Unit M2.3
(All About) Strain

<u>Readings</u>:

CDL 4.8, 4.9, 4.10

16.001/002 -- *"Unified Engineering"*
Department of Aeronautics and Astronautics
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LEARNING OBJECTIVES FOR UNIT **M2.3**

Through participation in the lectures, recitations, and work associated with Unit M2.3, it is intended that you will be able to………

- ….**explain** the concept and types of strains and how such is manifested in materials and structures
- ….**use** the various ways of **describing** states of strain
- ….**describe** the relationship between strain and displacement in a body
- ….**apply** the concept of compatibility to the state of strain

We've just talked about how a solid continuum carries load via stress. Now we need to describe how such a continuum deforms. For this, we need to introduce

The Concept of Strain

Definition: **Strain** is the deformation of the continuum at a point

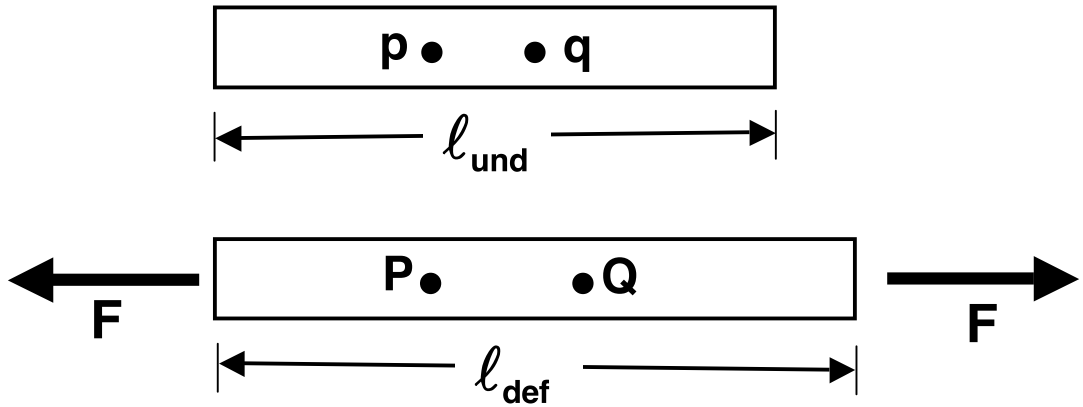
or

the percentage deformation of an infinitesimal element

To explore this concept, we need to think about the physical reality of how items deform:

1. Elongation

Figure M2.3-1 **Example of one-dimensional elongation**



Consider the change in length, $\Delta\ell$:

$$\Delta\ell = \ell_{deformed} - \ell_{undeformed}$$

(Note: $\Delta\ell$ can be positive or negative)

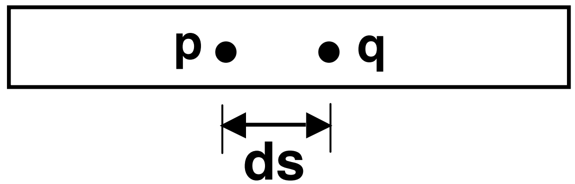
Reference this to the original length:

$$Elongation = E = \frac{\ell_{deformed} - \ell_{undeformed}}{\ell_{undeformed}}$$

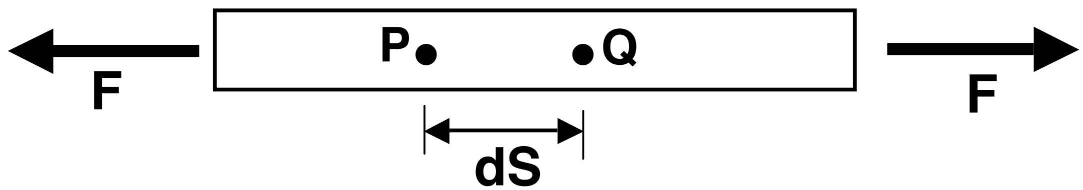
--> Now consider the infinitesimal:

(Note: small letters pertain to undeformed;
CAPITAL LETTERS to deformed)

<u>undeformed</u> length of infinitesimal: $ds = p - q$



<u>deformed</u> length of infinitesimal: $dS = P - Q$



Thus:

$$E = \frac{(P - Q) - (p - q)}{(p - q)}$$

$$\Rightarrow E = \frac{(P - Q)}{(p - q)} - \frac{(p - q)}{(p - q)} = \frac{dS}{ds} - 1$$

