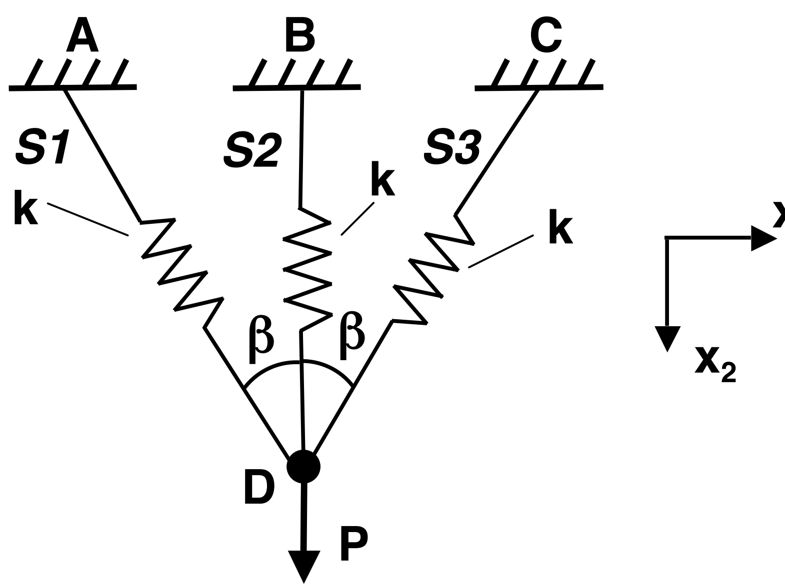
UNIFIED HANDOUT
MATERIALS AND STRUCTURES - #M-2
Fall, 2008

Notes **on Displacement Compatibility**

In considering displacement compatibility, the order of the solution approach and the associated assumptions that one uses changes the overall outcome of solution. It does not, however, affect the basic *concept* of compatibility -- it is the model that changes while compatibility is valid within the assumptions associated with the model.

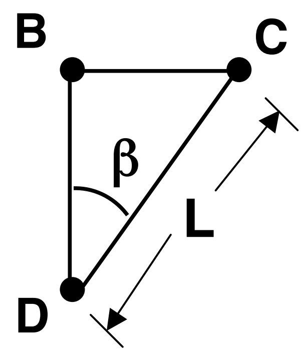
To demonstrate this, consider the example from Unit M1.5 on page 11 where three springs are each hooked at one end to a wall and at another end to each other with a load applied at the common point of contact. This is represented in the following figure:



A key factor in considering this configuration is the reality and principle of ***symmetry***. This configuration is geometrically symmetric about the BD line including symmetry of load application. Thus, it must behave the same way on either side of this *line of symmetry*. Therefore, the problem needs to only be considered and solved about onehalf of this line of symmetry. That solution gives the solution for the other side of the line. This is a manifestation of the *principle of symmetry*.

So consider the two springs S2 and S3 and what happens as point D moves in the x_2-direction. We are not interested in how far it moves for a given load P, at this point, only what must happen with regard to displacement *compatibility* between the springs. We do know that because of symmetry, the point D will deform only in the x_2-direction (Springs S3 and S1 exert equal and opposite resistance to movement in the x_1-direction resulting in no displacement in that direction.)

Consider the relationship between the lengths of S2 and S3 and the length between their attachment point to the wall (at points B and C, respectively).

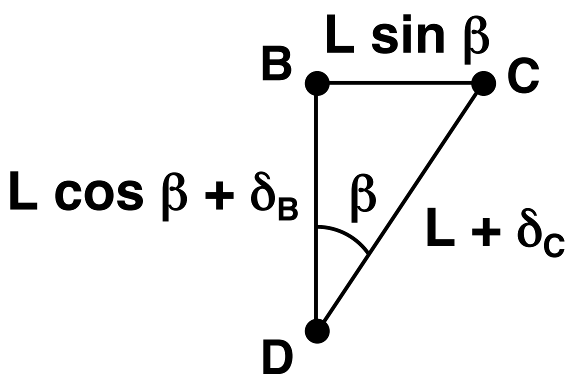


If the length of S3 (from D to C) is L, then the length of S2 (from D to B) is ($L \cos \beta$), and the length along the wall (from B to C) is ($L \sin \beta$).

Now, our solution for displacement depends on our assumptions and the "order" of our solutions........... Consider different such solutions.

1. __Zeroth Order Solution__ -- *angle does not change*

In this case, it is assumed that the angle does not change after displacement and is still β. Label the deformation of S2 along its length as δ_B, and that of S3 along its length as δ_C. With these definitions, the new length of S2 is the original length plus the deformation: ($L \cos \beta + \delta_B$) ; and the new length of S3 is the original length plus the deformation: $(L + \delta_C)$. The attachment points stay the same, so the distance from B to C is still $(L \sin\beta)$. This is represented in the new triangular geometry below.



