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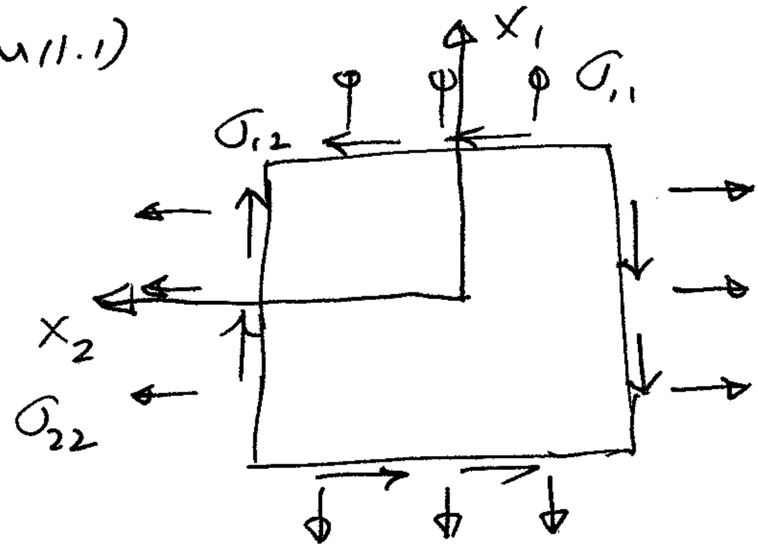
11/6/08

Unified Engineering

Problem Set 10 - Week 11
Fall, 2008

SOLUTIONS

M18 (M11.1)



$\sigma_{11} = -24$ ksi
$\sigma_{22} = 16$ ksi
$\sigma_{12} = -8$ ksi

(a) For plane stress the principal stresses are the roots of the equation:

$$\tau^2 - \tau(\sigma_{11} + \sigma_{22}) + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0$$

We do this relative to the original loading axes. Using these stresses gives:

$$\tau^2 - \tau(-24\,ksi + 16\,ksi) + [(-24\,ksi)(16\,ksi) - (-8\,ksi)^2] = 0$$

$$\Rightarrow \tau^2 - (-8\,ksi)\tau - 448\,(ksi)^2 = 0$$

Solve via the quadratic formula:

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{for } ax^2 + bx + c = 0$$

Here: $a = 1$; $b = +8\,ksi$ $c = -448\,(ksi)^2$

$$\Rightarrow \tau = \frac{-(+8\,ksi) \pm \sqrt{(+8\,ksi)^2 - 4(1)(-448)(ksi)^2}}{2(1)}$$

$$= \frac{-8 \pm \sqrt{1856}}{2}\,ksi$$

$$= \frac{-8 \pm 43.1}{2}\,ksi$$

$$\Rightarrow \tau = -25.6\,ksi,\ 17.5\,ksi$$

$$\Rightarrow \boxed{\begin{array}{l} \sigma_I = -25.6\,ksi \\ \sigma_{II} = +17.5\,ksi \end{array}}$$

to find the associated directions, use the expression:

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{12}}{\sigma_{11}-\sigma_{22}}\right)$$

$$\Rightarrow \theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2(-8\,ksi)}{-24\,ksi-(16\,ksi)}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{-16}{-40}\right)$$

$$= \frac{1}{2}\tan^{-1}(0.4)$$

$$\Rightarrow \theta_p = \frac{1}{2}(21.8^\circ)$$

$\Rightarrow \theta_p = +10.9^\circ$ for σ_I (check manually)

with σ_{II} rotated 90° from that

so:
$$\boxed{\begin{aligned}\theta_{p_I} &= +10.9^\circ \\ \theta_{p_{II}} &= +100.9^\circ = -79.1^\circ\end{aligned}}$$

Check the angles via the transformation equations for shear and that shear stress goes to zero (definition of principal axes and stresses):

$$\tilde{\sigma}_{12} = -\sin\theta\cos\theta\,\sigma_{11} + \cos\theta\sin\theta\,\sigma_{22} + (\cos^2\theta - \sin^2\theta)\,\sigma_{12}$$

for $\theta = +10.9^\circ$:

$$0 \stackrel{?}{=} -(+0.189)(0.982)(-24\,ksi) + (0.982)(+0.189)(+16\,ksi) + (0.964 - 0.036)(-8\,ksi)$$

$$\Rightarrow 0 \stackrel{?}{=} 4.45\,ksi + 2.97\,ksi - 7.42\,ksi$$

✓ (YES)

(same as for -79.1° with signs switched)

(b) Maximum shear stress(es) occur along planes/directions that are at 45° to the principal axes.

