Unit M3.2
General Stress-Strain Behavior

Readings:
A & J 3
CDL 5.1, 5.2, 5.4, 5.10

16.001/002 -- “Unified Engineering”
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LEARNING OBJECTIVES FOR UNIT **M3.2**

Through participation in the lectures, recitations, and work associated with Unit M3.2, it is intended that you will be able to………

- ….**explain** the meaning of the elasticity and compliance tensors and **analyze** their mathematical details
- ….**describe** the behavior of a material in terms of constitutive response
- ….**discuss** engineering/elastic constants, their measurement, and their relationship to tensors
- ….**employ** a continuum version of the constitutive law of elasticity

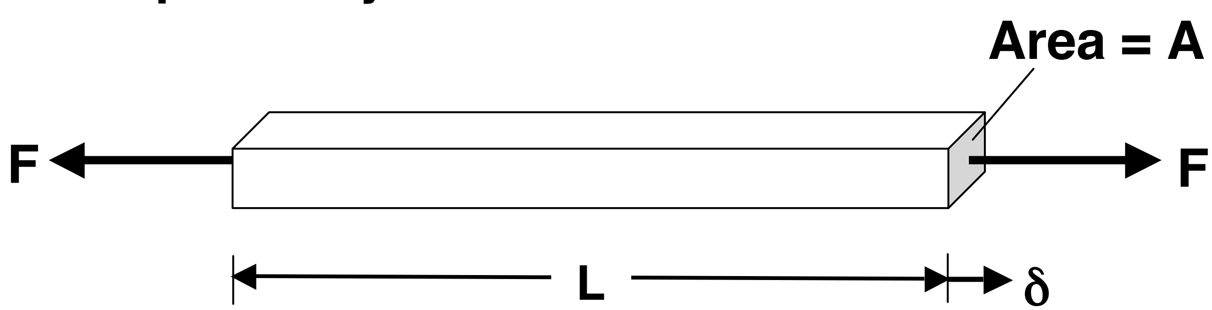
Will look at the model to relate stress and strain and consider how we manipulate this ***mathematically*** and determine the properties/characterization ***experimentally***.

To orient ourselves, let's first look at how actual materials behave and examine…

Uniaxial Stress-Strain

Consider a…

Figure M3.2-1 **Bar pulled by a force F**



such that there is only uniaxial (σ_{11}) stress:

$$\sigma_{11} = \frac{F}{A}$$

and the relative elongation is:

$$\varepsilon_{11} = \frac{\delta}{\ell}$$

Such a load is generally applied via a testing machine to obtain σ versus ε experimentally (*recall truss experiment*).

Different types of material exhibit different stress-strain behavior. There are two general categories:

Figure M3.2-2a **Illustration of brittle behavior (e.g., glass, ceramics)**

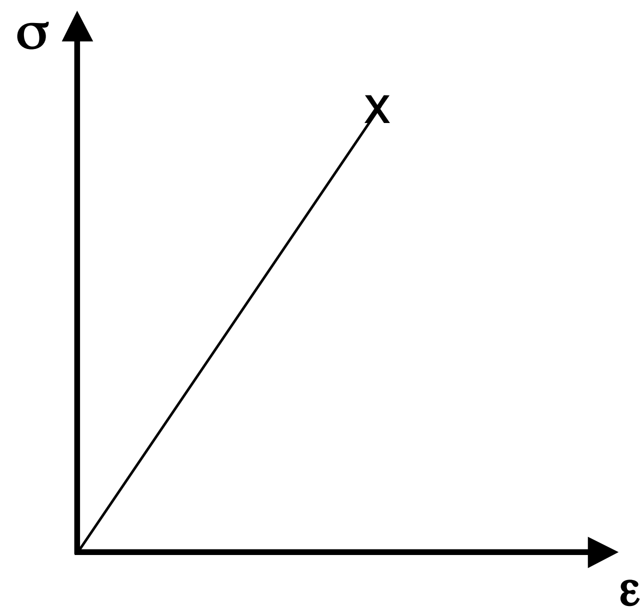
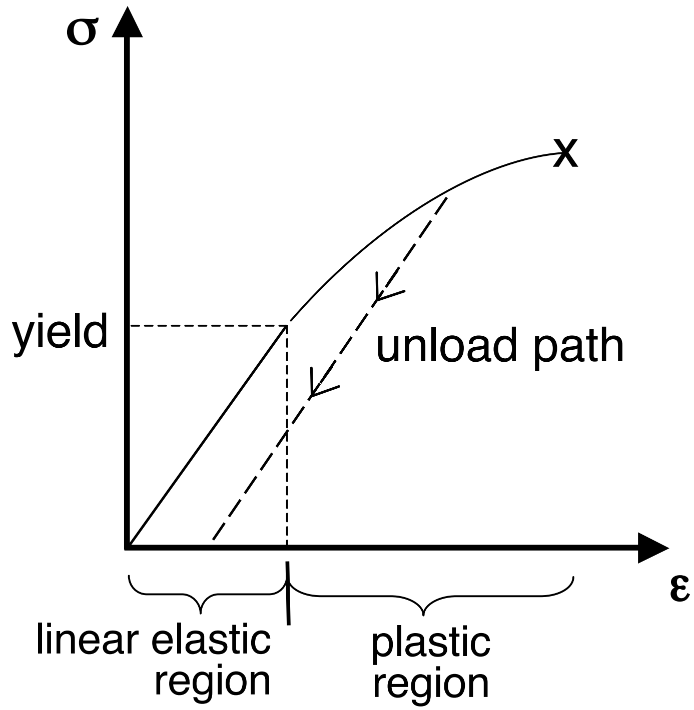


