Unit M2.2

(All About) Stress

Readings:

CDL 4.2, 4.3, 4.4

16.001/002 -- *"Unified Engineering"*
Department of Aeronautics and Astronautics
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LEARNING OBJECTIVES FOR UNIT **M2.2**

Through participation in the lectures, recitations, and work associated with Unit M2.2, it is intended that you will be able to………

- ….**explain** the concept and types of stress and how such is manifested in materials and structures
- ….**use** the various ways of **describing** states of stress
- ….**apply** the concept of equilibrium to the state of stress

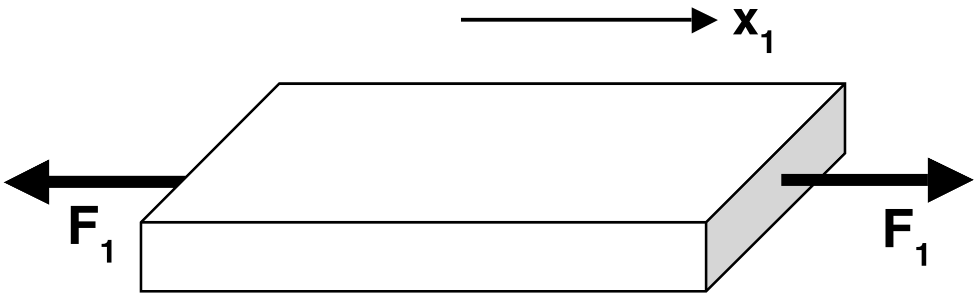
Thus far we've talked about loads, both external and internal. But when we talk about how a structure carries a load internally, we need to consider not just how this is carried in an aggregate sense, but how it is carried point to point. For this we need to introduce…..

The Concept of Stress

Definition: Stress is the measure of the intensity of force acting at a point

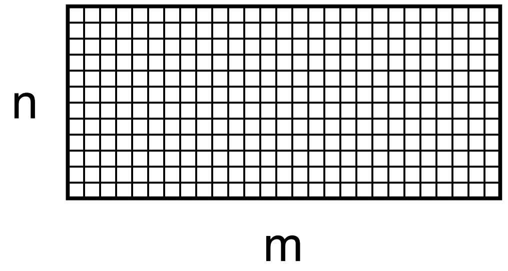
--> Think about this as follows:

Figure M2.2-1 **Representation of a bar with a colinear force acting in the x_1 direction**



Now divide the x_1 - force surface into a *n x m* grid

Figure M2.2-2 **Representation of bar cross-section divided into *n x m* grid**



$$\Delta F_1 = \frac{F_1}{n \cdot m}$$

$$\Delta A_{(1)} = \frac{A_{(1)}}{n \cdot m} \longleftarrow \text{on (1) surface}$$

A = total area
Δ = individual block area

The stress is the intensity, thus let *n* and *m* go to infinity (so $\Delta A \rightarrow 0$)

$$\underline{\sigma}_1 = \lim_{\substack{n \to \infty \\ m \to \infty}} \frac{\Delta F_1}{\Delta A_{(1)}}$$

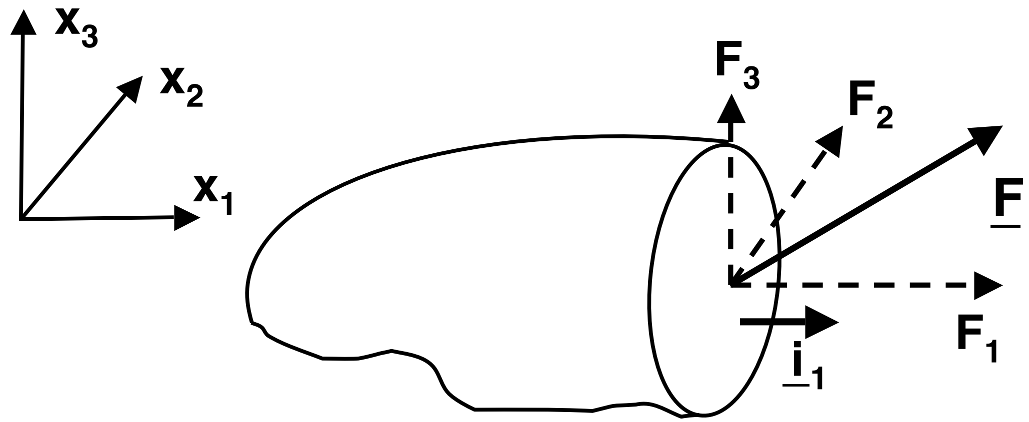
Units: $\left[\frac{Force}{(length^2)}\right]$