So: direction of maximum shear stress:

θ_{p_I} $+45^\circ$ from $x_1 = +55.9^\circ$

$\theta_{p_{II}}$ $+45^\circ$ from $x_1 = -34.1^\circ$

$$\boxed{\text{Directions of maximum shear} = +55.9^\circ, -34.1^\circ}$$

The value of the maximum shear stress(es) in the $x_1 - x_2$ plane is:

$$\frac{|\sigma_I - \sigma_{II}|}{2} = \frac{|-25.6\,ksi - (17.5\,ksi)|}{2}$$

$\Rightarrow$ $\boxed{\text{value of maximum shear stress} = 21.6\,ksi}$

(this takes on a sign of + and −)

<u>NOTE</u>: This value can also be determined by using the stress transformation equation for σ_{12} and the direction of maximum shear relative to the original loading axes. So we:

$$\tilde{\sigma}_{12} = -\sin\theta\cos\theta\,\sigma_{11} + \cos\theta\sin\theta\,\sigma_{22} + (\cos^2\theta - \sin^2\theta)\,\sigma_{12}$$

with:

$\sigma_{11} = -24\,ksi$

$\sigma_{22} = +16\,ksi$ and $\theta = +55.9°, -34.1°$

$\sigma_{12} = -8\,ksi$

$\rightarrow$ There are two other maximum shear stresses out of the $x_1 - x_2$ plane. For the case of plane stress, the out-of-plane principal stress is zero (i.e. $\sigma_{III} = 0$).

So we have:

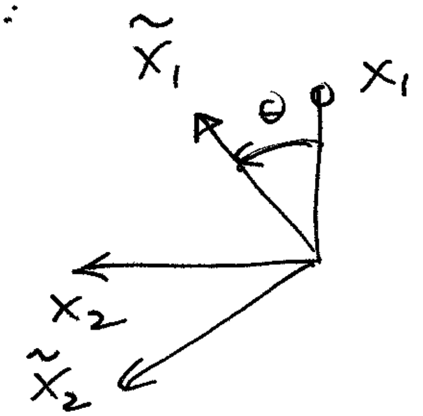
$$|\tau_{max}| = \left|\frac{\sigma_I - \sigma_{III}}{2}\right| = \frac{25.6\,ksi}{2} = \boxed{12.8\,ksi}$$

This is a plane at 45° to the $x_1 - x_2$ plane rotated about the x_2-axis and also:

$$|\tau_{max}| = \left|\frac{\sigma_{II} - \sigma_{III}}{2}\right| = \frac{17.5\text{ ksi}}{2} = \boxed{8.8\text{ ksi}}$$

This is a plane at 45° to the $x_1 - x_2$ plane rotated about the x_1-axis

(c) The direction of the axes for reference relative to the original loading axes (call θ):



does <u>not</u> change the basic stress state and thus the principal stresses and maximum shear stresses **do not change**

