Unit M1.2
(All About) Mechanical Equilibrium

Readings:

CDL 1.6

16.001/002 -- *"Unified Engineering"*
Department of Aeronautics and Astronautics
Massachusetts Institute of Technology

LEARNING OBJECTIVES FOR UNIT **M1.2**

Through participation in the lectures, recitations, and work associated with Unit M1.2, it is intended that you will be able to………

- ….**apply** the principle/concept of equilibrium to **determine** the applied and transmitted forces and moments, and related motion, for a particle, set of particles, or body
- ….**model** a body/system and external forces and moments acting on such
- ….**apply** the concept of equipollent force systems to **model** a set of forces

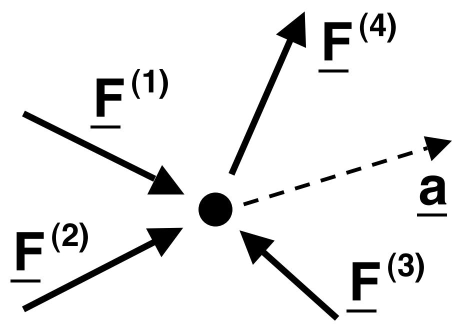
The concept of equilibrium is the first of the 3 Great Concepts of solid mechanics we will consider.

For a structure, need to consider the general case. But let's build up to this by considering the simplest case.

Equilibrium of a particle

A particle is a body whose mass is concentrated at a point

Figure M1.2-1 **Forces acting on a particle produce acceleration**



Newton's law gives:

$$\sum_i \underline{F}^{(i)} = m\underline{a}$$

Rewrite this as:

$$\underbrace{\sum_i \underline{F}^{(i)}}_{\text{external forces}} - \underbrace{m\underline{a}}_{\text{inertial forces}} = 0$$

--> **"D'Alembert's Principle"**
- *ma* treated as "inertial force"

--> In <u>many</u> cases in structures, we have:

$$\underline{a} = 0 \quad \text{(\underline{Statics})}$$

So this reduces to:

$$\sum \underline{F}^{(i)} = 0$$

using vector notation:

$$\sum \underline{F}^{(i)} = \sum F_m^{(i)} \underline{i}_m = 0$$

This can be written as:

$$\boxed{\sum_i F_m^{(i)} = 0} \quad \text{(\underline{Static}) Equilibrium}$$

which represents three equations:

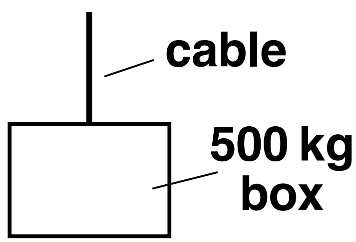
3 scalar equations

$$\left\{\begin{array}{ll} \sum_i F_1^{(i)} = 0 & \text{sum of forces in } x_1 = 0 \\ \sum_i F_2^{(i)} = 0 & \text{sum of forces in } x_2 = 0 \\ \sum_i F_3^{(i)} = 0 & \text{sum of forces in } x_3 = 0 \end{array}\right\}$$

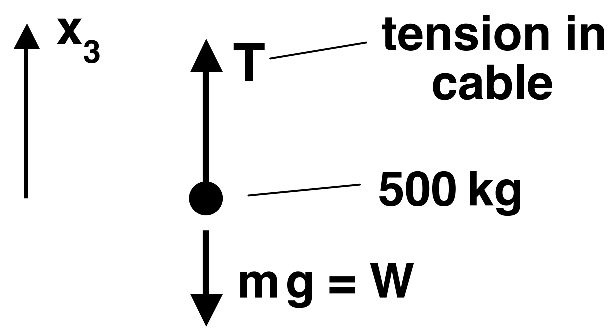
--> Example of static equilibrium versus general case:

Elevator…find cable tension

Figure M1.2-2 **Case of no motion**



Model as point mass:



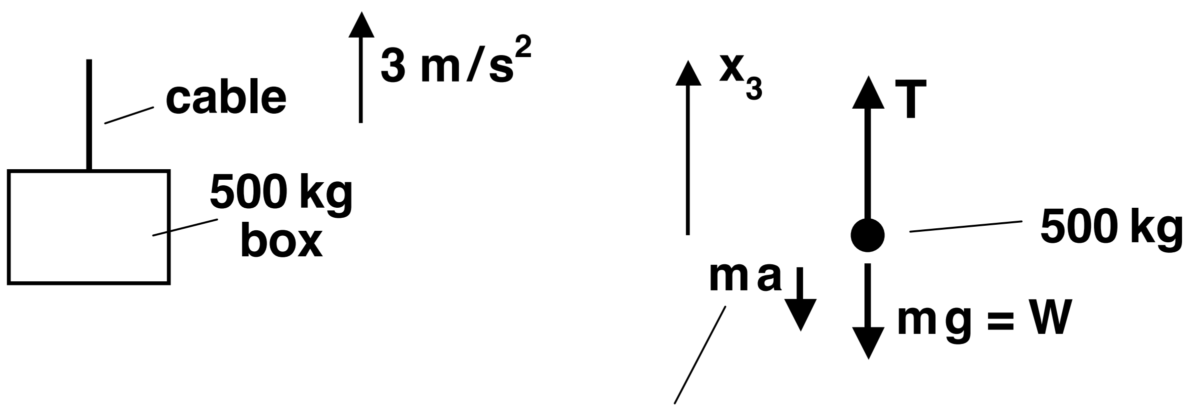
$$\sum F = 0 \Rightarrow \uparrow + \qquad \text{(give \underline{positive} direction)}$$

$$\Rightarrow T - 500 kg \left(9.8 \frac{m}{s^2}\right) = 0$$

$$\Rightarrow T = \underline{\underline{4900 \ N}}$$

(Note: Work/answers not correct without units!)

Figure M1.2-3 **Case of accelerating upwards at 3 m/s²**



inertial force (a = 3 m/s²) in
opposite direction from acceleration

Same model but now also model acceleration as inertial force

$$\sum F = 0$$

$$T - 500kg\ (9.8\ \frac{m}{s^2}) - 500\left(3\ \frac{m}{s^2}\right) = 0$$

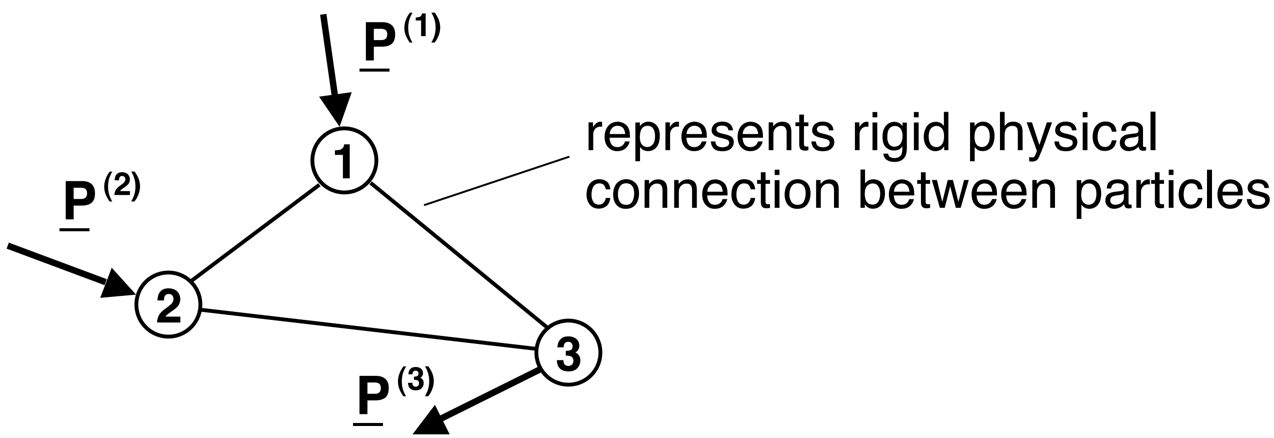
$$\Rightarrow T = \underline{\underline{6400\ N}}$$

Let's move up one level and consider…

Equilibrium of a System of Particles

Look at…

Figure M1.2-4 **A system of particles with forces, $\underline{P}^{(i)}$, acting on particle *i***



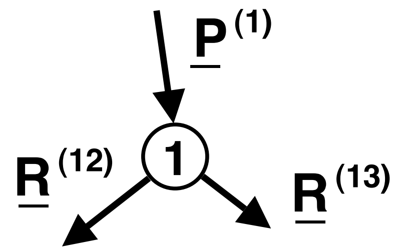
"Isolate" particle 1…

--> What other forces act on particle 1 <u>besides</u> external force $\underline{P}^{(1)}$