Where $\underline{\sigma}_1$ is the stress at a point on the x_1 - *face*

- magnitude σ_1
- direction $\underline{i}_1$

--> Consider a more generalized case

Figure M2.2-3 **Cross-section of general body subjected to internal force**



Force $\underline{F}$ is acting on x_1 - *face* ($\underline{i}_1$ is normal to plane of face)

Resolve force into three components:

$$\underline{F} = F_1 \underline{i}_1 + F_2 \underline{i}_2 + F_3 \underline{i}_3$$

Then take the limit as the force on the face is carried by a smaller set of areas (bigger grid) until we have:

$$\lim_{\Delta A \to 0} \frac{\Delta \underline{F}}{\Delta A_{(1)}} = \underline{\sigma}_1$$

This gives:

$$\underline{\sigma}_1 = \underset{\substack{\| \\ \frac{\Delta F_1}{\Delta A_1}}}{\sigma_{11}}\, \underline{i}_1 + \underset{\substack{\| \\ \frac{\Delta F_2}{\Delta A_1}}}{\sigma_{12}}\, \underline{i}_2 + \underset{\substack{\| \\ \frac{\Delta F_3}{\Delta A_1}}}{\sigma_{13}}\, \underline{i}_3$$

where $\underline{\sigma}_1$ is the stress vector and the σ_{mn} are the components acting in the three orthogonal directions

We can do the same on the x_2 and x_3 faces and get:

$$\underline{\sigma}_2 = \sigma_{21} \underline{i}_1 + \sigma_{22} \underline{i}_2 + \sigma_{23} \underline{i}_3 \quad \text{(on } \underline{i}_2 \text{ - face)}$$

$$\underline{\sigma}_3 = \sigma_{31} \underline{i}_1 + \sigma_{32} \underline{i}_2 + \sigma_{33} \underline{i}_3 \quad \text{(on } \underline{i}_3 \text{ - face)}$$

Note that using tensor format we can write this as:

$$\boxed{\underline{\sigma}_m = \sigma_{mn} \underline{i}_n}$$

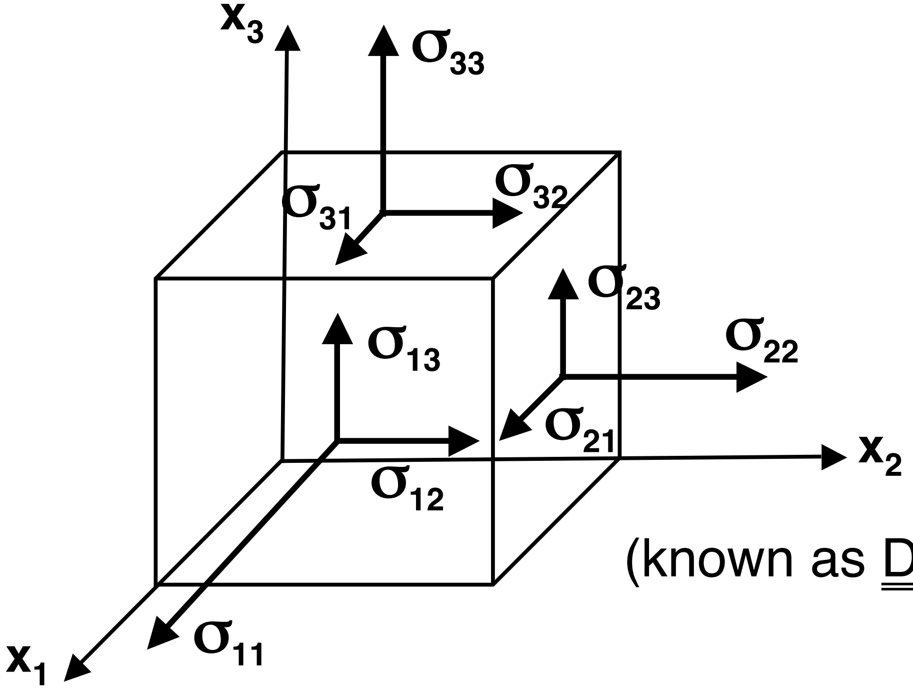
This leads us to consider the…..