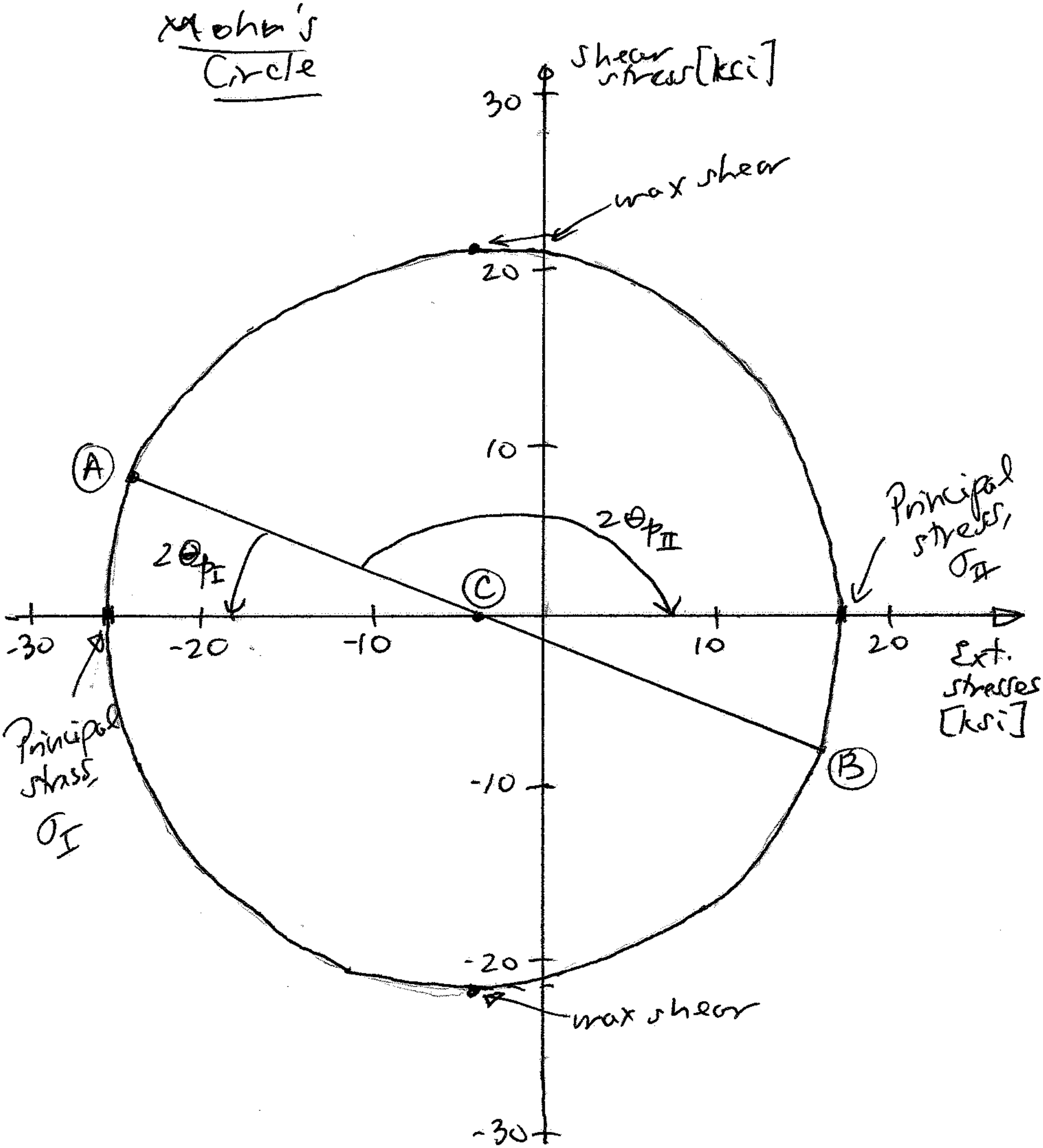
And ... the principal directions and planes of the maximum stresses **stay the same** relative to the loading axes $x_1 - x_2$.

→ However, the angle relative to the initial axis system, θ from the original loading axes must be properly subtracted/added to get the direction(s) relative to these axes.

(d) Draw the Mohr's circle as specified in the instructions (see associated figure)

① Plot $\sigma_{11}, -\sigma_{12}$ (−24 ksi, +8 ksi) as Point Ⓐ
② Plot σ_{22}, σ_{21} (16 ksi, −8 ksi) as Point Ⓑ
③ Connect Ⓐ and Ⓑ
④ Draw circle of diameter of the line Ⓐ-Ⓑ about the point where this line crosses the horizontal axis (denote this point as Ⓒ):

$$\text{point } C = \text{midpoint} = \frac{\sigma_{11} + \sigma_{22}}{2} = \frac{-24\,\text{ksi} + 16\,\text{ksi}}{2} = -4\,\text{ksi}$$


Mohr's Circle
Shear stress [ksi]
30
max shear
20
10
A
$2\theta_{p_I}$
$2\theta_{p_{II}}$
Principal stress, σ_{II}
C
-30
-20
-10
10
20
Ext. stresses [ksi]
Principal stress, σ_I
B
-10
-20
max shear
-30

→ Principal stresses and directions (2-D)
(part (a))

The intersection of the circle with the horizontal axis gives the two values of the principal stresses. By sight these are of the same value corresponding to the results of part (a): -24.8 ksi + 16.8 ksi.
One can be more formal by finding the circle diameter $\left(= 2\sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}}\right)$ and

then adding and subtracting half of this from the midpoint value (point Ⓒ = -4.0 ksi).