Figure M3.2-2b **Illustration of ductile behavior (e.g., metals)**



--> In elastic region, return path is to origin

--> In plastic region, return path is parallel to original linear portion ⇒ permanent strain remains

--> Let's concentrate on the linear portion and model that behavior.

We look to work done in 1676 proposed by Hooke and consider:

(Generalized) Hooke's Law

Hooke said that force and displacement and also stress and strain are linearly related:

$$\boxed{\sigma = E\varepsilon} \text{ --Hooke's Law}$$

(also think of F = kx)

Thus, the slope of the uniaxial stress-strain response in the linear region is:

$$\frac{\sigma}{\varepsilon} = E \longleftarrow \text{Modulus of Elasticity}$$

Units: $\left[force / length^2 \right]$

$\left[psi \right]$ $\left[Pa \right]$

$\hat{M} \left(10^6 \right)$ $\hat{G} \left(10^9 \right)$

Note: $\sigma = \dfrac{F}{A}, \quad \varepsilon = \dfrac{\delta}{\ell}$

$$\Rightarrow \frac{F / A}{\delta / \ell} = E \quad \Rightarrow \delta = \frac{F\ell}{AE}$$ (as we've seen before)

We need to generalize this concept in order to relate general stress (a second-order tensor) to general strain (a second-order tensor). We arrive at…..

--> Generalized Hooke's Law

$$\sigma_{mn} = E_{mnpq} \, \varepsilon_{pq}$$

the elasticity tensor

This is a *fourth-order* tensor which is needed to related two second-order tensors

Write out for a sample case (m = 1, n = 1)

$$\begin{aligned} \sigma_{11} = \; & E_{1111}\varepsilon_{11} + E_{1112}\varepsilon_{12} + E_{1113}\varepsilon_{13} && \text{(p = 1, sum on q)} \\ & + E_{1121}\varepsilon_{21} + E_{1122}\varepsilon_{22} + E_{1123}\varepsilon_{23} && \text{(p = 2, sum on q)} \\ & + E_{1131}\varepsilon_{31} + E_{1132}\varepsilon_{32} + E_{1133}\varepsilon_{33} && \text{(p = 3, sum on q)} \end{aligned}$$

(sum on p)

Collect like terms (e.g. $\varepsilon_{mn} = \varepsilon_{nm}$) to get:

$$\begin{aligned} \sigma_{11} = \; & E_{1111}\varepsilon_{11} + E_{1122}\varepsilon_{22} + E_{1133}\varepsilon_{33} \\ & + 2E_{1112}\varepsilon_{12} + 2E_{1113}\varepsilon_{13} + 2E_{1123}\varepsilon_{23} \end{aligned}$$

(Note: recall 2's for tensorial strain)

But in order to consider this 3-D stress-strain relation in its entirety, we need to consider the…

Elasticity and Compliance Tensors

E_{mnpq} is the "Elasticity Tensor"

How many components does this appear to have?