Stress Tensor and Stress Types

σ_{mn} is the stress tensor

and this has particular meaning.

Figure M2.2-4 **Infinitesimal element (a cube) representing a very small piece of a body**



(known as Differential Element)

σ_{mn} tells the face and direction of the stress

stress acts in n-direction

stress acts on face with normal vector in the m-direction

Note: one important convention

- If face has "**positive normal**", positive stress is in **positive** direction
- If face has "**negative normal**", positive stress is in **negative** direction

--> Also note that there are two types of stress:

- Acts normal to the face = Normal/extensional stress
- Acts in-plane of face = Shear/stress

What do these terms mean?

Normal/extensional -- extends element

and

Shear -- caused angular changes (think of deck of cards)

So there are (at first) **9** components of the stress tensor

σ_{11}

σ_{22}

σ_{33}

extensional

σ_{12} σ_{21}

σ_{23} σ_{32}

σ_{31} σ_{13}

shear

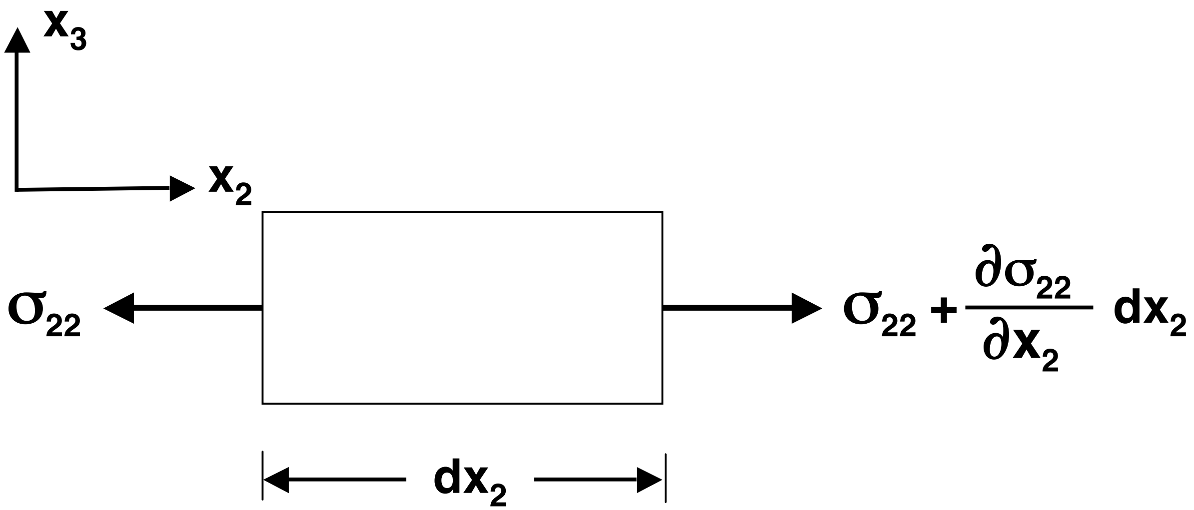
But not all of these are independent due to the

Symmetry of Stress Tensor

One needs to consider the Moment Equilibrium of the differential element

--> Note important notation/convention:

Figure M2.2-5 Consider x_2 faces and stresses acting on such



- σ_{22} acts on one side
- length of element in x_2 direction is *dx_2*
- on other side, stress is: $\sigma_{22} + \frac{\partial \sigma_{22}}{\partial x_2} dx_2$

$$\sigma_{22} + \underbrace{\frac{\partial \sigma_{22}}{\partial x_2}}_{\text{rate of change of } \sigma_{22} \text{ with respect to } x_2} \underbrace{dx_2}_{\text{infinitesimal length over which change is occurring}}$$