Directions (angles) can be measured via a protractor and half the angle from line Ⓐ-Ⓑ to the horizontal axis of the two point (2 directions)

→ Maximum shear stresses (part (b))

These are the upper and lower "reaches" of the circle along the vertical direction. It can be read off to be just about 21.0 ksi (and -21.0 ksi).
This is very close to the value calculated in part (b) of ± 21.6 ksi. Note that the value found via Mohr's circle is exactly the radius of the circle $\left(= \left|\frac{\sigma_{I} - \sigma_{II}}{2}\right| = 21.6 \text{ ksi}\right)$.

The direction(s) of the maximum shear stress(es) are 90° on Mohr's circle from that of the principal stresses or these two associated lines are perpendicular. Divide by 2, since this is twice the rotation angle, and that is 45° added on to $\theta_{p_{I,II}}$ as in (b).

<u>NOTE</u>: Only the maximum shear stress in the x_1-x_2 plane can be determined since Mohr's circle only allows rotation in the x_1-x_2 plane.

M19 (M11.2) Begin by writing out the transformation equations for in-plane strain (as for the case of plane stress):

$$\tilde{\epsilon}_{11} = \cos^2\theta\, \epsilon_{11} + \sin^2\theta\, \epsilon_{22} + 2\cos\theta \sin\theta\, \epsilon_{12}$$

$$\tilde{\epsilon}_{22} = \sin^2\theta\, \epsilon_{11} + \cos^2\theta\, \epsilon_{22} - 2\cos\theta \sin\theta\, \epsilon_{12}$$

$$\tilde{\epsilon}_{12} = -\sin\theta \cos\theta\, \epsilon_{11} + \cos\theta \sin\theta\, \epsilon_{22} + (\cos^2\theta - \sin^2\theta)\, \epsilon_{12}$$

Considering this, one can see that no matter what direction an elongation strain is measured, one cannot get a result for the shear strain ϵ_{12} -- ϵ_{11} and ϵ_{22} are involved in these expressions. So:

measuring <u>one</u> <u>elongation strain</u> is $\boxed{\text{not sufficient}}$

Further consideration of these transformation equations shows that knowing three strains (ϵ_{11}, ϵ_{22}, ϵ_{12}) and an angle of rotation allows one to characterize the state of strain in any direction. Working with this, one can say:

① Any $\underline{\underline{3}}$ strains fully characterize the state of in-plane strain of a body along with the knowledge of the strain directions.

② ~~M~~easuring $\underline{\underline{3}}$ elongation strains and using the knowledge of their directions will yield the shear strain in one set of axes.

Demonstrate this via the equations:

1. Measure $\tilde{\epsilon}_{11}$ and have θ
2. Measure ϵ_{11}
3. Measure ϵ_{22}
4. Use all in the first transformation equation:

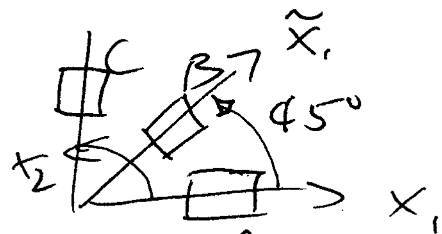
$$\tilde{\epsilon}_{11} = \cos^2\theta\, \epsilon_{11} + \sin^2\theta\, \epsilon_{22} + 2\cos\theta \sin\theta\, \epsilon_{12}$$

and solve for ϵ_{12}:

$$\boxed{\epsilon_{12} = \frac{\tilde{\epsilon}_{11} - \cos^2\theta\, \epsilon_{11} - \sin^2\theta\, \epsilon_{22}}{2\cos\theta \sin\theta}}$$

So: $\boxed{\underline{\underline{3}} \text{ measurements}}$ are needed

Note 1: Such a "shear strain gage" normally measures elongational strain along axes of 0°, 45°, and 90°:



So in the ϵ_{12} equation, $\theta = 45°$. If the reading for ϵ_{11} is A; for $\tilde{\epsilon}_{11}$ is B; for ϵ_{22} is C; then:

$$\epsilon_{12} = B - \tfrac{1}{2}(A+C)$$

CAUTION!! This is tensorial shear strain.