m, n, p, q = 1, 2, 3

$\Rightarrow 3 \times 3 \times 3 \times 3 = \underline{\underline{81}}$ components

But there are several symmetries:

1. Since $\sigma_{mn} = \sigma_{nm}$ (equilibrium/energy considerations)

   $\Rightarrow E_{mnpq} = E_{nmpq}$ *(symmetry in switching <u>first two</u> <u>indices</u>)*

2. Since $\varepsilon_{pq} = \varepsilon_{qp}$ (geometrical considerations)

   $\Rightarrow E_{mnpq} = E_{mnqp}$ *(symmetry in switching <u>last two</u> <u>indices</u>)*

3. From thermodynamic considerations (1st law of thermo)

   $\Rightarrow E_{mnpq} = E_{pqmn}$ *(symmetry in switching <u>pairs of</u> <u>indices</u>)*

Note that:

- since $\sigma_{mn} = \sigma_{nm}$, the apparent 9 equations for stress are only 6
- 2's come out automatically since $\varepsilon_{pq} = \varepsilon_{qp}$ and $E_{mnpq} = E_{mnqp}$ terms like $E_{mnpq}\,\varepsilon_{pq} + E_{mnqp}\,\varepsilon_{qp} = 2E_{mnpq}\,\varepsilon_{pq}$
  (factor of 2!)

--> as we saw for case of m = 1, n = 1

--> don't put them in $\underset{\sim}{\varepsilon}$

With these symmetrics, the resulting full 3-D stress-strain equations are (in matrix form):

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 2E_{1123} & 2E_{1113} & 2E_{1112} \\ E_{1122} & E_{2222} & E_{2233} & 2E_{2223} & 2E_{2213} & 2E_{2212} \\ E_{1133} & E_{2233} & E_{3333} & 2E_{3323} & 2E_{3313} & 2E_{3312} \\ E_{1123} & E_{2223} & E_{3323} & 2E_{2323} & 2E_{1323} & 2E_{1223} \\ E_{1113} & E_{2213} & E_{3313} & 2E_{1323} & 2E_{1313} & 2E_{1213} \\ E_{1112} & E_{2212} & E_{3312} & 2E_{1223} & 2E_{1213} & 2E_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}$$

Note: 2's come naturally here

or:

$$\underset{\sim}{\sigma} = \underset{\sim}{E} \; \underset{\sim}{\varepsilon}; \qquad \sigma_{mn} = E_{mnpq} \, \varepsilon_{pq}$$

This results in $\underline{\underline{21}}$ independent components of the elasticity tensor

- Along diagonal (6)
- Upper right half of matrix (15)

[don't worry about 2's]

The components of the E_{mnpq} can be placed into <u>3 groups</u>:

- Extensional strains to extensional stresses

E_{1111} E_{1122}

E_{2222} E_{1133}

E_{3333} E_{2233}

e.g., $\sigma_{11} = \ldots E_{1122} \, \varepsilon_{22} \ldots$

- Shear strains to shear stresses

E_{1212} E_{1213}

E_{1313} E_{1323}

E_{2323} E_{2312}

e.g., $\sigma_{12} = \ldots 2E_{1223} \, \varepsilon_{23} \ldots$

- Coupling terms: extensional strains to shear stresses *or* shear strains to extensional stresses

$$\begin{matrix} E_{1112} & E_{2212} & E_{3312} \\ E_{1113} & E_{2213} & E_{3313} \\ E_{1123} & E_{2223} & E_{3323} \end{matrix}$$

e.g., $\sigma_{12} = \ldots E_{1211}\ \varepsilon_{11} \ldots$

$\sigma_{11} = \ldots 2E_{1123}\ \varepsilon_{23} \ldots$

A material which behaves in this manner is "fully" anisotropic

Need to consider a "companion" to the elasticity tensor…

The Compliance Tensor

Just as there is a general relationship between stress and strain:

$$\sigma_{mn} = E_{mnpq}\ \varepsilon_{pq}$$

there is an inverse relationship between strain and stress:

$$\varepsilon_{mn} = S_{mnpq}\, \sigma_{pq}$$

where: S_{mnpq} is the **compliance** **tensor**

Using matrix notation:

$$\underset{\sim}{\sigma} = \underset{\sim}{E}\, \underset{\sim}{\varepsilon}$$

$$\Rightarrow \quad \underset{\sim}{E}^{-1}\, \underset{\sim}{\sigma} = \underset{\sim}{\varepsilon}$$