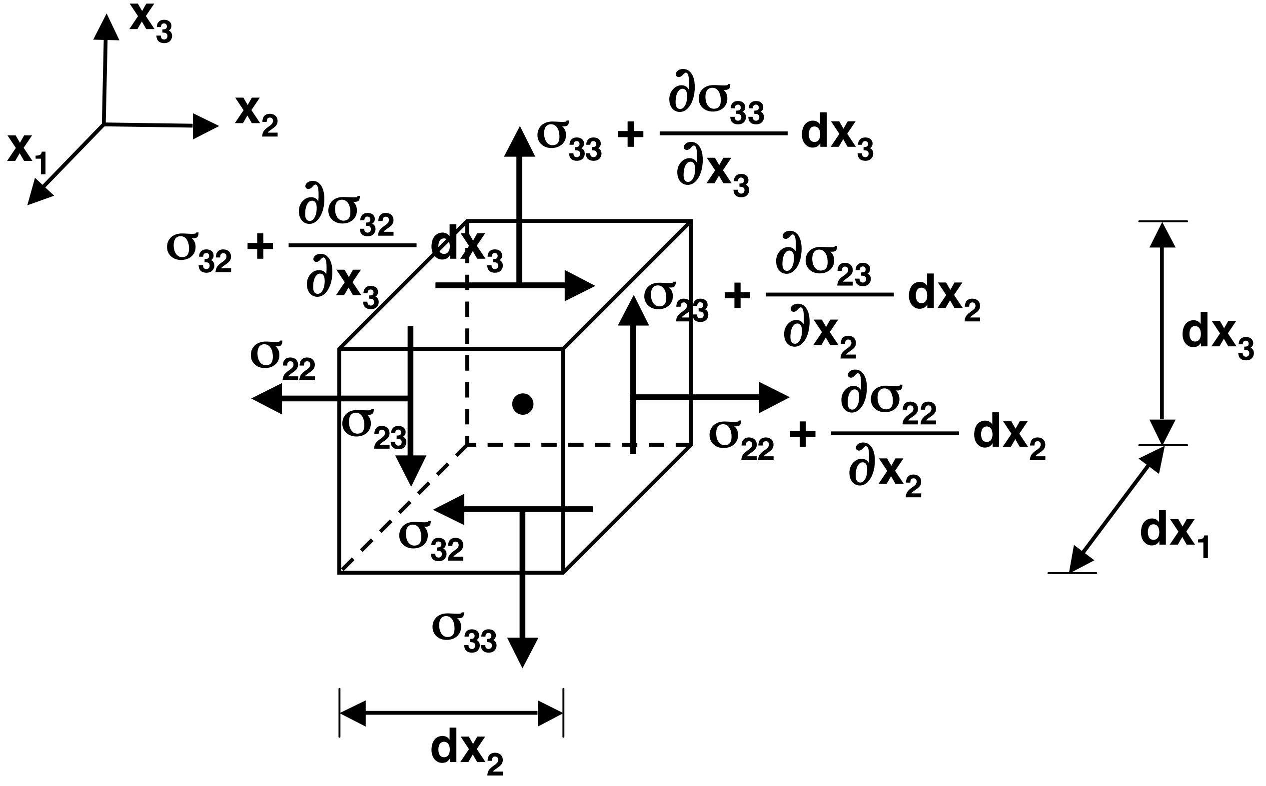
base value

σ_{22} is field variable: $\sigma_{22}\left(x_1,\ x_2,\ x_3\right)$

Do the same with all stresses (and all directions!)

Let's, using this concept, consider moment equilibrium about the x_1 axis
(Thus, any stresses with a subscript of 1 do not contribute since they act parallel to x_1 or have no moment arm)

Figure M2.2-6 **Differential stress element under gradient stress field**



Take moments about center of cube: ↺+

$$\sum M = 0 \quad \circlearrowleft +$$

--> Note that σ_{22}, σ_{33} and associated stresses act through *m* (no moment arm,) so they don't contribute. This leaves us with:

$$\sigma_{23}\left(dx_1\, dx_3\right)\frac{dx_2}{2} - \sigma_{32}\left(dx_1 dx_2\right)\frac{dx_3}{2}$$

$$+ \left(\sigma_{23} + \frac{\partial \sigma_{23}}{\partial x_2}\, dx_2\right)\left(dx_1 dx_3\right)\frac{dx_2}{2}$$

$$- \left(\sigma_{32} + \frac{\partial \sigma_{32}}{\partial x_3}\, dx_3\right)\left(dx_1 dx_2\right)\frac{dx_3}{2} = 0$$

(<u>Note</u>: Form is
(stress) (area) (moment arm))

Canceling out common $dx_1\, dx_2\, dx_3$ and 1/2 leaves:

$$\sigma_{23} - \sigma_{32} + \sigma_{23} + \frac{\partial \sigma_{23}}{\partial x_2}\, dx_2 - \sigma_{32} + \frac{\partial \sigma_{32}}{\partial x_3}\, dx_3 = 0$$

The two derivatives are “higher order terms” (HOT’s) which we can disregard to first order. Why?