The base assumption is that the angle β does not change. By trigonometric relationships we get:

$$(L + \delta_C)\cos\beta = (L\cos\beta + \delta_B)$$

This gives:

$$L\cos\beta + \delta_C\cos\beta = L\cos\beta + \delta_B$$

and finally yields:

$$\boxed{\delta_C \cos\beta = \delta_B}$$ ***Zeroth Order Solution***

This is the same as the lecture notes on page 12. Note that there is a slight inconsistency since geometrically one must still have that:

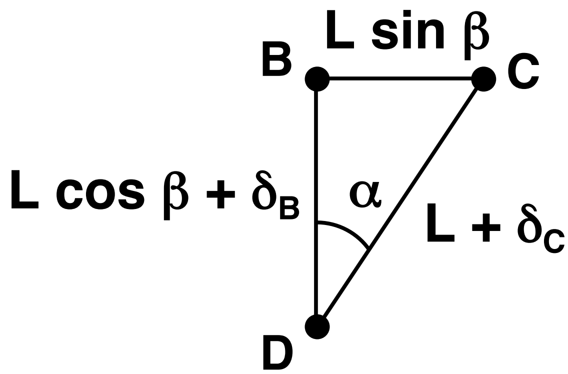
$$(L + \delta_C)\sin\beta = L\sin\beta$$

This can only be true if the term $(\delta_C \sin\beta)$ is sufficiently small that it can be neglected.

This inconsistency can be better addressed by considering a higher order mode. The current model ignores any rotation of the spring S3 and thus change in the angle β. To account for this, proceed to the next level and consider the......

2. **First Order Solution** -- *displacements are small but angle changes*

We start with the same initial geometry, and the deformed geometry is quite similar (e.g. still a triangle), **except** the angle is now different. This is represented in the triangular geometry for this case.



Call the new angle α. One can note that since the lengths of the springs have increased, the angle will have decreased. From the geometry in this case, we have:

$$(L + \delta_C) \sin \alpha = L \sin \beta \tag{1}$$

and

$$(L + \delta_C) \cos \alpha = L \cos \beta + \delta_B \tag{2}$$

The first of these gives an expression for the angle, α:

$$\alpha = \sin^{-1} \left[\{ L / (L + \delta_C) \} \sin \beta \right] \tag{3}$$

The trigonometric relations indicate that:

$$\text{if} \quad \alpha = \sin^{-1} x \qquad \text{then:} \quad \cos \alpha = \left(1 - x^2\right)^{1/2} \tag{4}$$

Here we have:

$$x = \{ L / (L + \delta_C) \} \sin \beta \tag{5}$$

Put this expression in the second equation for $\cos \alpha$ in Equation (4) and then use this in the overall expression of Equation (2):

$$(L + \delta_C) \left[1 - \{ L / (L + \delta_C)^2 \} \sin^2 \beta \right]^{1/2} = L \cos \beta + \delta_B \tag{6}$$

This can be manipulated to get the following expression:

$$(L + \delta_C)\left[\{(L + \delta_C)^2 - L^2 \sin^2 \beta\} / (L + \delta_C)^2\right]^{1/2} = L \cos \beta + \delta_B \qquad (7)$$

The term $(L + \delta_C)$ cancels out in the expression when taking it inside the expression in brackets. This leaves:

$$\left[(L + \delta_C)^2 - L^2 \sin^2 \beta\right]^{1/2} = L \cos \beta + \delta_B \qquad (8)$$

Square both sides and perform operations:

$$L^2 + 2L\delta_C + \delta_C^2 - L^2 \sin^2 \beta = L^2 \cos^2 \beta + 2L \cos \beta \, \delta_B + \delta_B^2 \qquad (9)$$

then get:

$$L^2 + 2L\delta_C + \delta_C^2 = L^2 (\cos^2 \beta + \sin^2 \beta) + 2L \cos \beta \, \delta_B + \delta_B^2 \qquad (10)$$

Recalling that $(\cos^2 \beta + \sin^2 \beta) = 1$ and using this in the above while eliminating L^2 from both sides gives:

$$2L\delta_C + \delta_C^2 = 2L \cos \beta \, \delta_B + \delta_B^2 \qquad (11)^*$$

This equation is the key to define compatibility for the general deformation case where angles can change. For the first order solution, we **NOW** make the **<u>first</u> <u>order</u> approximation**. This is that deformations are small and *higher order terms* (e.g. squared) *are ignored*. With this assumption the governing equation $(11)^*$ becomes:

$$\boxed{\delta_C = \delta_B \cos \beta}$$

First Order Solution

3. <u>**Higher Order Solution**</u> -- *allow large deformations*

The <u>next</u> <u>order</u> solution would involve not assuming that the deformations are small and retaining higher order terms in the derivation previously performed. This leaves the full nonlinear Equation $(11)^*$. The solutions comes from there.

The key in all this is **<u>CONSISTENCY</u>**. One must look at the resulting solution (actual numbers) to determine if the result(s) is (are) **consistent** with the initial assumption(s) or with the level of fidelity one wants (or is mandated by the accuracy/fidelity of the original data).