Unit M2.4
Stress and Strain Transformations

<u>Readings</u>:

CDL 4.5, 4.6, 4.7, 4.11, 4.12, 4.13

CDL 4.14, 4.15

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LEARNING OBJECTIVES FOR UNIT **M2.4**

Through participation in the lectures, recitations, and work associated with Unit M2.4, it is intended that you will be able to………

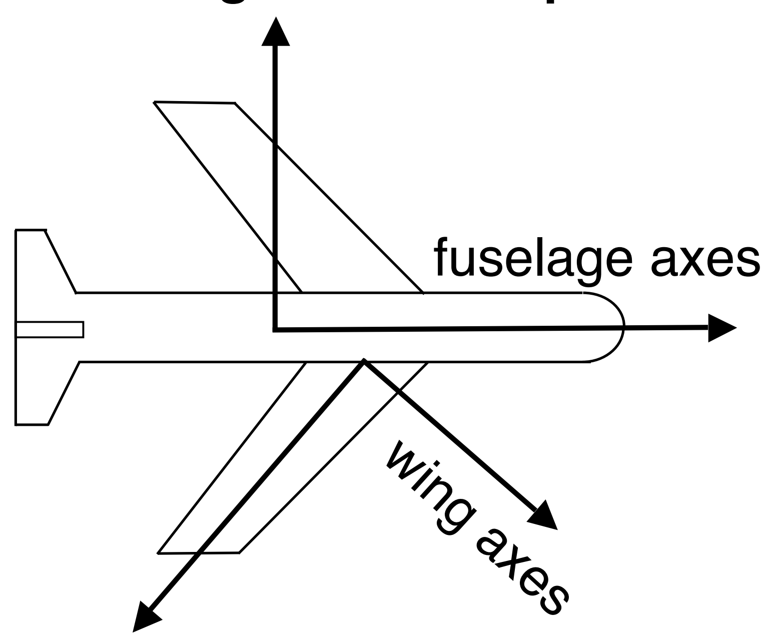
- ….**explain** the bases for the transformations of the states of stress, strain, and deformation
- ….**cite** the equations for 3-D transformations of stress, strain, and deformation
- ….**transform** the states of stress, strain, and deformation for any 2-D configuration
- ….**apply** the concepts associated with principal stress/strain/axes

Earlier, we learnt how to transform coordinates and forces. We may want to do the same with stresses and strains. Why?…..Recall the

Motivation

--> We may want to describe the behavior (stress and strain) of a structure with reference to more than one set of axes:

Figure M2.4.1 **Example of loading axes on airplane**

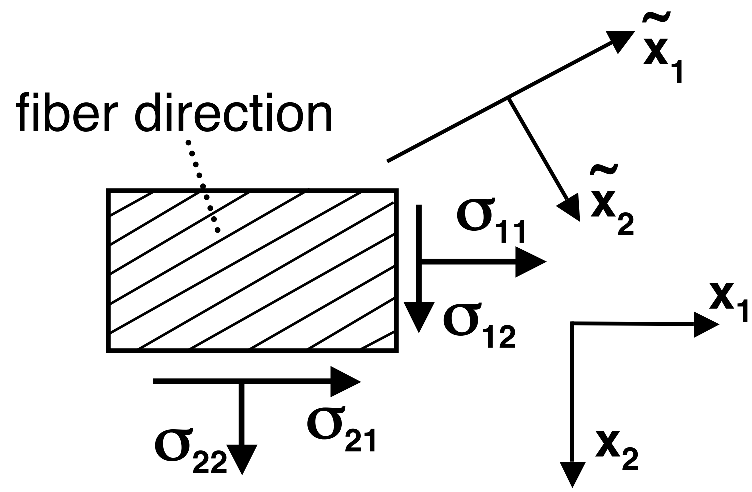


Different "loading axes" for wing and fuselage.

Definition: Loading axes are axes along which loading is applied/oriented

Also, wing axes not oriented with fluid flow axes (transformation needed here)

Figure M2.4.2 **Example of a fibrous composite and different axes**



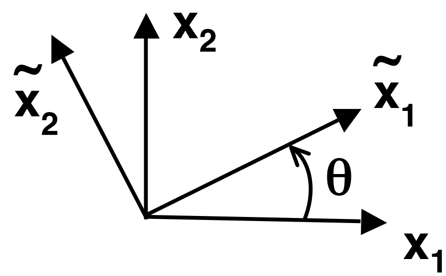
Know stresses (or strains) along loading axes but want to know stresses in axis system referenced to fibers.