(differential) (differential) = very small

So we’re left with:

$$\sigma_{23} - \sigma_{32} + \sigma_{23} - \sigma_{32} = 0$$

$$\Rightarrow \sigma_{23} = \sigma_{32} \text{ !}$$

symmetry!

Similar moment equilibrium about the other two axes yields:

(about x_2) $\sigma_{13} = \sigma_{31}$

(about x_3) $\sigma_{12} = \sigma_{21}$

Thus, in general:

$$\boxed{\sigma_{mn} = \sigma_{nm}}$$

Stress tensor is $\underline{\underline{\text{symmetric}}}$

This leaves us with 6 independent components of the stress tensor

σ_{11}
σ_{22}
σ_{33}

Extensional

$\sigma_{23} = \sigma_{32}$ rotation about x_1
$\sigma_{13} = \sigma_{31}$ rotation about x_2
$\sigma_{12} = \sigma_{21}$ rotation about x_3

Shear

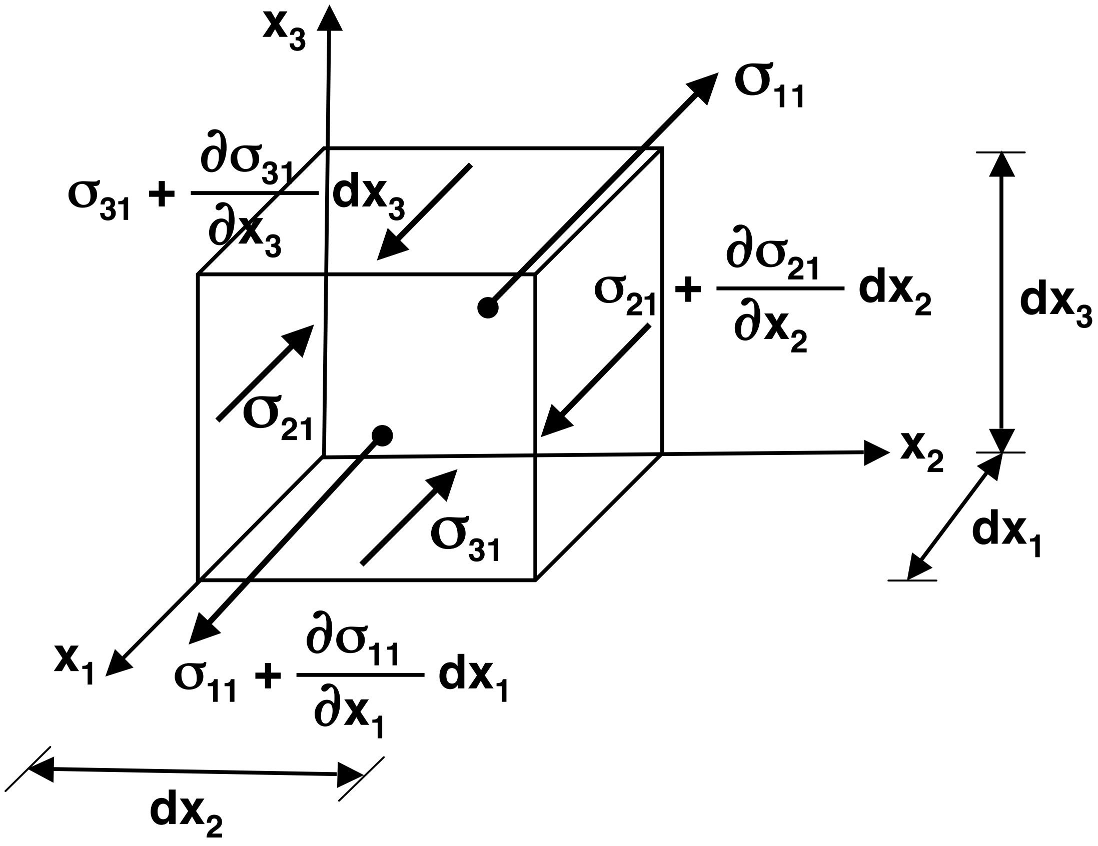
Thus far we've just defined the stress tensor and we've used moment equilibrium to show the symmetry of the stress tensor. But there are other relations between these stress components. As in most structural problems, we always apply equilibrium. This will give us the….