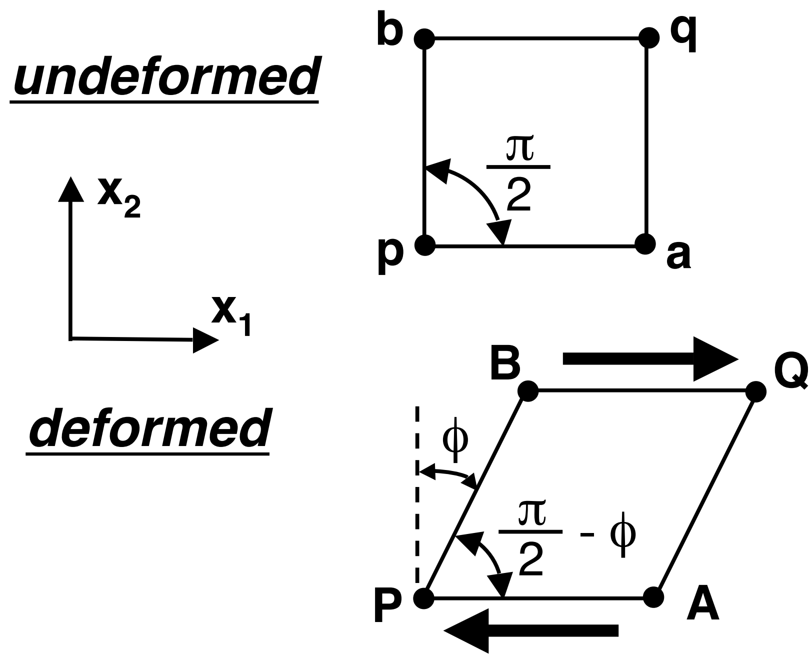
We will return to this.

The other way in which a body can deform is via…..

2. Shear

This produces an angle change in the body (with no elongations for pure shear)

Figure M2.3-2 **Illustration of shear deformation of the infinitesimal element**



Consider the change in angle:

$$\Delta\angle = \angle_{deformed} - \angle_{undeformed}$$

Would at first make sense.

But, by convention, a reduction in angle is positive shear. So:

$$\Delta\angle = \angle_{undeformed} - \angle_{deformed}$$

In this case:

$$\Delta\angle = \left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \phi\right)\right] = \phi$$

Also note that by keeping this in radians, this is already a nondimensional quantity. [**Units**: Nondimensional…

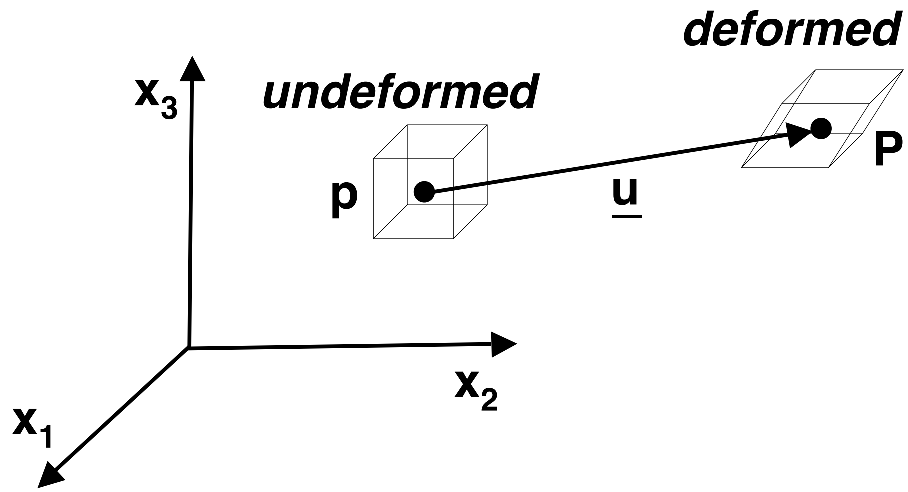
$$\frac{length}{length} = \text{"}strain\text{"}; \quad \mu strain = 10^{-6}$$]

These give us the basic concepts of strain and that there are two types: elongation and shear, but to deal with the full three-dimensional configuration, we need to deal with the….

Strain Tensor and Strain Types

In going from the undeformed (small letters) to the deformed (capital letters) body, we can define a displacement vector, $\overline{u}$, for any point P.

Figure M2.3-3 **Displacement vector from undeformed to deformed body**



$$\underline{u} = u\,\underline{i}_1 + v\,\underline{i}_2 + w\,\underline{i}_3$$

The overall displacement will have contributions from 4 basic parts:

1. Pure translation (3 directions)
2. Pure rotation (3 planes)
3. Elongation (3 axes/directions)
4. Shear (3 planes)

So we have components of strain.