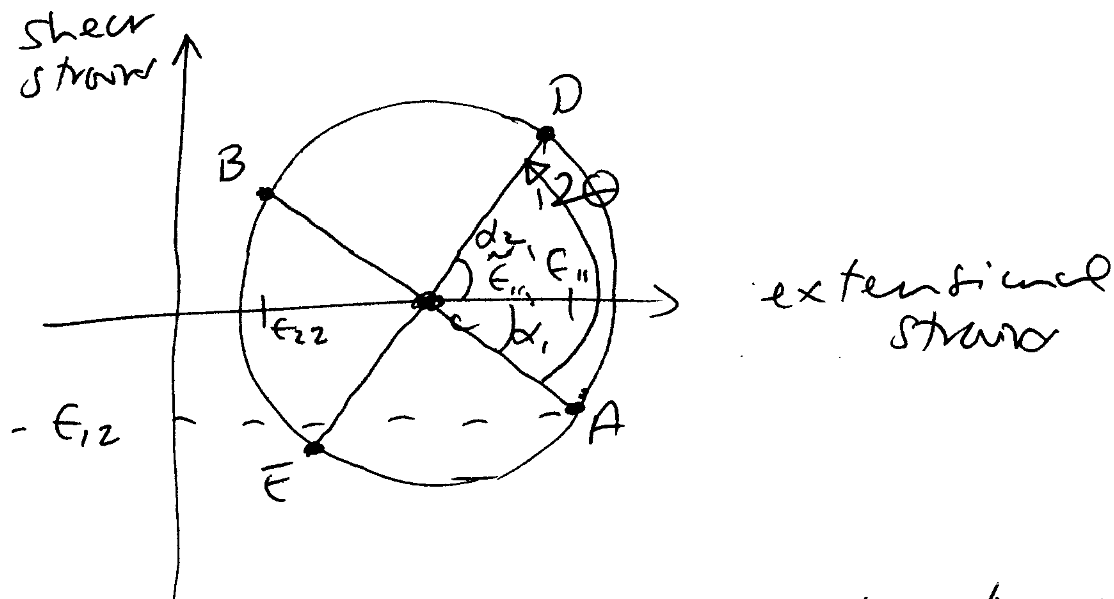
For engineering shear strain, there is a factor of 2: $\gamma_{12} = 2B - (A+C)$

Note 2: The same reasoning can be followed through using the Mohr's circle as a basis. Again, three readings (and associated directions) of longitudinal strain are needed to characterize the circle.

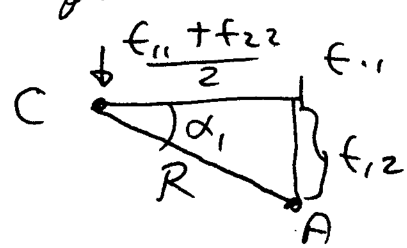
How? Consider the invariants first. The pertinent one is that which gives the midpoint: $\frac{\epsilon_{11} + \epsilon_{22}}{2}$

Measuring a second $\tilde{\epsilon}_{11}$ gives an $\tilde{\epsilon}_{22}$

Now draw the circle:



Considering geometry one can show the angle to principal strains is:



$\Rightarrow R\cos\alpha_1 = \epsilon_{11} - \frac{\epsilon_{11}+\epsilon_{22}}{2} = \frac{\epsilon_{11}-\epsilon_{22}}{2}$

and $R\sin\alpha_1 = \epsilon_{12}$

In a similar way:

$$R\cos\alpha_2 = \frac{\tilde{\epsilon}_{11} - \tilde{\epsilon}_{22}}{2}$$

factor of 2 [illegible] Mohr circle

Finally: $\alpha_2 = 2\theta - \alpha_1$

$\Rightarrow \cos\alpha_2 = \cos(\theta - \alpha_1) = \cos 2\theta \cos\alpha_1 + \sin 2\theta \sin\alpha_1$