Stress Equations of Equilibrium

We can still apply the three equations of force equilibrium. This will give us three relations among the stress components

Let's do this with the x_1 - direction. Thus, we must consider all stresses with a second subscript of 1 (implies acts in x_1 - direction):

Figure M2.2-7 Infinitesimal stress element and all stresses acting in x_1 - direction



Now, $\sum F_1 = 0$ ↙+

form is (stress) (area)

$$i_1 - face: \left(\overset{①}{\cancel{\sigma_{11}}} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1 \right) \left(dx_2\, dx_3 \right) - \left(\overset{①}{\cancel{\sigma_{11}}} \right) \left(dx_2\, dx_3 \right)$$

$$i_2 - face: \left(\overset{②}{\cancel{\sigma_{21}}} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2 \right) \left(dx_1\, dx_3 \right) - \left(\overset{②}{\cancel{\sigma_{21}}} \right) \left(dx_1\, dx_3 \right)$$

$$i_3 - face: \left(\overset{③}{\cancel{\sigma_{31}}} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3 \right) \left(dx_1\, dx_2 \right) - \left(\overset{③}{\cancel{\sigma_{31}}} \right) \left(dx_1\, dx_2 \right)$$

$$+ \underbrace{f_1}_{\text{body force}} \underbrace{dx_1\, dx_2\, dx_3}_{\text{(acts over entire volume)}} = 0$$

Canceling out terms and dividing through by the common $dx_1\, dx_2\, dx_3$ gives:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + f_1 = 0$$ x_1 - direction

Similar work in the other two directions yields:

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = 0$$ x_2 - direction

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0$$ x_3 - direction

These are the <u>three</u> equations of stress equilibrium.
These can be summarized in tensor form as:

$$\boxed{\frac{\partial \sigma_{mn}}{\partial x_m} + f_n = 0}$$

These are the key items about stress, but we also need to talk a bit more about…

(More) Stress Notation

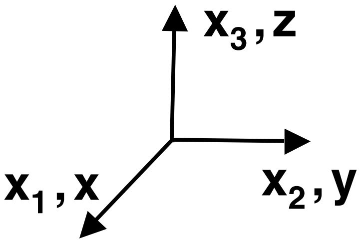
Thus far we've used tensor notation but other notations are used and one must be able to "converse" in all of these

Although these are a member of different notations, the most important in engineering is:

--> Engineering Notation

Here, the subscripts are the directions *x, y, z* rather than x_1, x_2, x_3

Tensor	Engineering
x_1	x
x_2	y
x_3	z



In using the subscripts, only one subscript is used on the extensional stresses. Thus:

Tensor	Engineering	
σ_{11}	σ_x	
σ_{22}	σ_y	
σ_{33}	σ_z	
σ_{23}	σ_{yz}	$= \tau_{yz}$
σ_{13}	σ_{xz}	$= \tau_{xz}$
σ_{12}	σ_{xy}	$= \tau_{xy}$

} sometimes used for shear stresses

Finally, there is a

--> Matrix Notation

it is sometimes convenient to represent the stress tensor in matrix form:

$$\underset{\sim}{\sigma}_{mn} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

↖ symmetric matrix

Finally we want to consider the case of

Two-Dimensional Stress

In many cases, the stresses of importance are in-plane (i.e., two dimensional). There is a reduction in equations, stress components and considerations.

The assumption is to get the "out-of-plane" stresses to zero (they may be nonzero, but they are negligible).

By convention, the "out-of-plane" stresses are in the x_3 - direction. Thus:

$$\sigma_{33} = 0$$

$$\sigma_{13} = 0$$

$$\sigma_{23} = 0$$

This leaves only σ_{11}, σ_{22} and σ_{12} as nonzero.

This is known as **Plane Stress**

(also add the condition that $\frac{\partial}{\partial x_3} = 0$)

Many useful structural configurations can be modeled this way.
more later…(and especially in 16.20)

Now that we know all about stress, we need to turn to the companion concept of <u>strain</u>.