Let's first consider the mathematical form of the transformation:

Tensorial Form

We learnt in Unit U4 that to transform from one axis system to another, we need the direction cosines:

Figure M2.4.3 **Two different axis systems**



$$\ell_{\tilde{n}m} = \text{(direction) cosine of angle from } \tilde{x}_n \text{ to } x_m$$

$$\tilde{x}_n = \ell_{\tilde{n}m} \, x_m$$

--> Axes and forces are first-order tensors (1 subscript) and require 1 direction cosine for transformation.

--> Stresses and strains are second-order tensors (2 subscripts) and require 2 direction cosines for transformation.

Thus:

$$\tilde{\sigma}_{mn} = \ell_{\tilde{m}p} \, \ell_{\tilde{n}q} \, \sigma_{pq}$$

$$\tilde{\varepsilon}_{mn} = \ell_{\tilde{m}p} \, \ell_{\tilde{n}q} \, \varepsilon_{pq}$$

and for displacement:

$$\tilde{u}_m = \ell_{\tilde{m}p} \, u_p$$

These are the tensor equations to transform stress and strain from the x_m - system to the $\tilde{x}_n$ - system.

(we won't write this out in full until we go to two dimensions)

**Remember from before the

IMPORTANT CONCEPT: The axis system in which we describe a quantity (or set thereof) does not change the quantity (or set thereof), only its description.

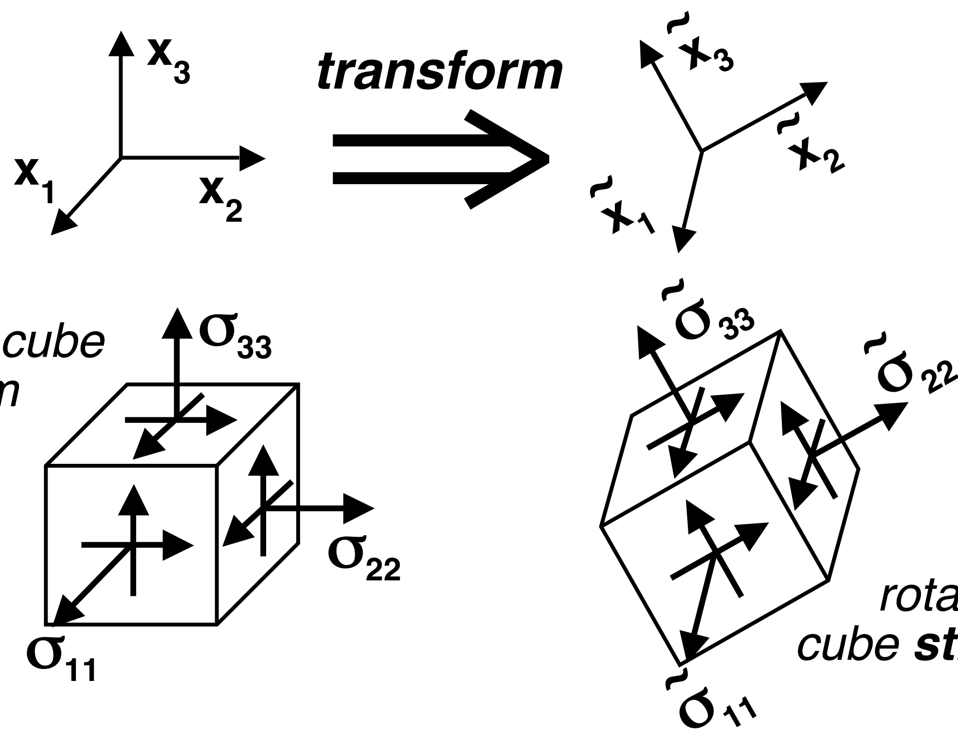
So, the stress state and the strain state do not change, we just describe them differently. In order to see this let's consider the physical bases for these transformations.

Physical Bases

--> Stress

We are looking at the same stress state as referenced to two sets of coordinate axes/systems.