inverse

with: $\underset{\sim}{\varepsilon} = \underset{\sim}{S}\, \underset{\sim}{\sigma}$

comparing the two gives:

$$\boxed{\underset{\sim}{E}^{-1} = \underset{\sim}{S}}$$

$$\Rightarrow \underset{\sim}{E}\, \underset{\sim}{S} = \underset{\sim}{I}$$

⇒The compliance matrix is the inverse of the elasticity matrix

<u>Note</u>: the same symmetries apply to S_{mnpq} as to E_{mnpq}

Meaning of the tensors and their components:

- Elasticity term E_{mnpq}: amount of stress (σ_{mn}) caused by/related to the deformation/strain (ε_{pq})
- Compliance term S_{mnpq}: amount of strain (ε_{mn}) caused by the stress (σ_{pq})

--> Final note…<u>Transformations</u>

These are $\underline{\underline{\text{fourth}}}$ order tensors and thus require $\underline{\underline{4}}$ direction cosines to transform:

$$\tilde{E}_{mnpq} = \ell_{\tilde{m}r}\, \ell_{\tilde{n}s}\, \ell_{\tilde{p}t}\, \ell_{\tilde{q}u}\, E_{rstu}$$

$$\tilde{S}_{mnpq} = \ell_{\tilde{m}r}\, \ell_{\tilde{n}s}\, \ell_{\tilde{p}t}\, \ell_{\tilde{q}u}\, S_{rstu}$$

Not all materials require all 21 components to describe their behavior. We therefore consider…..

Classes of Stress-Strain Behavior

(e.g., anisotropy, orthotropy, isotropy)

--> **Good reference for this:**

Bisplinghoff, Mar, and Pian, "Statics of Deformable Solids", Addison, Wesley, 1965, Ch. 7.

Start out by making a table of the classes of material stress-strain behavior and the associated number of *independent* components of E_{mnpq}:

Class of Stress-Strain Behavior	# of Independent Components of E_{mnpq}
Anisotropic	21
Monoclinic	13
Orthotropic	9
Tetragonal	6
"Transversely Isotropic"*	5
Cubic	3
Isotropic	2

Composite Laminates

Useful Engineering Materials

Basic Composite Ply

Metals (on average)

Consider some key cases:

--> Anisotropic

- 21 independent components
- Nonorthogonal crystals
- Currently no useful engineering materials

Why, then, do we bother with anisotropy?

Two reasons:

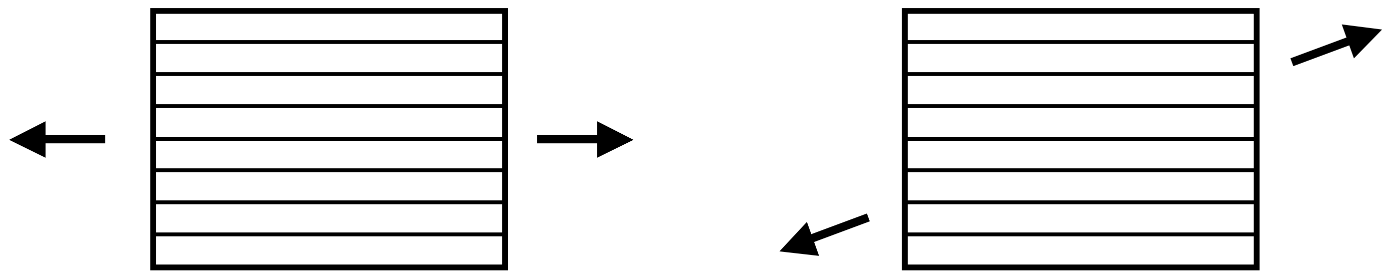
1. Someday, we may have useful fully anisotropic materials (certain crystals now behave that way) Also, 40-50 years ago, people only worried about isotropy

2. It may not always be convenient to describe a structure (i.e., write the governing equations) along the principal material axes, but may use loading axes.

In these other axis systems, the material may *appear to* have "more" elastic components. **But it really doesn't.**

(you can't "create" elastic components just by describing a material in a different axis system, the inherent properties of the material stay the same).