- reaction (internal) forces in connection
- inertial forces

So for particle 1 draw a "Free Body Diagram" (will discuss this more in Unit M1.5)

Figure M1.2-5 **Free Body Diagram for Particle 1**

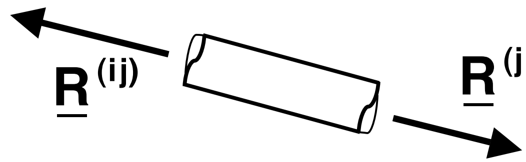


<u>Notes</u>:

- $\underline{R}^{(ij)}$ represents reaction in connection between particles *i* and *j*
- Draw reactions initially in <u>tension</u> (+) (convenient convention). If they come out (–), they are in compression, but you "take care of this" through sign

- Newton's law of action-reaction says $\underline{R}^{(ij)} = -\underline{R}^{(ji)}$ (must point in opposite directions but are of equal magnitude)

Figure M1.2-6

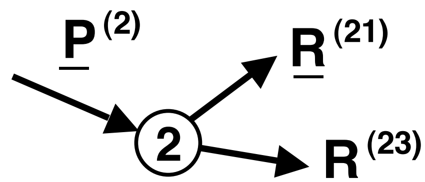


So apply equilibrium:

$$\underline{P}^{(1)} + \underline{R}^{(12)} + \underline{R}^{(13)} = m_1 \, \underline{a}_1 = 0$$ ← Statics

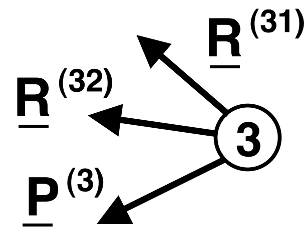
Similarly for particles 2 and 3:

Figure M1.2-7 **Free Body Diagram for Particle 2**



$$\underline{P}^{(2)} + \underline{R}^{(21)} + \underline{R}^{(23)} = m_2 \, \underline{a}_2 = 0$$

Figure M1.2-8 **Free Body Diagram for Particle 3**



$$\underline{P}^{(3)} + \underline{R}^{(31)} + \underline{R}^{(32)} = m_3 \, \underline{a}_3 = 0$$

Recalling $\underline{R}^{(ij)} = -\underline{R}^{(ji)}$ and summing these gives:

$$\underline{P}^{(1)} + \underline{P}^{(2)} + \underline{P}^{(3)} = \sum_i m_i \, a_i = 0 \quad \leftarrow \text{Statics}$$

For statics, this is simply:

$$\sum_i \underline{P}^{(i)} = 0 \quad \text{(as before)}$$

--> Is this enough?

We have dealt with no <u>linear</u> acceleration. What else can a system of particles/body do?

<u>ROTATE</u>

So there is a second set of equilibrium requirements…..

Moment Equilibrium

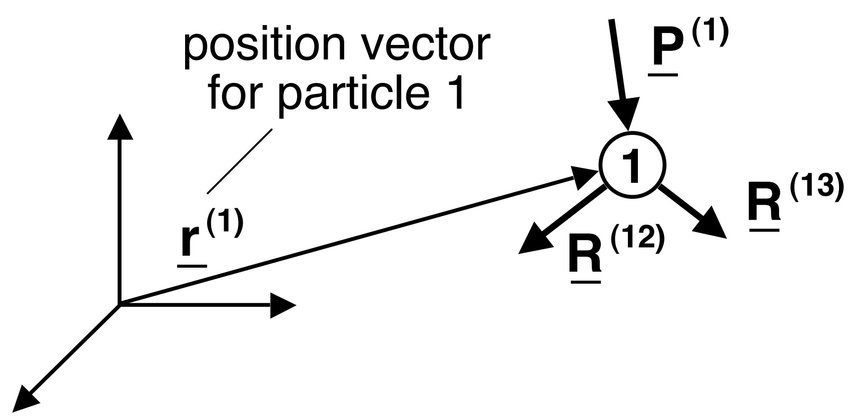
For there to be no rotation, the moments about a (any) point/axis must be zero

--> Why any?