Figure M2.4.4 **Infinitesimal cubes of stress in two axis systems**



So the transformation of stresses is based on equilibrium
(we'll prove it when we go to 2-D)

--> Strain

Here we are looking at the same physical deformation as referenced to two different sets of coordinate axis/systems.

Thus, the geometry stays the same

and

The transformation of strain is based on geometry

The transformation laws are generally in 3-D and their bases can be proven in 3-D. However, it is easier to consider these items by looking at their….

Two-Dimensional Forms

And there are numerous problems which we deal with in 2-D (e.g., plane stress)

Since we have written the tensorial forms of the transformation equations in 3-D, we can do the same in 2-D (just use Greek subscripts!):

$$\left.\begin{aligned}\tilde{\sigma}_{\alpha\beta} &= \ell_{\tilde{\alpha}\theta}\,\ell_{\tilde{\beta}\tau}\,\sigma_{\theta\tau} \\ \tilde{\varepsilon}_{\alpha\beta} &= \ell_{\tilde{\alpha}\theta}\,\ell_{\tilde{\beta}\tau}\,\varepsilon_{\theta\tau}\end{aligned}\right\}$$ 2-D forms for stress and strain

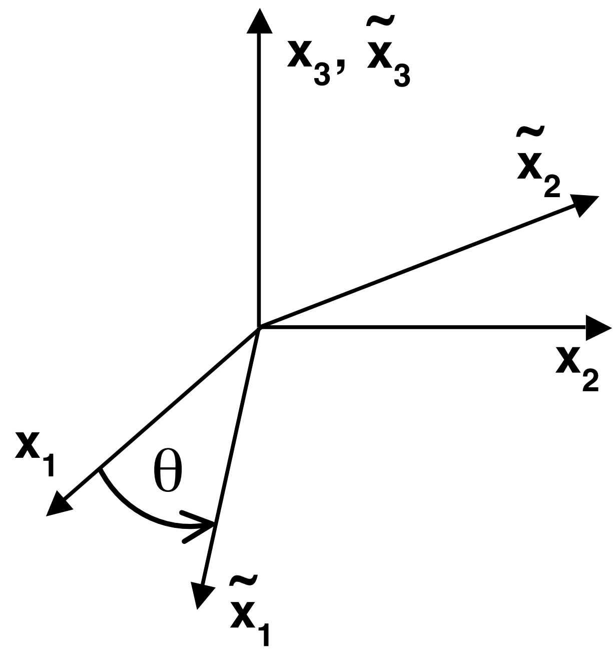
Writing the first out in full:

--> <u>Stress</u>

$$\begin{aligned}
\tilde{\sigma}_{11} &= \cos^2\theta\,\sigma_{11} + \sin^2\theta\,\sigma_{22} + 2\cos\theta\,\sin\theta\,\sigma_{12} \\
\tilde{\sigma}_{22} &= \sin^2\theta\,\sigma_{11} + \cos^2\theta\,\sigma_{22} - 2\cos\theta\,\sin\theta\,\sigma_{12} \\
\tilde{\sigma}_{12} &= -\sin\theta\cos\theta\,\sigma_{11} + \cos\theta\,\sin\theta\,\sigma_{22} \\
&\qquad + \left(\cos^2\theta - \sin^2\theta\right)\sigma_{12}
\end{aligned}$$

where θ is angle from x_m to $\tilde{x}_m$ axes (θ +CCW)

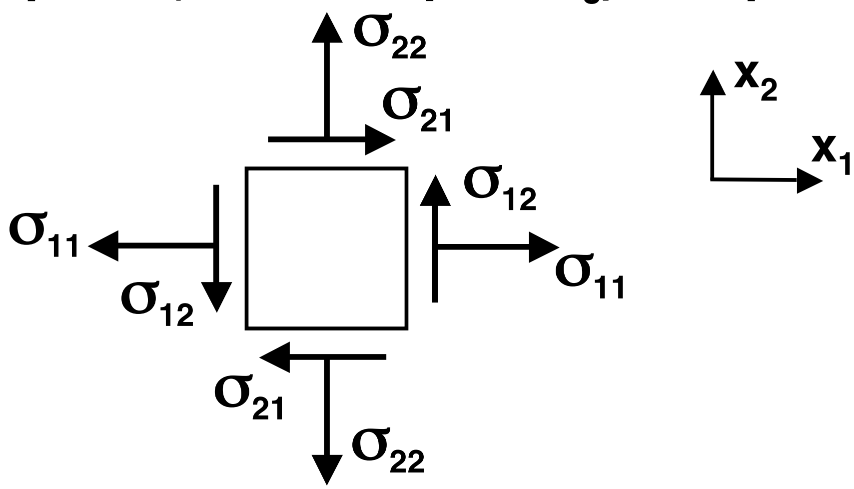
Figure M2.4.5 **Illustration of axis transformation**