--> For elongation, need to specify changes of length of three sides of body (so do relative to axes):

ε_{11} = relative elongation in x_1 – direction

ε_{22} = relative elongation in x_2 – direction

ε_{33} = relative elongation in x_3 – direction

--> For shear, need to specify changes in angles of three sides of body (use planes defined by axes):

ε_{12} + ε_{21} = total angle change in $x_1 - x_2$ plane

ε_{13} + ε_{31} = total angle change in $x_1 - x_3$ plane

ε_{23} + ε_{32} = total angle change in $x_2 - x_3$ plane

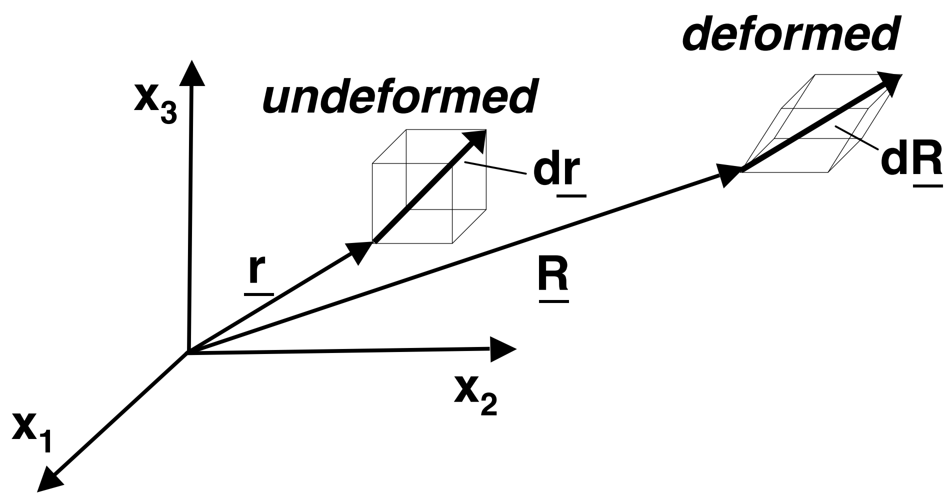
Relate to displacement via strain-displacement relations:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Let's see how we get this….

Formally, the strain tensor is defined by considering the diagonals of the deformed and undeformed elements.

Figure M2.3-4 Position vectors to deformed and undeformed element and the associated diagonals



$\underline{r}$ = position vector to undeformed element

$\underline{R}$ = position vector to deformed element

$d\underline{r}$ = diagonal of undeformed element

$d\underline{R}$ = diagonal of deformed element

Take the squares of the diagonals:

$$(ds)^2 = d\underline{r} \cdot d\underline{r} \qquad (dS)^2 = d\underline{R} \cdot d\underline{R}$$

Formal Definition of Strain Tensor

$$\boxed{(dS)^2 - (ds)^2 = 2\varepsilon_{mn}\, dx_m dx_n}$$

$$\underbrace{(dS)^2 - (ds)^2}_{\text{change in magnitude}} = \underbrace{2\varepsilon_{mn}\, dx_m dx_n}_{\text{factor of 2 for angular changes!}}$$

ε_{mn} = Strain Tensor

Ref: Bisplinghoff, Mar and Pian, Statics of Deformable Solids, Ch. 5.

But what good does this do us?

This general definition is needed for the most general case with "large strains", but in many (most engineering) cases we can consider….

Small Strains (vs. Large Strains)

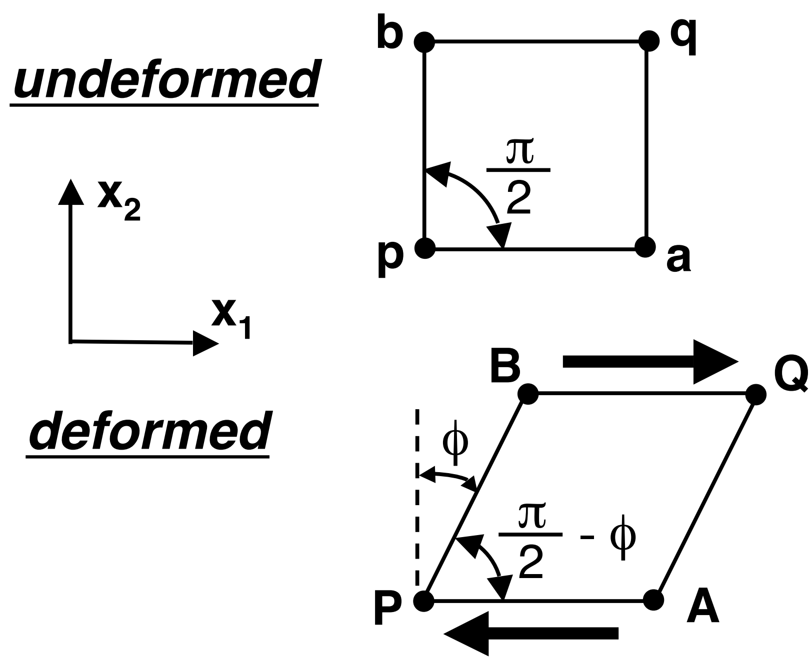
With small deformations in most structures, we can put limits on strains such that:

changes of length $<$ 10%
changes of angles $<$ 5%

Good for range of most "engineering materials"

In such cases, higher order terms become negligible and we can equate:

- extensional strain with elongation
- shear strain with angular change



for <u>small strains</u>:

elongation

$$\varepsilon_{11} \cong E_{11} = \frac{|PA| - |pa|}{|pa|}$$

where: E_{11} = elongation in x_1 - direction

$$\varepsilon_{22} \cong E_{22} = \frac{|PB| - |pb|}{|pb|}$$

and a similar drawing can be made to include x_3 so that:

$$\varepsilon_{33} \cong E_{33} = \frac{|PC| - |pc|}{|pc|}$$

In general:

$$\text{elongation strain} = \lim_{\text{element length} \to 0} \frac{\text{change in element length}}{\text{element length}}$$

shear:

$$\varepsilon_{12} \cong \frac{1}{2}\phi_{12} = \frac{1}{2}\left[\angle apb - \angle APB\right]$$

where: ϕ_{12} = angular change in x_1 - x_2 plane

And again, drawings to include x_3 will give:

$$\varepsilon_{13} \cong \frac{1}{2}\phi_{13} = \frac{1}{2}\left[\angle apc - \angle APC\right]$$

$$\varepsilon_{23} \cong \frac{1}{2}\phi_{23} = \frac{1}{2}\left[\angle bpc - \angle BPC\right]$$

In general:

shear strain = 1/2 (angular change)

--> we now have a definition of strain and can deal with the most useful case of "small strain". But we have not yet defined formally how strain and displacement are related, so we need the:

Strain - Displacement Relations

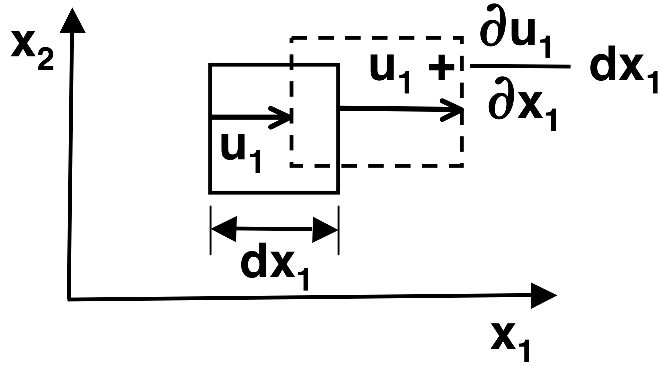
Consider first extensional strains.

We know:

$\varepsilon_{11} \cong$ elongation in x_1

$$\cong \frac{\ell_{def} - \ell_{und}}{\ell_{und}}$$

Figure M2.3-5 **Unit (infinitesimal) element of length dx_1**



u_1 is a <u>field</u> <u>variable</u> = $u_1 (x_1, x_2, x_3)$

$\Rightarrow$ u_1 is displacement of left-hand side

$$\left(u_1 + \frac{\partial u_1}{\partial x_1} \; dx_1 \right)$$ is displacement of right-hand side

way u_1 changes with x_1 (for $\frac{\partial u_1}{\partial x_1}$)

infinitesimal length in x_1 - direction (for dx_1)

We see:

$$\ell_{undeformed} = dx_1$$

$$\ell_{deformed} = dx_1 + \left(u_1 + \frac{\partial u_1}{\partial x_1} dx_1 \right) - u_1$$

$$= dx_1 + \frac{\partial u_1}{\partial x_1} dx_1$$

So:

$$\varepsilon_{11} = \frac{dx_1 + \frac{\partial u_1}{\partial x_1} dx_1 - dx_1}{dx_1}$$

$$\Rightarrow \boxed{\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}}$$

Similarly: (pictures in x_2 and x_3 directions)