Figure M3.2-3 **Example of unidirectional composite (transversely isotropic) in two different axis systems**



No shear / extension coupling

Shears <u>with</u> <u>regard</u> <u>to</u> loading axis but still no <u>inherent</u> shear/extension coupling

In order to describe full behavior, need to do …TRANSFORMATIONS

--> <u>Orthotropic</u>

- Limit of current useful engineering materials
- Needed for composite analysis
- <u>No</u> coupling terms in the *<u>principal axes of the material</u>*

$$E_{1112},\ E_{1113},\ E_{1123},\ E_{2212},\ E_{2213},\ E_{2223},$$

$$E_{3312},\ E_{3313},\ E_{3323} = 0$$

– No shear strains arise when extensional stress is applied and vice versa)

e.g., $E_{1112} = 0$

(total of 9 terms are now zero)

– No extensional strains arise when shear stress is applied (and vice versa)

(same terms become zero as for previous condition)

– Shear strains (stresses) in one plane do <u>not</u> cause shear strains (stresses) in another plane

$$E_{1223},\ E_{1213},\ E_{1323} = 0$$

With these additional terms being zero, we end up with $\underline{\underline{9}}$ independent components:

(21 - 9 - 3 = 9)

and the resulting equations are:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{1122} & E_{2222} & E_{2233} & 0 & 0 & 0 \\ E_{1133} & E_{2233} & E_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2E_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2E_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2E_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}$$

For other cases, no more terms become zero, but the terms are not Independent.

--> $\underline{\underline{9}}$ independent components

(3 orthogonal axes with different responses along each)

(e.g., orthogonal crystals, woods, composites)

For example, consider….

--> Isotropic

- No further zero terms (after orthotropic)
- Components of elasticity tensor are related
- Only 2 independent constants
    - $E_{1111} = E_{2222} = E_{3333}$
    - $E_{1122} = E_{1133} = E_{2233}$
    - $E_{2323} = E_{1313} = E_{1212}$
    - And there is one other equation relating E_{1111} , E_{1122} and E_{2323}
- Behavior of most metals, polymers

--> elastic response the same in all directions

To better consider these cases we need to discuss:

(Measurement of) Engineering Constants

It is important to remember that generalized Hooke's Law is a *model* of the stress-strain response. So the components must be measured experimentally.

The components of the tensors cannot be directly measured, but we characterize materials by their…

"Engineering Constants"

(*or*, Elastic Constants)

What we can physically measure for orthotropic materials (inclusive), there are 3 types:

1. Longitudinal (Young's) (Extensional) Modulus: relates extensional strain in the direction of loading to stress in the direction of loading.

(3 of these)

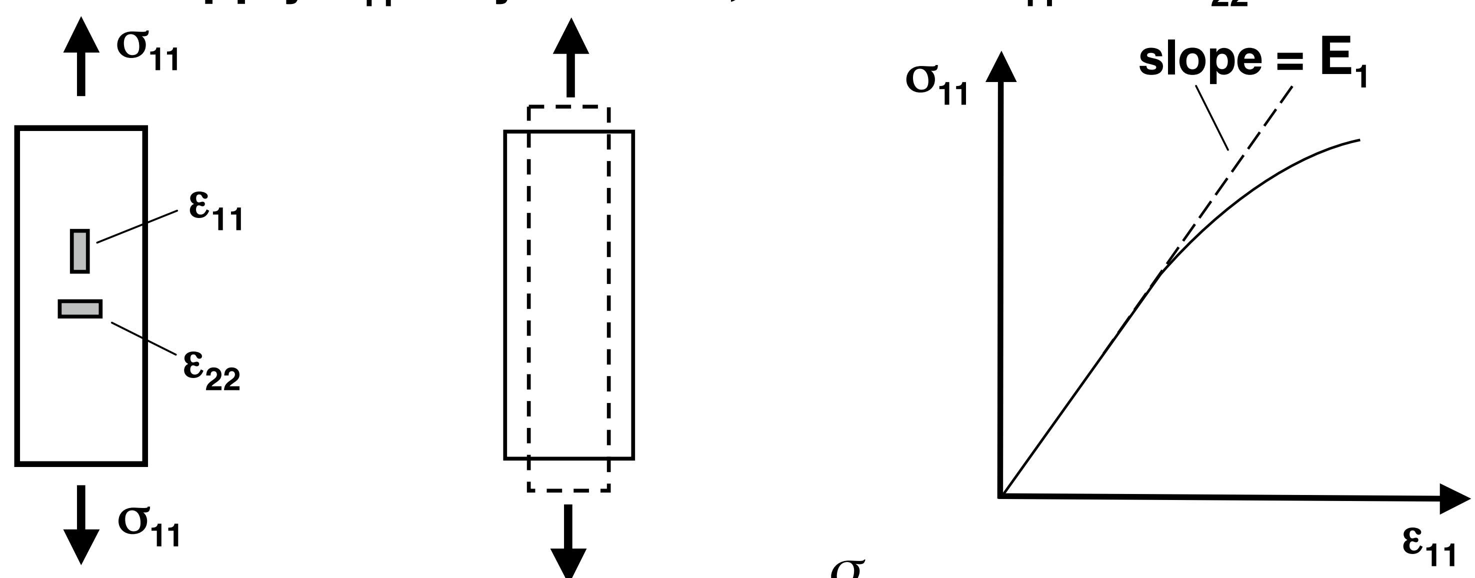
2. Poisson's Ratio: relates extensional strain in the loading direction to extensional strain in another direction.
   (6 of these…only 3 are independent)