Let's look at the physical basis for this and show where one of these equations comes from

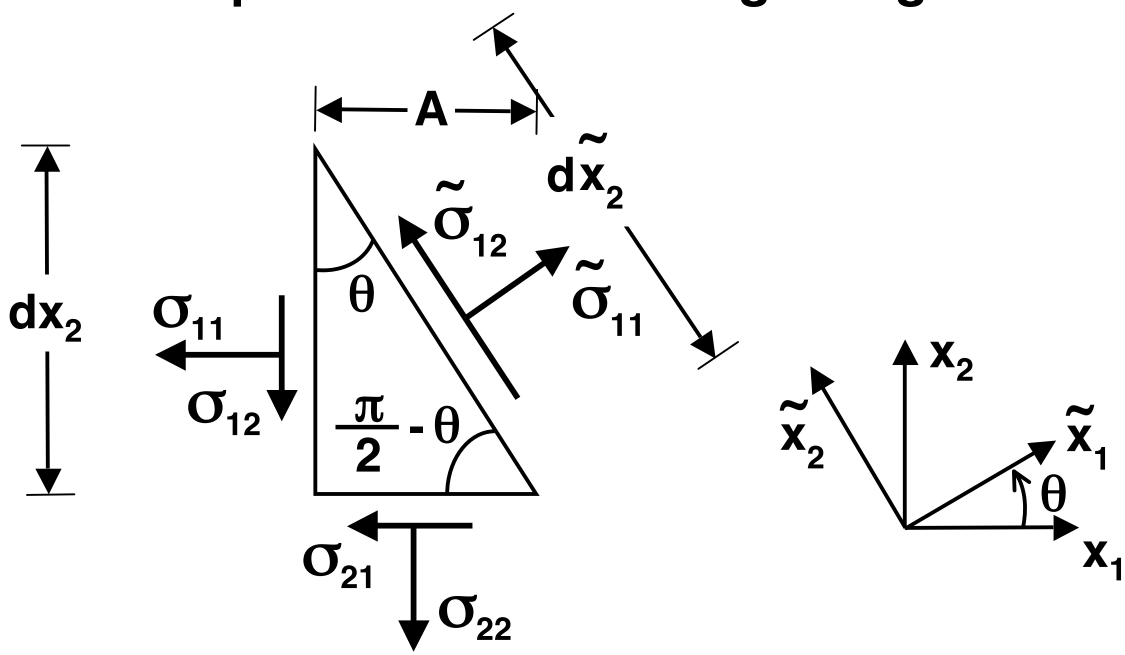
Consider a….

Figure M2.4.6 **Unit square (of unit depth dx_3) in equilibrium**



Now cut this diagonally or at some angle θ (more generally):

Figure M2.4.7 **Cut of unit square and axes acting along each face**



- Align $\tilde{x}_1$ perpendicular to cut face; $\tilde{x}_2$ parallel to cut face
- This is still of unit depth dx_3
- This must still be in equilibrium

Take $\sum F_{\tilde{1}} = 0$:

form = (stress) (length) (depth)

and can see $dx_2 = d\tilde{x}_2 \cos\theta$

$A = d\tilde{x}_2 \sin\theta$

$$(\tilde{\sigma}_{11})(d\tilde{x}_2)\, dx_3 - (\sigma_{11} \cos\theta)(d\tilde{x}_2 \cos\theta)\, dx_3 - (\sigma_{22} \sin\theta)(d\tilde{x}_2 \sin\theta)\, dx_3$$
$$- (\sigma_{12} \sin\theta)(d\tilde{x}_2 \cos\theta)\, dx_3 - (\sigma_{21} \cos\theta)(d\tilde{x}_2 \sin\theta)\, dx_3 = 0$$

