UNIFIED HANDOUT
MATERIALS AND STRUCTURES - #M-8
Fall, 2008

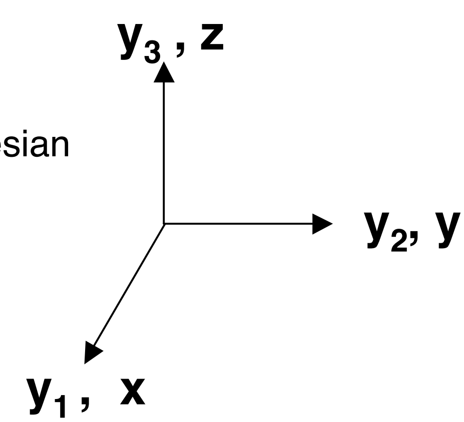
Review of Stress, Strain and Elasticity

NOTATION REVIEW (*e.g., for strain*)

Engineering		Contracted		Engineering "Tensor"		Tensor
ε_x	=	ε_1	=	ε_{xx}	=	ε_{11}
ε_y	=	ε_2	=	ε_{yy}	=	ε_{22}
ε_z	=	ε_3	=	ε_{zz}	=	ε_{33}
γ_{yz}	=	ε_4	=	$2\ \varepsilon_{yz}$	=	$2\ \varepsilon_{23}$
γ_{xz}	=	ε_5	=	$2\ \varepsilon_{xz}$	=	$2\ \varepsilon_{13}$
γ_{xy}	=	ε_6	=	$2\ \varepsilon_{xy}$	=	$2\ \varepsilon_{12}$

EQUATIONS OF ELASTICITY

Right-handed rectangular Cartesian coordinate system



15 equations/15 unknowns

1. Equilibrium (3)

$$\left.\begin{aligned} \frac{\partial \sigma_{11}}{\partial y_1} + \frac{\partial \sigma_{21}}{\partial y_2} + \frac{\partial \sigma_{31}}{\partial y_3} + f_1 &= 0 \\ \frac{\partial \sigma_{12}}{\partial y_1} + \frac{\partial \sigma_{22}}{\partial y_2} + \frac{\partial \sigma_{32}}{\partial y_3} + f_2 &= 0 \\ \frac{\partial \sigma_{13}}{\partial y_1} + \frac{\partial \sigma_{23}}{\partial y_2} + \frac{\partial \sigma_{33}}{\partial y_3} + f_3 &= 0 \end{aligned}\right\} \quad \frac{\partial \sigma_{mn}}{\partial y_m} + f_n = 0$$

2. Strain-Displacement (6)

$$\varepsilon_{11} = \frac{\partial u_1}{\partial y_1} \qquad \varepsilon_{21} = \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y_2} + \frac{\partial u_2}{\partial y_1}\right)$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial y_2} \qquad \varepsilon_{31} = \varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y_3} + \frac{\partial u_3}{\partial y_1}\right)$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial y_3} \qquad \varepsilon_{32} = \varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial y_3} + \frac{\partial u_3}{\partial y_2}\right)$$

$$\Bigg\} \quad \varepsilon_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial y_n} + \frac{\partial u_n}{\partial y_m}\right)$$

3. Stress-Strain (6)

$$\sigma_{mn} = E_{mnpq}\,\varepsilon_{pq}$$

Generalized Hooke's Law:

- Anisotropic:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 2E_{1123} & 2E_{1113} & 2E_{1112} \\ E_{1122} & E_{2222} & E_{2233} & 2E_{2223} & 2E_{2213} & 2E_{2212} \\ E_{1133} & E_{2233} & E_{3333} & 2E_{3323} & 2E_{3313} & 2E_{3312} \\ E_{1123} & E_{2223} & E_{3323} & 2E_{2323} & 2E_{1323} & 2E_{1223} \\ E_{1113} & E_{2213} & E_{3313} & 2E_{1323} & 2E_{1313} & 2E_{1213} \\ E_{1112} & E_{2212} & E_{3312} & 2E_{1223} & 2E_{1213} & 2E_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}$$

- Orthotropic:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{1122} & E_{2222} & E_{2233} & 0 & 0 & 0 \\ E_{1133} & E_{2233} & E_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2E_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2E_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2E_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix}$$

Compliance Form: $\varepsilon_{mn} = S_{mnpq}\,\sigma_{pq}$

where: $\underset{\sim}{E}^{-1} = \underset{\sim}{S}$

DEFINITION OF ENGINEERING CONSTANTS

1. Longitudinal (Young's) (Extensional) Moduli:

$$E_{mm} = \frac{\sigma_{mm}}{\varepsilon_{mm}}$$ due to σ_{mm} applied *only* *(no summation on m)*

2. Poisson's Ratios:

$$\nu_{nm} = -\frac{\varepsilon_{mm}}{\varepsilon_{nn}}$$ due to σ_{nn} applied *only* *(for $n \neq m$)*

Reciprocity: $\nu_{nm} E_m = \nu_{mn} E_n$ (no sum)
($m \neq n$)

3. Shear Moduli:

$$G_{mn} = \frac{\sigma_{mn}}{2\varepsilon_{mn}}$$ due to σ_{mn} applied *only*

"ENGINEERING" STRESS-STRAIN EQUATIONS *(using contracted notation)*

• Orthotropic form

In terms of ENGINEERING CONSTANTS (using *contracted notation*):

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_6} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

• Isotropic form

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}$$

with: $G = \frac{E}{2(1 + \nu)}$

TRANSFORMATIONS

$$\tilde{\sigma}_{mn} = \ell_{\tilde{m}p} \, \ell_{\tilde{n}q} \, \sigma_{pq}$$

$$\tilde{\varepsilon}_{mn} = \ell_{\tilde{m}p} \, \ell_{\tilde{n}q} \, \varepsilon_{pq}$$

$$\tilde{x}_m = \ell_{\tilde{m}p} \, x_p$$

$$\tilde{u}_m = \ell_{\tilde{m}p} \, u_p$$

$$\tilde{E}_{mnpq} = \ell_{\tilde{m}r} \, \ell_{\tilde{n}s} \, \ell_{\tilde{p}t} \, \ell_{\tilde{q}u} \, E_{rstu}$$

where: $\ell_{\tilde{m}n}$ = cosine of angle from $\tilde{y}_m$ to y_n