Pure moment…magnitude does not change as point changes!

--> Consider each particle and the moments acting on it. Take moments about some "convenient" point:

Figure M1.2-9 **Moment for Particle 1**



See that

$$\underline{r}^{(1)} \times \left(\underline{P}^{(1)} + \underline{R}^{(12)} + \underline{R}^{(13)} \right) = 0$$

Similarly for particles 2 and 3:

$$\underline{r}^{(2)} \times \left(\underline{P}^{(2)} + \underline{R}^{(21)} + \underline{R}^{(23)}\right) = 0 \qquad \text{Particle 2}$$

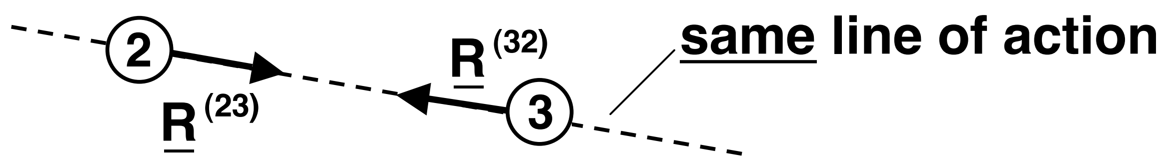
$$\underline{r}^{(3)} \times \left(\underline{P}^{(3)} + \underline{R}^{(31)} + \underline{R}^{(32)}\right) = 0 \qquad \text{Particle 3}$$

--> again use Newton's Law of Action-Reaction

$$\left(\underline{R}^{(ij)} = -\underline{R}^{(ji)}\right)$$

--> and we must further note that:

Figure M1.2-10 **Pairs Act Along the Same Line of Action**



Thus, the perpendicular distance from the point about which we take the moment to the line of action is the same.

So using both pieces of info:

$$\underline{r}^{(1)} \times \underline{R}^{(13)} + \underline{r}^{(3)} \times \underline{R}^{(31)} = 0$$

(⇒ pairs give equal but opposite moments!)

So take the three equilibrium equations, add them, and use this general concept:

$$\underline{r}^{(i)} \times \underline{R}^{(ij)} + \underline{r}^{(j)} \times \underline{R}^{(ji)} = 0$$

to get:

$$\boxed{\sum_n \underline{r}^{(n)} \times \underline{P}^{(n)} = 0}$$

(+ any inertial considerations)

or $\sum M^{(n)} = 0$

moment on particle (n) due to force on (n)

=> Equilibrium requires the moments of the external forces to be zero

What about for a general body?

Particles vs. Body/Equilibrium of a Body

A body can be considered to be a system of particles. In the limit, a body is an infinite set of infinitesimal masses/particles.

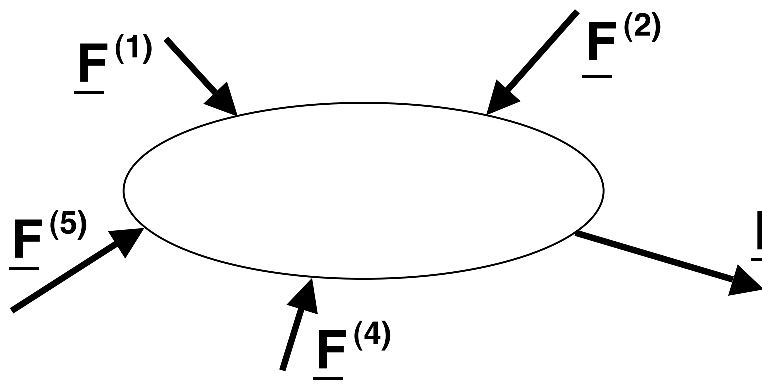
So can generalize the equations of equilibrium into two sets:

General Equations of Equilibrium:

$$\sum_n \underline{F}^{(n)} = 0$$ **Force Equilibrium**

$$\sum_n \underline{r}^{(n)} \times \underline{F}^{(n)} = 0$$ **Moment Equilibrium**

Figure M1.2-11 **Forces acting on a general body**



--> Deals with external forces only.
--> Gives two necessary and sufficient conditions for equilibrium (6 scalar equations total)