Use this and cancel out dx_3:

$$\tilde{\sigma}_{11}\, d\tilde{x}_2 = \sigma_{11} \cos\theta\, d\tilde{x}_2 \cos\theta + \sigma_{22} \sin\theta\, d\tilde{x}_2 \sin\theta$$
$$+ \sigma_{12} \sin\theta\, d\tilde{x}_2 \cos\theta + \sigma_{21} \cos\theta\, d\tilde{x}_2 \sin\theta$$

Canceling out $d\tilde{x}_2$ and combining:

$$\tilde{\sigma}_{11} = \cos^2\theta\, \sigma_{11} + \sin^2\theta\, \sigma_{22} + 2 \cos\theta \sin\theta\, \sigma_{12}$$

<u>same</u> as via tensorial form!

This can be done for $\tilde{\sigma}_{22}$ and $\tilde{\sigma}_{12}$ and could be done in 3-D as well!

Note: Transformation equations for engineering notation form of stresses are the same (but can't write it tensorially)

Now let's consider…

--> Strain

For tensor notation, the 2-D (and 3-D) transformation equations have the exact same form:

$$\tilde{\varepsilon}_{11} = \cos^2\theta\, \varepsilon_{11} + \sin^2\theta\, \varepsilon_{22} + 2\cos\theta \sin\theta\, \varepsilon_{12}$$

$$\tilde{\varepsilon}_{22} = \sin^2\theta\, \varepsilon_{11} + \cos^2\theta\, \varepsilon_{22} - 2\cos\theta \sin\theta\, \varepsilon_{12}$$

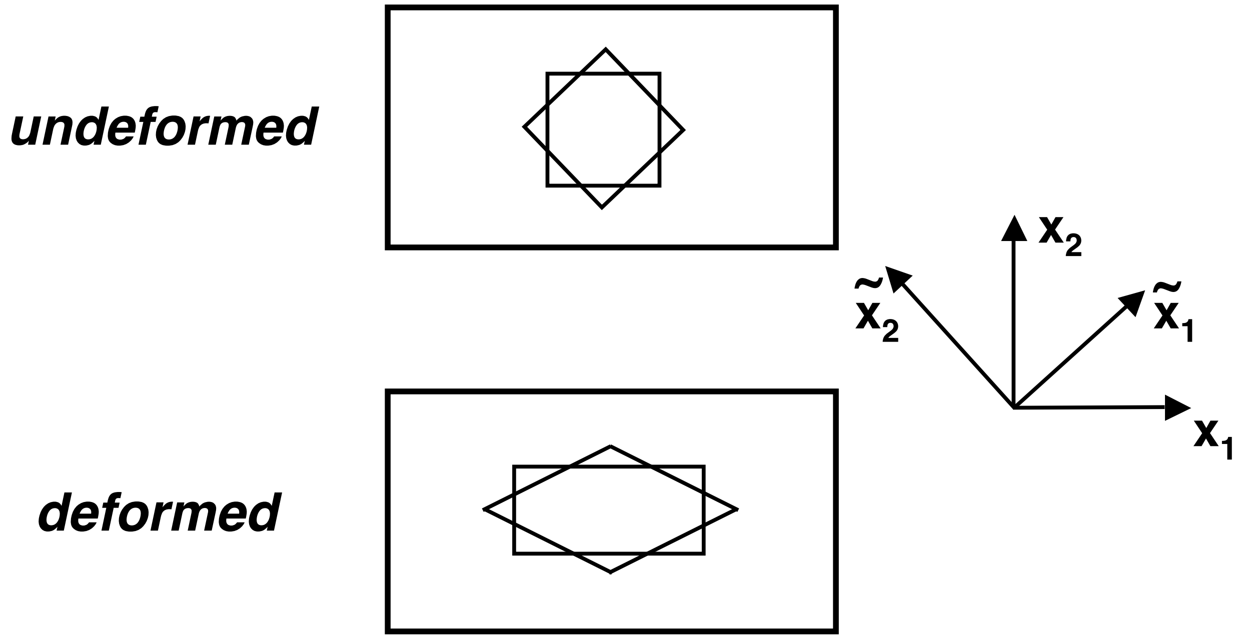
$$\tilde{\varepsilon}_{12} = -\sin\theta \cos\theta\, \varepsilon_{11} + \cos\theta \sin\theta\, \varepsilon_{22} + \left(\cos^2\theta - \sin^2\theta\right) \varepsilon_{12}$$

where θ is angle from x_n to $\tilde{x}_m$ axes (θ +CCW)

Using geometry, these equations can be shown physically.