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

In general: *extensional* strain is equal to the *rate of change of displacement*

Now consider shear strains

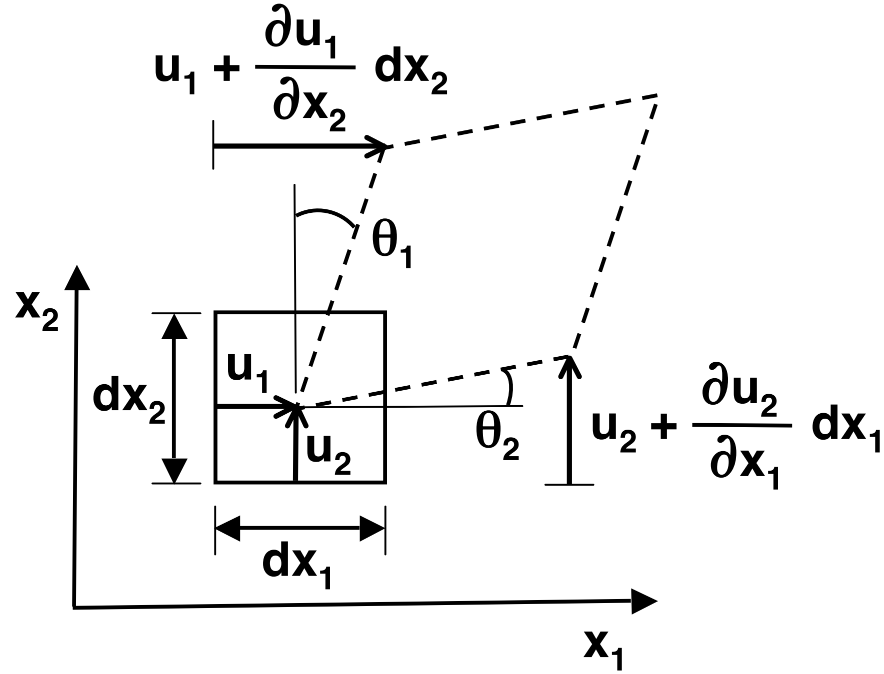
We know:

$$\varepsilon_{12} \cong \frac{1}{2} \text{ angle change in } x_1 - x_2 \text{ plane} \cong \frac{1}{2}\phi_{12}$$

$$\cong \frac{1}{2}\left\{\angle_{undef} - \angle_{def}\right\}$$

$$= \frac{1}{2}\left\{\frac{\pi}{2} - \left(\frac{\pi}{2} - \phi\right)\right\}$$

Figure M2.3-6 Unit (infinitesimal) element dx_1 by dx_2 in the x_1 - x_2 plane



- Using the field variables $u_1\ (x_1, x_2, x_3)$ and $u_2\ (x_1, x_2, x_3)$
- Assume small angles such that: $\tan \theta \cong \theta$
- Start with

$$\phi = \theta_1 + \theta_2$$

$$\theta_1 = \frac{\left(u_1 + \frac{\partial u_1}{\partial x_2} dx_2\right) - u_1}{dx_2} = \frac{\partial u_1}{\partial x_2}$$

$$\theta_2 = \frac{\left(u_2 + \frac{\partial u_2}{\partial x_1} dx_1\right) - u_2}{dx_1} = \frac{\partial u_2}{\partial x_1}$$

Thus:

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \varepsilon_{21}$$

Recall symmetry of strain tensor

Similarly: (pictures in x_1 - x_3 and x_2 - x_3 planes)

$$\varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

and $$\varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

These can be written in general tensor form as:

$$\boxed{\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

Strain-Displacement Relations

where:

$$\underline{u} = u_1\,\underline{i}_1 + u_2\,\underline{i}_2 + u_3\,\underline{i}_3$$

with 6 independent components:

extensional	shear
ε_{11}	$\varepsilon_{12} = \varepsilon_{21}$
ε_{22}	$\varepsilon_{13} = \varepsilon_{31}$
ε_{33}	$\varepsilon_{23} = \varepsilon_{32}$

<u>Note</u>: These relations are developed for <u>small displacements</u> only. As displacements get large, must include higher order terms.

It looks like we're done, but not quite. There is one more concept known as:

<u>Compatibility</u>

One cannot *<u>independently</u>* describe <u>3</u> displacement fields $\{u_1\ (x_1, x_2, x_3), u_2\ (x_1, x_2, x_3), u_3\ (x_1, x_2, x_3)\}$ by <u>6</u> strains

The strains must be related by equations in order for them to be "<u>compatible</u>".