3. Shear Modulus: relates shear strain in the plane of shear
   (3 of these)

Examples:

1. Longitudinal Modulus

Figure M3.3a **Apply σ_{11} only to a bar, measure ε_{11} and ε_{22}:**



$$E_1 = \frac{\sigma_{11}}{\varepsilon_{11}} = E_x$$

generally:

1) E_{11} or E_{xx} or E_1 or E_x

2) E_{22} or E_{yy} or E_2 or E_y

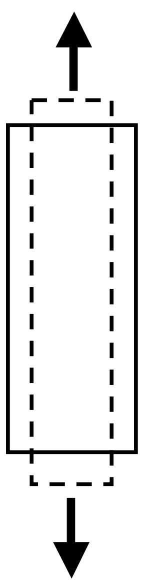
3) E_{33} or E_{zz} or E_3 or E_z

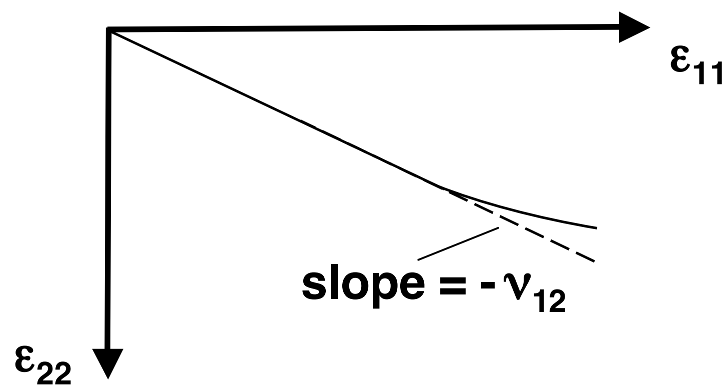
In general: $E_{mm} = \dfrac{\sigma_{mm}}{\varepsilon_{mm}}$ due to σ_{mm} applied *only*

(no summation on m)

2. Poisson's Ratios (***negative*** ratios)

Figure M3.3b **Consider ratio of two longitudinal strains**





$$\nu_{12} = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = \nu_{xy}$$

negative ratio of ε_{22} to ε_{11} with σ_{11} applied only!

In general: ν_{mn}

loading in m - direction

strain in n - direction

generally:

1) ν_{12} or ν_{xy}: (negative of) ratio of ε_{22} to ε_{11} due to σ_{11}
2) ν_{13} or ν_{xz}: (negative of) ratio of ε_{33} to ε_{11} due to σ_{11}
3) ν_{23} or ν_{yz}: (negative of) ratio of ε_{33} to ε_{22} due to σ_{22}
4) ν_{21} or ν_{yx}: (negative of) ratio of ε_{11} to ε_{22} due to σ_{22}
5) ν_{31} or ν_{zx}: (negative of) ratio of ε_{11} to ε_{33} due to σ_{33}
6) ν_{32} or ν_{zy}: (negative of) ratio of ε_{22} to ε_{33} due to σ_{33}

In general: $\nu_{nm} = -\frac{\varepsilon_{mm}}{\varepsilon_{nn}}$ due to σ_{nn} applied *only*