For a "baseline feeling," consider….

Figure M2.4.8 **Two squares drawn on a bar, one at 45° relative angle to the other**



One appears stretched/elongated, yet they are the same deformation field!

Use geometry!

Note (on engineering notation): the transformation equations change due to the factor of 2 in the shear strain. **BE CAREFUL!**

There are some important aspects associated with stress/strain transformations. The most important of these is:

Principal Stresses/Strains/Axes

There is a set of axes into which any state of stress (or strain) can be resolved such that there are no shear stresses (or strains). These are known as the principal axes of stress (or strain) and the resolved set of stresses (or strains) are known as the *principal stresses* (or *strains*).

Can see this readily in 2-D by considering the $\tilde{\sigma}_{12}\ (or\ \tilde{\varepsilon}_{12})$ equation:

$$\tilde{\sigma}_{12} = -\sin\theta\cos\theta\ \sigma_{11} + \sin\theta\cos\theta\ \sigma_{22} + \left(\cos^2\theta - \sin^2\theta\right)\sigma_{12}$$

Set $\tilde{\sigma}_{12}$ to 0 and solve for θ_p. Then use θ_p in equations for $\tilde{\sigma}_{11}$ and $\tilde{\sigma}_{22}$.

↖ principal

More generally in 3-D can determine the principal stresses as the root of the equations:

$$\begin{vmatrix} (\sigma_{11} - \tau) & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & (\sigma_{22} - \tau) & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & (\sigma_{33} - \tau) \end{vmatrix} = 0$$

This has three roots (for τ) which we label (in decreasing numerical order):

$\sigma_I, \sigma_{II}, \sigma_{III}$ -- **THE PRINCIPAL STRESSES**

The three directions (axes) along which these principal stresses act can be found via:

$$(\sigma_{11} - \sigma_I)\,\ell_{\tilde{I}1} + \sigma_{12}\,\ell_{\tilde{I}2} + \sigma_{13}\,\ell_{\tilde{I}3} = 0$$

$$\sigma_{12}\,\ell_{\tilde{I}1} + (\sigma_{22} - \sigma_I)\,\ell_{\tilde{I}2} + \sigma_{23}\,\ell_{\tilde{I}3} = 0$$

$$\ell_{\tilde{I}1}^2 + \ell_{\tilde{I}2}^2 + \ell_{\tilde{I}3}^2 = 1$$

where $\ell_{\tilde{I}n}^2$ are the direction cosines between the axis along which σ_I acts and the original axes x_n

--> Similar equations can be written for the case of σ_{II} and σ_{III}.

The resulting equations have…

- eigenvalues (roots) -- these are the principal stresses
- eigenvectors -- these are the principal axes

--> write this out explicitly for 2-D:

$$\begin{vmatrix} \sigma_{11} - \tau & \sigma_{12} \\ \sigma_{12} & \sigma_{22} - \tau \end{vmatrix} = 0$$

$$\Rightarrow \sigma_{11}\sigma_{22} - (\sigma_{11} + \sigma_{22})\tau + \tau^2 - \sigma_{12}^2 = 0$$

$$\Rightarrow \boxed{\tau^2 - \tau(\sigma_{11} + \sigma_{22}) + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0}$$

Solve via quadratic formula. Roots are σ_I and σ_{II}

To get θ_p use:

$$\sigma_I = \cos^2\theta_p\,\sigma_{11} + \sin^2\theta_p\,\sigma_{22} + 2\cos\theta_p\,\sin\theta_p\,\sigma_{12}$$

and solve for θ_p

or use:

$$\tilde{\sigma}_{12} = 0 = -\sin\theta\cos\theta\,\sigma_{11} + \sin\theta\cos\theta\,\sigma_{22} + \left(\cos^2\theta - \sin^2\theta\right)\sigma_{12}$$