Can derive by: (e.g., ε_{11}, ε_{22}, ε_{12})

- take second partial of each

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} = \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} \qquad \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2}$$

$$\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{1}{2}\left(\frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2}\right)$$

- substitute first two in latter to get:

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0$$

In general this can be written in tensor form:

$$\boxed{\frac{\partial^2 \varepsilon_{nk}}{\partial x_m \partial x_\ell} + \frac{\partial^2 \varepsilon_{m\ell}}{\partial x_n \partial x_k} - \frac{\partial^2 \varepsilon_{n\ell}}{\partial x_m \partial x_k} - \frac{\partial^2 \varepsilon_{mk}}{\partial x_n \partial x_\ell} = 0}$$

gives <u>6</u> equations (3 conditions)

Are we done? NO…we again need to address…

(More) Strain Notation

Just as in the case of stress, we also need to be familiar with other notations, particularly

--> Engineering Notation

The subscript changes are the same, but there is a fundamental difference with regard to strain

Engineering shear strain = total angle change

Tensorial shear strain = 1/2 angular change

BEWARE: The factor of 2

--> always ask: tensorial or engineering shear strain?

Thus:

Tensor	Engineering
ε_{11}	ε_{x}
ε_{22}	ε_{y}
ε_{33}	ε_{z}
ε_{12}	1/2 ε_{xy}
ε_{13}	1/2 ε_{xz}
ε_{23}	1/2 ε_{yz}

In addition, γ (gamma) is often used for the shear strains:

$$\gamma_{xy} = \gamma_{yx} = \varepsilon_{xy} = \varepsilon_{yx}$$

$$\gamma_{xz} = \gamma_{zx} = \varepsilon_{xz} = \varepsilon_{zx}$$

$$\gamma_{yz} = \gamma_{zy} = \varepsilon_{yz} = \varepsilon_{zy}$$

Finally, can also use….

--> Matrix Notation

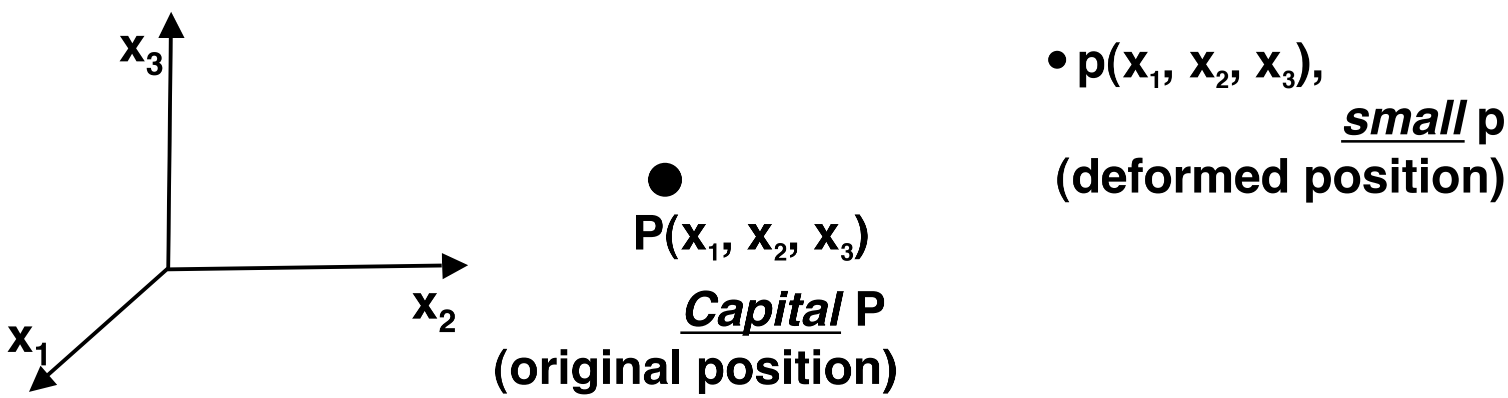
$$\underset{\sim}{\varepsilon}_{mn} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

symmetric matrix

Finally…

Deformation/Displacement Notation

Figure M2.3-7 **Displacement Notation**



$$u_m = p(x_m) - P(x_m)$$

--> Compare notations

Tensor	Engineering	Direction in Engineering
u_1	u	x
u_2	v	y
u_3	w	z