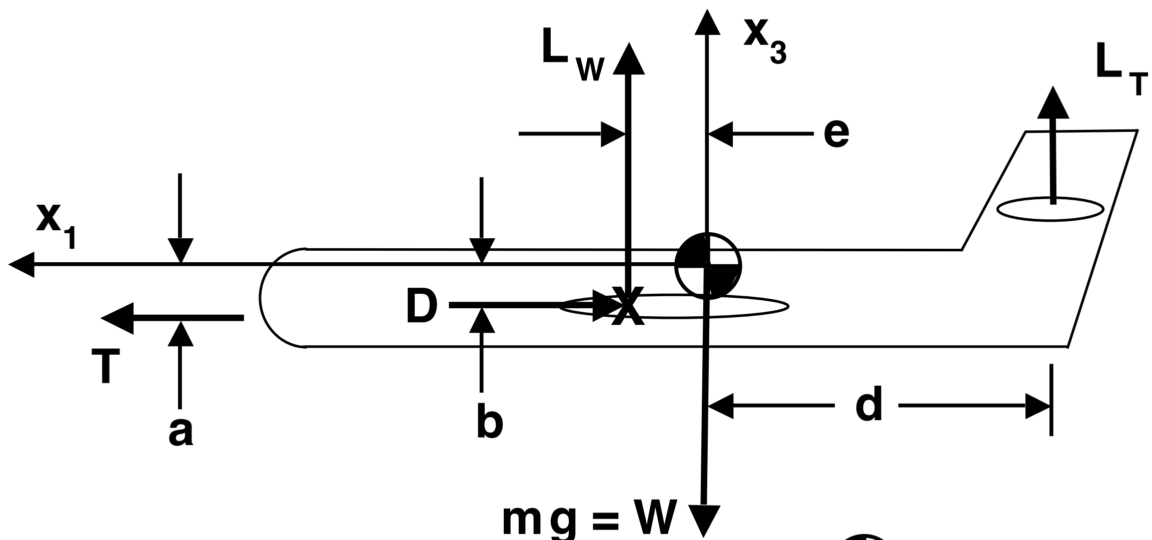
Let's now consider these concepts in two-dimensions (2-D) so that we can more easily represent (and comprehend) them. In addition, many force systems can be well-represented as a:

Planar Force System:

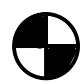
This is a force system where all forces act in one plane.

Example: Airplane in straight and level flight

Figure M1.2-12 **A model of an airplane**



All forces in x_1 - x_3 plane

 = center of gravity

X = aerodynamic center

Forces are:

$mg = W$	Weight
L_w	Lift at "aerodynamic center" of wing
D	Drag
T	Thrust
L_T	Lift on tail (could be downwards for "trim")

This is a "Free Body Diagram"

Important dimensions/distances:

a - distance thrust acts off c.g. line (in x_3)

b - distance drag acts off c.g. line *(a-c* to *c-g* in x_3*)*

e - distance wing lift acts off c.g. *(a-c* to *c-g* in x_1*)*

d - distance tail lift acts off c.g. *(*in x_1*)*

Write out equations of equilibrium:

$$\sum F_1 = 0 \quad \overset{+}{\longleftarrow} \quad \Rightarrow \quad T - D = 0 \qquad \text{(A1)}$$

show + direction!

$$\sum F_2 = 0 \qquad 0 = 0 \qquad \text{(A2)}$$

$$\sum F_3 = 0 \quad \uparrow + \quad L_w + L_T - W = 0 \qquad \text{(A3)}$$

$$\sum M_1 = 0$$

no forces out of x_1 - x_3 plane

$$\Rightarrow 0 = 0 \qquad \text{(A4)}$$

$$\sum M_2 = 0 \quad \circlearrowleft + \quad -Ta + Db - L_w e + L_T d = 0 \qquad \text{(A5)}$$

use c.g.

indicate point/axis

$$\sum M_3 = 0 \qquad 0 = 0 \qquad \text{(A6)}$$

(no forces out of x_1 - x_3 plane)

Note that 3 equations $\left(\sum F_2, \sum M_1, \sum M_3\right)$ are automatically satisfied since this is a planar system.

So two force equations (along axis which define plane) and one moment equation (about axis out of plane) remain for equilibrium.

Notes:

1. If aerodynamic center is behind c.g., tail lift must be negative for trim. Look at

   $\sum M_2 = 0$ a and b are small

   $$\Rightarrow -Le + L_T d = 0$$

   But if *e* is negative => L_T must be negative!
2. Always check to see that units are consistent!