(no shear stress in principal axes)

use the following trigonometric identities

$$\cos^2\theta = \frac{1}{2}\left(1 + \cos 2\theta\right)$$

$$\sin^2\theta = \frac{1}{2}\left(1 - \cos 2\theta\right)$$

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

$$\Rightarrow 0 = -\sigma_{11}\frac{1}{2}\sin 2\theta + \sigma_{22}\frac{1}{2}\sin 2\theta + \sigma_{12}\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta - \frac{1}{2} + \frac{1}{2}\cos 2\theta\right)$$

gives:

$$\frac{1}{2}\sin 2\theta\left(\sigma_{11} - \sigma_{22}\right) = \sigma_{12}\left(\cos 2\theta\right)$$

$$\Rightarrow \tan 2\theta = \frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}$$

$$\Rightarrow \theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}\right)$$

Could use this result in transformation equations for $\tilde{\sigma}_{11}$ and $\tilde{\sigma}_{22}$ to get σ_I and σ_{II} (optionally)

--> There are ways to check work when doing transformation because of….

Invariants

(Invariant --> doesn't vary [with axis system])

• $(\sigma_{11} + \sigma_{22} + \sigma_{33})$ is a constant!

Thus, sum of normal stresses is invariant

So: $(\sigma_I + \sigma_{II} + \sigma_{III}) = (\sigma_{11} + \sigma_{22} + \sigma_{33})$

--> Prove in 2-D:

$$\tilde{\sigma}_{11} + \tilde{\sigma}_{22} = \cos^2\theta\,\sigma_{11} + \sin^2\theta\,\sigma_{22} + 2\cos\theta\,\sin\theta\,\sigma_{12}$$
$$+ \sin^2\theta\,\sigma_{11} + \cos^2\theta\,\sigma_{22} - 2\cos\theta\,\sin\theta\,\sigma_{12}$$

$$= \sigma_{11}\underbrace{\left(\cos^2\theta + \sin^2\theta\right)}_{=1} + \sigma_{22}\underbrace{\left(\sin^2\theta + \cos^2\theta\right)}_{=1}$$
$$+ \sigma_{12}\underbrace{\left(2\cos\theta\,\sin\theta - 2\cos\theta\,\sin\theta\right)}_{=0}$$

So:

$$\tilde{\sigma}_{11} + \tilde{\sigma}_{22} = \sigma_{11} + \sigma_{22}$$ Q. E. D.

(doesn't vary with θ!)

Note: all these same concepts can be done for strain:

- principal strains
- principal axes
- invariants

(use same equations)

Associated with the concept of principal stresses/strains/axes, is the concept of

Maximum (Extreme) Shear Stresses/Strains/Planes

These are planes along which the value(s) of the shear stresses/strains are maximized.

(important in failure considerations)

--> Can see what direction this is relative to the principal stresses/strains by considering:

$$\tilde{\sigma}_{12} = -\sin\theta\cos\theta\ \sigma_I + \sin\theta\cos\theta\ \sigma_{II}$$

(recall $\sigma_{12} = 0$ in principal axes)

Maximize, take derivative:

$$\frac{\partial\sigma_{12}}{\partial\theta} = 0 = \sigma_I\left(\sin^2\theta - \cos^2\theta\right) + \left(-\sin^2\theta + \cos^2\theta\right)\sigma_{II}$$

$$\Rightarrow 0 = \left(\sigma_I - \sigma_{II}\right)\left(\sin^2\theta - \cos^2\theta\right)$$

$$\Rightarrow \sin^2\theta = \cos^2\theta \Rightarrow \underline{\underline{\theta = 45°}}$$

Can generalize to 3-D and show that maximum shear stresses/strains occur on planes oriented 45° to the principal axes

Values are:

$$\frac{\sigma_I - \sigma_{II}}{2}, \quad \frac{\sigma_{II} - \sigma_{III}}{2}, \quad \frac{\sigma_I - \sigma_{III}}{2} \quad \text{(use } \tilde{\sigma}_{12} \text{ equation)}$$

There is one final concept to look at with regard to transformations. Before calculators/computers, use of geometry was made to do stress and strain calculations. Done via:

<u>Mohr's Circle</u>

(see handout M-7 for description)

- Based on invariants (radius of circle is sum of normal stresses/strains)
- Easy to see principal stresses/strains, important angles, maximum shear stresses/strains
- Relatively simple for 2-D, can be extended (not easily to 3-D)