in both systems:

$$\underline{r} = x_1 \underline{i}_1 + x_2 \underline{i}_2 + x_3 \underline{i}_3 = x_m \underline{i}_m$$

and

$$\underline{r} = \tilde{x}_1 \tilde{\underline{i}}_1 + \tilde{x}_2 \tilde{\underline{i}}_2 + \tilde{x}_3 \tilde{\underline{i}}_3 = \tilde{x}_n \tilde{\underline{i}}_n$$

--> To relate $x_m$ to $\tilde{x}_n$, let's take the dot product of both sides with $\tilde{\underline{i}}_1$:

$$\tilde{\underline{i}}_1 \cdot x_m \underline{i}_m = \tilde{\underline{i}}_1 \cdot \tilde{x}_n \tilde{\underline{i}}_n$$

use Kronecker delta: $( \tilde{\underline{i}}_m \cdot \tilde{\underline{i}}_n = \delta_{\tilde{m}\tilde{n}} )$

$$\Rightarrow \tilde{\underline{i}}_1 \cdot x_m \underline{i}_m = \tilde{x}_n \delta_{\tilde{1}\tilde{n}}$$

But $\delta_{\tilde{1}\tilde{n}}$ is non zero only if $\tilde{n} = \tilde{1}$. Thus:

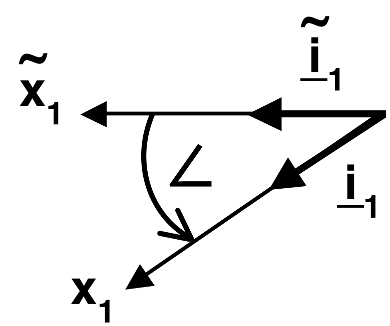
$$\tilde{x}_1 = x_1 \tilde{\underline{i}}_1 \cdot \underline{i}_1 + x_2 \tilde{\underline{i}}_1 \cdot \underline{i}_2 + x_3 \tilde{\underline{i}}_1 \cdot \underline{i}_3 \quad (*)$$

Recall definition of dot product:

$$\tilde{\underline{i}}_1 \cdot \underline{i}_1 = \underbrace{\left|\tilde{\underline{i}}_1\right|}_{1} \underbrace{\left|\underline{i}_1\right|}_{1} \cos\left(\widehat{\tilde{x}_1 \, x_1}\right)$$

$$= \cos \widehat{\tilde{x}_1 \, x_1}$$

angle from $\tilde{x}_1$ axis to $x_1$ axis = $\angle \tilde{x}_1 x_1$



Generalizing get:

$$\boxed{\ell_{\tilde{n}m} = \cos \widehat{\tilde{x}_n\, x_m} = \underline{\tilde{i}}_n \cdot \underline{i}_m}$$ = <u>Direction Cosine</u>

So (*) can be written using direction cosines and indicial notation:

$$\tilde{x}_1 = \ell_{\tilde{1}m}\, x_m$$

Similarly:

$$\tilde{x}_2 = \ell_{\tilde{2}m}\, x_m$$

$$\tilde{x}_3 = \ell_{\tilde{3}m}\, x_m$$

We have a <u>free</u> <u>index</u> which ranges over the values 1, 2, 3, so these 3 equations can be represented as:

$$\boxed{\tilde{x}_n = \ell_{\tilde{n}m}\, x_m}$$

Can also show the reverse

$$\boxed{x_m = \ell_{m\tilde{n}}\, \tilde{x}_n}$$

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And can transform forces, etc. via:

$$\tilde{F}_n = \ell_{\tilde{n}m} F_m$$

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