Question: What is this L_w?

Is there really only one for a vector on a wing?

NO

Air moves by and pressure differential over entire area causes lift.

So how do we get L_w?

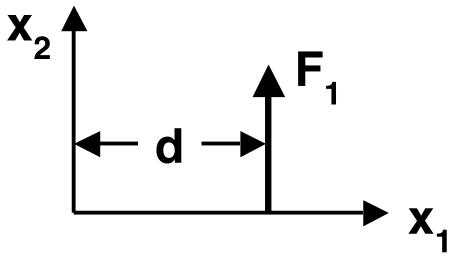
We need the concept of…..

Equipollent forces

--> Definition: Two force systems are equipollent (equally powerful) if they have the same total force and total moment about the same arbitrary point.

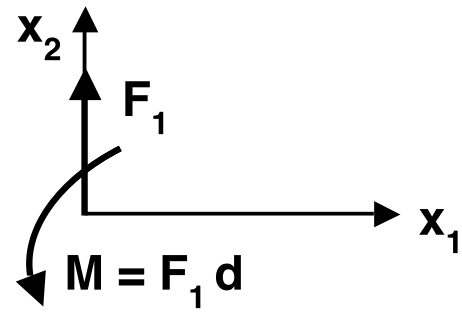
Figure M1.2-13 Example of equipollent force systems

(A) **Force at location d**



equipollent to

(B) **Force and pure moment at origin**



Why?

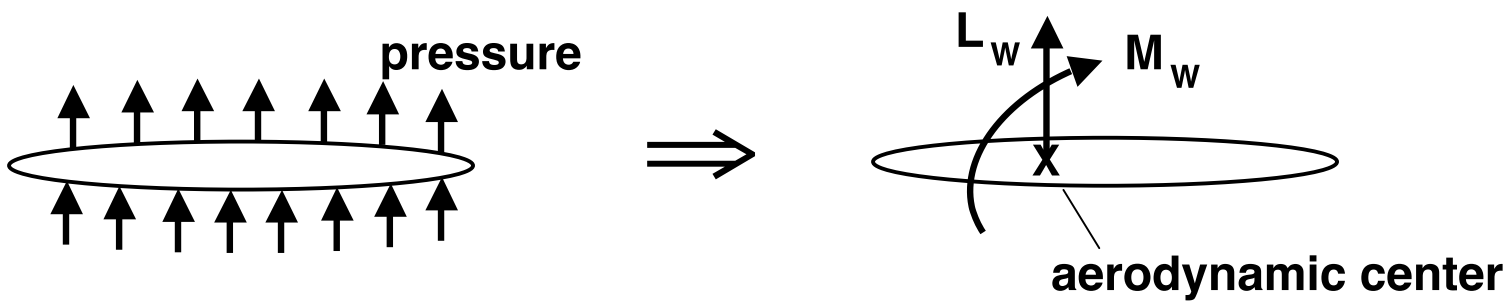
(A)		(B)
$F_A = F_1$	$=$	$F_B = F_1$
$M_A = F_1 d$	$=$	$M_B = F_1 d$

Not equilibrium $\left(\sum = 0\right)$ but equipollence (sums the same)

--> Generalize to 3-D:

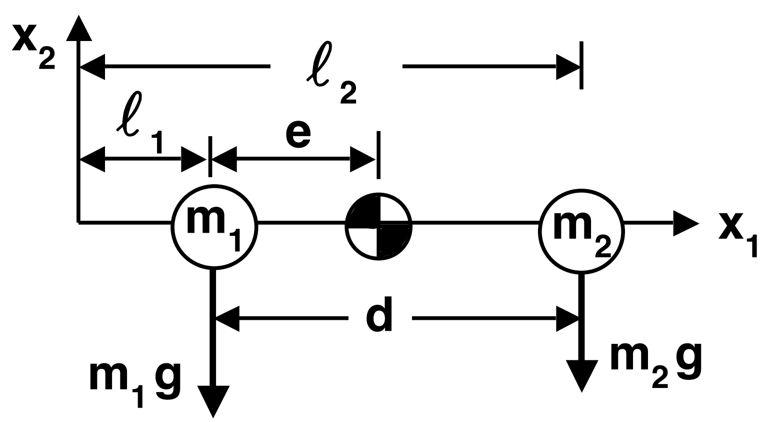
Must have $\sum F_i$ (exterior) and $\sum M_i$ (exterior) the same for both cases.