(for $n \neq m$)

Important: $\nu_{nm} \neq \nu_{mn}$

However, these are not all independent. There are relations known as "reciprocity relations" (3 of them) between Poisson's ratios and extensional moduli:

$$\nu_{21} E_{11} = \nu_{12} E_{22}$$

$$\nu_{31} E_{11} = \nu_{13} E_{33}$$

$$\nu_{32} E_{22} = \nu_{23} E_{33}$$

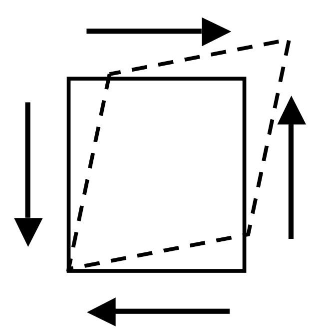
$\Longrightarrow$ **only 3 Poisson's ratios are independent!**

3. Shear Moduli

--> Apply σ_{12} only and measure ε_{12}

$$G_{12} = \frac{\sigma_{12}}{2\varepsilon_{12}} = G_{xy}$$

factor of 2 because it is an engineering constant and thus use engineering strain



generally:

1) G_{12} or G_{xy} or G_6: contribution of $(2)\varepsilon_{12}$ to σ_{12}
2) G_{13} or G_{xz} or G_5: contribution of $(2)\varepsilon_{13}$ to σ_{13}
3) G_{23} or G_{yz} or G_4: contribution of $(2)\varepsilon_{23}$ to σ_{23}

<u>In general</u>: $G_{mn} = \dfrac{\sigma_{mn}}{2\varepsilon_{mn}}$ due to σ_{mn} applied *<u>only</u>*

factor of 2 here since it relates physical quantities

$$\frac{\text{shear stress}}{\text{shear deformation (angular change)}} \Rightarrow G_{mn} = \frac{\tau_{mn}}{\gamma_{mn}}$$

--> one can think about doing each case separately and measuring the effects. Since this is linear, one can use superposition and thereby get the overall effect. This gives the stress-strain relations with the engineering constant (compliance format)

Orthotropic

In material principal axes, there is no coupling between extension and shear and no coupling between planes of shear, so the following constants remain:

$$\underbrace{\begin{matrix} E_1 \\ E_2 \\ E_3 \end{matrix}}_{3} \quad + \quad \underbrace{\begin{matrix} \nu_{12}, \nu_{21} \\ \nu_{13}, \nu_{31} \\ \nu_{23}, \nu_{32} \end{matrix}}_{3} \quad + \quad \underbrace{\begin{matrix} G_{12} \\ G_{13} \\ G_{23} \end{matrix}}_{3} \quad = \quad \underline{9}$$

(*same as E_{mnpq},* ***better be!***)

matrix form:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{S} \; \underset{\sim}{\sigma}$$

this is, in fact, the compliance matrix

Note: 2's now incorporated in engineering strain terms

Note: the reciprocity relations can be used to get the equations in the following form:

$$\varepsilon_1 = \frac{1}{E_1}\left[\sigma_1 - \nu_{12}\sigma_2 - \nu_{13}\sigma_3\right]$$

$$\varepsilon_2 = \frac{1}{E_2}\left[-\nu_{21}\sigma_1 + \sigma_2 - \nu_{23}\sigma_3\right]$$

$$\varepsilon_3 = \frac{1}{E_3}\left[-\nu_{31}\sigma_1 - \nu_{32}\sigma_2 + \sigma_3\right]$$

$$\gamma_{23} = \frac{1}{G_{23}}\sigma_{23}$$

$$\gamma_{13} = \frac{1}{G_{13}}\sigma_{13}$$

$$\gamma_{12} = \frac{1}{G_{12}}\sigma_{12}$$

Isotropic

As we get to materials with less elastic constants (< 9) than an orthotropic material, we no longer have any more zero terms in the elasticity or compliance matrix, but more nonzero terms are related.

For the isotropic case:

- All extensional moduli are the same:

  $E_1 = E_2 = E_3 = E$

- All Poisson's ratios are the same:

  $\nu_{12} = \nu_{21} = \nu_{13} = \nu_{31} = \nu_{23} = \nu_{32} = \nu$

- All shear moduli are the same:

  $G_4 = G_5 = G_6 = G$

- And, there is a relationship between E, ν and G:

$$G = \frac{E}{2(1 + \nu)}$$

Thus, there are only 2 independent constants.