(using trigonometric expansions)

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working this last item with what is known:

$$\frac{\tilde{\epsilon}_{11} - \tilde{\epsilon}_{22}}{2R} = \cos 2\theta \left(\frac{\epsilon_{11} - \epsilon_{22}}{2R}\right) + \sin 2\theta \, \frac{\epsilon_{12}}{R}$$

→ R cancels out

→ θ, ϵ_{11}, ϵ_{22}, $\tilde{\epsilon}_{11}$ are measured

→ $\tilde{\epsilon}_{22}$ results since $\epsilon_{11} + \epsilon_{22} = \tilde{\epsilon}_{11} + \tilde{\epsilon}_{22}$

$$\Rightarrow \tilde{\epsilon}_{22} = \epsilon_{11} + \epsilon_{22} - \tilde{\epsilon}_{11}$$

So: $2\tilde{\epsilon}_{11} - \epsilon_{11} - \epsilon_{22} = (\epsilon_{11} - \epsilon_{22}) \cos 2\theta + 2 \sin 2\theta \, \epsilon_{12}$

and solve for ϵ_{12}

for the case of $\theta = 45°$

and $\tilde{\epsilon}_{11} = B$

$\epsilon_{11} = A$

$\epsilon_{22} = C$

$\Rightarrow \cos 2\theta = 0$, $\sin 2\theta = 1$

giving: $2B - A - C = 2\epsilon_{12}$

$$\Rightarrow \epsilon_{12} = B - \tfrac{1}{2}(A + C)$$

as before!

M20 **(M11.3)** The following answers, as asked for in the problem statement, include a brief sentence on the primary functional requirement that needs to be met for each of the given cases. This includes the loads (e.g. tension, compression, shear, impact, cyclic, thermal, electrical) and five material properties that are most relevant to meeting this requirement. Note that the items listed are just ***some*** of the possible requirements, loads, and properties. (**NOTE:** Problem set answers will vary according to what the individual student indicates are the relevant loads and properties.)

(a) Compressor blades of a jet engine: Must maintain desired shape during operation of engine.

Types of loads:
1. Impact
2. Tension and Compression
3. Aerodynamic/Pressure
4. Wear

Material properties:
1. Strength - High
2. Modulus - High
3. Impact - High
4. Corrosion - High
5. Cyclic - High

(b) Cable used for towing large trucks: Must provide load-carrying capacity and resistance to environment for loads and items encountered in towing

Types of loads:
1. Tension (pulling, from bumps)
2. Impact
3. Thermal (due to baseline temperature from environment)
4. Wear

Material properties:
1. Strength - High
2. Abrasion and wear - High
3. Modulus - High
4. Corrosion - High
5. Price - Low

(c) Components of a truss for a radio tower: Must provide load-carrying capacity for loads that a radio tower undergoes.

Types of loads:
1. & 2. Tension and Compression (depending on design -- mainly compression due to gravity)
3. Assembly
4. Environmental (Thermal, Aerodynamic)

Material properties:
1. Corrosion - High
2. Modulus - High
3. Strength - Medium

4. Fabrication & Joining - High
5. Price - Low

(d) <u>Components of a space truss</u>: Must provide load-carrying capacity for loads that a space truss undergoes.

Types of loads:
1. Impact (docking)
2. Thermal (solar)
3. & 4. Tension and Compression (depending on design)
5. Cyclic

Material properties:
1. Thermal - High
2. Density - Low
3. Modulus - High
4. Joining - Medium
5. Longevity - High

(e) <u>Reentry shield on the space shuttle</u>: Must insulate the shuttle structure and its passengers from the extreme heat of reentry.

Types of loads:
1. Thermal
2. Cyclic
3. Impact

Material properties:
1. Thermal - High
2. Density - Low
3. Oxidation Resistance - High
4. Hardness - Medium
5. Strength - Medium

(f) <u>Tiles for a house floor</u>: Must provide an “aesthetic” and durable surface for a house floor.

Types of loads:
1. Impact
2. Compression
3. Thermal
4. Environmental

Material properties:
1. Price - Low
2. Availability - High
3. Hardness - Medium
4. Appearance - High
5. Finishing - High