Figure M1.2-14 **Example of wing**



Same total force and moment about aerodynamic center

Figure M1.2-15 **Example of center of gravity (of 2-mass system)**



$$\sum F_1 = 0$$

$$\sum F_2 = m_1 g + m_2 g$$

Find point where: $\sum M_3 = 0$

$$\Rightarrow \quad m_1 g e = m_2 g (e - d)$$

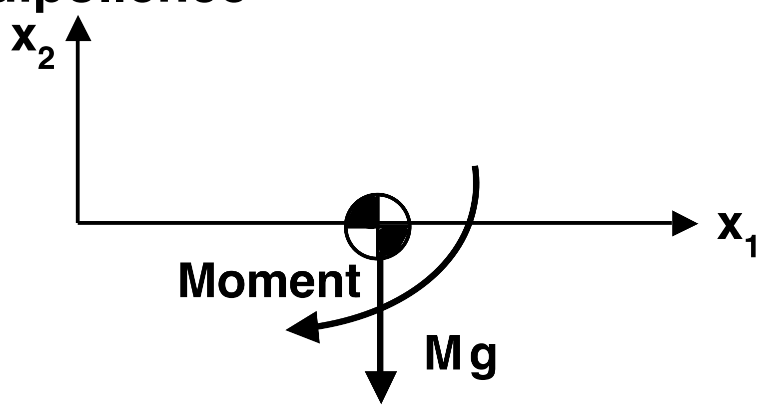
Solve for e

or can show by taking moments about origin:

$$\sum M_3 = 0 \; + \!\!\circlearrowright \quad \Rightarrow \quad m_1 g \ell_1 + m_2 g \ell_2$$

$$\Rightarrow m_1 g \ell_1 + m_2 g \ell_2 = \text{total moment}$$

Figure M1.2-16 **Place total weight and total moment at center of gravity for equipollence**



Recall for center of gravity in general:

$$x_{1c} = \frac{\sum m_1 \, x_1^{(i)}}{\sum m^{(i)}};$$

$$x_{2c} = \frac{\sum m^i \, x_2^{(i)}}{\sum m^{(i)}}$$

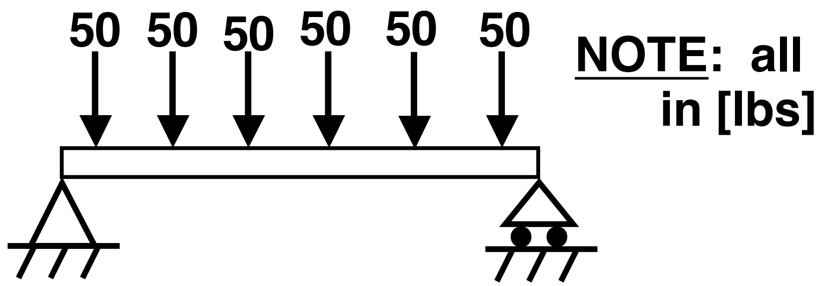
Important Note:

"Equipollent Force Systems" useful for idealizations. They produce:

(a) Same reactions for statics

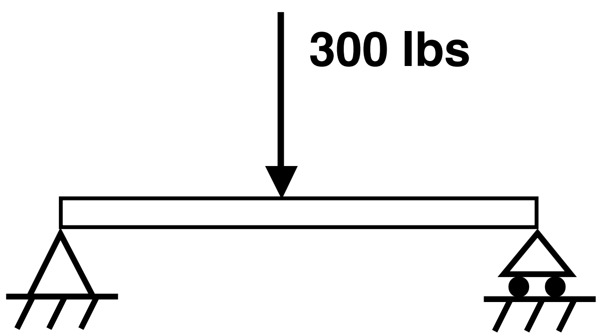
(b) Same rigid body motion for dynamics

BUT, will produce different deformations.



several small people on bench

vs.



one big person on bench (in middle)

Equipollent, but big guy causes bigger deflection

Thus, **equipollent** but **not** equivalent!

Next move on to see how external force are "reacted" in a body/through a body to achieve equilibrium.