This gives:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$$

Can also write this out in full form:

$$\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - \nu\sigma_2 - \nu\sigma_3\right)$$

$$\varepsilon_2 = \frac{1}{E}\left(-\nu\sigma_1 + \sigma_2 - \nu\sigma_3\right)$$

$$\varepsilon_3 = \frac{1}{E}\left(-\nu\sigma_1 - \nu\sigma_2 + \sigma_3\right)$$

$$\gamma_{23} = \frac{\sigma_{23}}{G}$$

$$\gamma_{13} = \frac{\sigma_{13}}{G}$$

$$\gamma_{12} = \frac{\sigma_{12}}{G}$$

As noted, there are only 2 independent elastic constants: E and ν

Sometimes express as/use Lamé's constants: μ and λ

$$\mu = \frac{E}{2(1 + \nu)} = G$$

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}$$

(can be derived by considering relationship between shear stress and principal stresses)

There is also another derived modulus known as the "bulk modulus", κ:

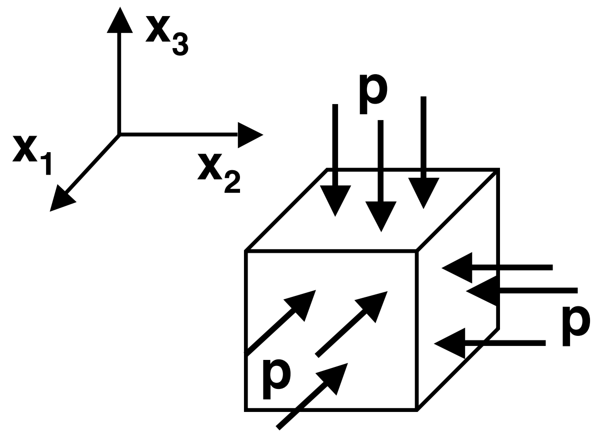
$$\kappa = \frac{3\lambda + 2\mu}{3} = \frac{E}{3(1 - 2\nu)}$$

The bulk modulus characterizes the <u>compressibility</u> of a material under *hydrostatic* stress/pressure

hydrostatic = same on all sides

(think of submerging cube in water)

Figure M3.4 **Illustration of hydrostatic stress/pressure**



The volume of the block changes from V to V′

$$\Rightarrow \Delta = \text{volumetric strain} = \frac{\Delta V}{V} = \frac{V' - V}{V}$$

And the bulk modulus, κ, relates stress to volumetric strain:

$$p = \kappa \Delta$$

Use this physical situation in the isotropic stress-strain equations:

$$\sigma_1 = \sigma_2 = \sigma_3 = -p$$

gives:

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon = \frac{-p}{E}(1 - 2\nu)$$

each side of the cube changes length related to strain from *L* to:

$$L' = L(1 - \varepsilon)$$

So the new volume is:

$$V' = (L')^3 = L^3(1 - \varepsilon)^3$$

$$= L^3(1 - 3\varepsilon + 3\varepsilon^2 - \varepsilon^3)$$

since ε is small (order of 0.01 - 0.02), we neglect higher order terms:

$$\Rightarrow V' = L^3(1 - 3\varepsilon)$$

Now:

$$\Delta V = V' - V = L^3(1 - 3\varepsilon - 1) = L^3(-3\varepsilon)$$

$$\Delta = \frac{\Delta V}{V} = -\frac{L^3(3\varepsilon)}{L^3} = -3\varepsilon$$

$$= 3\frac{p}{E}(1 - 2\nu)$$

Finally recalling that:

$$\kappa = \frac{p}{\Delta}$$

gives:

$$\kappa = \frac{E}{3(1 - 2\nu)}$$ Q.E.D.

In general, for all cases once we test and characterize the behavior, this gives us the compliance form of the equations. If we want to get the components of the tensors:

- convert to compliance tensor format
- invert compliance matrix to get elasticity matrix and thus components of elasticity tensor

This unit has been devoted to establishing the ***model*** for stress-strain response of a material on a **macroscopic** basis. But there are reasons based on the **microstructure** that certain materials behave certain ways. We thus need to look at ***the structure of materials*** to look at the physical basis for elastic properties.