PAL
11/20/08

# Unified Engineering

## Week 13 - Problem Set 12

## Fall, 2008

## SOLUTIONS

M23 (M13.1)

→ From the work on M22 from the previous week, it was found that there are 9 independent compliance components as for the elasticity case and these are expressed in the equations:

$$\epsilon_{11} = S_{1111}\sigma_{11} + S_{1122}\sigma_{22} + S_{1133}\sigma_{33}$$

$$\epsilon_{22} = S_{1122}\sigma_{11} + S_{2222}\sigma_{22} + S_{2233}\sigma_{33}$$

$$\epsilon_{33} = S_{1133}\sigma_{11} + S_{2233}\sigma_{22} + S_{3333}\sigma_{33}$$

$$\epsilon_{23} = 2S_{2323}\sigma_{23}$$

$$\epsilon_{13} = 2S_{1313}\sigma_{13}$$

$$\epsilon_{12} = 2S_{1212}\sigma_{12}$$

(a) Now use the stress-strain relations for the orthotropic case using engineering constants from the lecture notes (Unit M3.2, p. 29):

$$\epsilon_1 = \frac{1}{E_1}\left[\sigma_1 - \nu_{12}\sigma_2 - \nu_{13}\sigma_3\right]$$

$$\epsilon_2 = \frac{1}{E_2}\left[-\nu_{21}\sigma_1 + \sigma_2 - \nu_{23}\sigma_3\right]$$

$$\epsilon_3 = \frac{1}{E_3}\left[-\nu_{31}\sigma_1 - \nu_{32}\sigma_2 + \sigma_3\right]$$

$$\gamma_{23} = \frac{1}{G_{23}}\sigma_{23}$$

$$\gamma_{13} = \frac{1}{G_{13}}\sigma_{13}$$

$$\gamma_{12} = \frac{1}{G_{12}}\sigma_{12}$$

To go from tensorial to engineering notation for stress and strain, note that

| | |
|---|---|
| $\epsilon_{11} = \epsilon_1$ | $\sigma_{11} = \sigma_1$ |
| $\epsilon_{22} = \epsilon_2$ | $\sigma_{22} = \sigma_2$ |
| $\epsilon_{33} = \epsilon_3$ | $\sigma_{33} = \sigma_3$ |
| $2\epsilon_{23} = \gamma_{23}$ | $\sigma_{23} = \sigma_{23}$ |
| $2\epsilon_{13} = \gamma_{13}$ | $\sigma_{13} = \sigma_{13}$ |
| $2\epsilon_{12} = \gamma_{12}$ | $\sigma_{12} = \sigma_{12}$ |

→ Using these relations with the previous two sets of equations results in:

$$\boxed{\begin{aligned}
S_{1111} &= 1/E_1 \\
S_{1122} &= -\nu_{12}/E_1 = \frac{-\nu_{21}}{E_2} && \leftarrow \text{via reciprocity} \\
S_{1133} &= -\nu_{13}/E_1 = -\nu_{31}/E_3 && \leftarrow \text{via reciprocity} \\
S_{2222} &= 1/E_2 \\
S_{2233} &= -\nu_{23}/E_2 = -\nu_{32}/E_3 && \leftarrow \text{via reciprocity} \\
S_{3333} &= 1/E_3 \\
S_{2323} &= 1/(4G_{23}) \\
S_{1313} &= 1/(4G_{13}) \\
S_{1212} &= 1/(4G_{12})
\end{aligned}}$$

$\left.\begin{matrix} S_{2323} \\ S_{1313} \\ S_{1212} \end{matrix}\right\} (*)$

(*) <u>NOTE</u> factor of 4 in these cases.

Look at one particular case:

$$\epsilon_{23} = 2 S_{2323} \sigma_{23} \quad \text{and} \quad \gamma_{23} = \frac{1}{G_{23}} \sigma_{23}$$

along with $2\epsilon_{23} = \gamma_{23}$

$$\Rightarrow 2\epsilon_{23} = 4 S_{2323}\sigma_{23} = \gamma_{23} = \frac{1}{G_{23}}\sigma_{23}$$

$$\Rightarrow 4 S_{2323} = \frac{1}{G_{23}}$$

Finally: $\boxed{S_{2323} = 1/(4G_{23})}$

.... <u>same</u> for other two cases

(b) The compliance matrix is the inverse of the elasticity matrix and vice versa:

$$\underset{\sim}{E} = \underset{\sim}{S}^{-1}$$

For this orthotropic case, the compliance matrix is:

$$\underset{\sim}{S} = \begin{bmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 0 \\ S_{1122} & S_{2222} & S_{2233} & 0 & 0 & 0 \\ S_{1133} & S_{2233} & S_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2S_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2S_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2S_{1212} \end{bmatrix}$$

→ Using the relations found between the components of the compliance tensor and the engineering constants, results in the following expression of the compliance matrix:

$$\underset{\sim}{S} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_1 & -\nu_{13}/E_1 & 0 & 0 & 0 \\ -\nu_{21}/E_2 & 1/E_2 & -\nu_{23}/E_2 & 0 & 0 & 0 \\ -\nu_{31}/E_3 & -\nu_{32}/E_3 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/(2G_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/(2G_{13}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/(2G_{12}) \end{bmatrix}$$

(eq. 1)

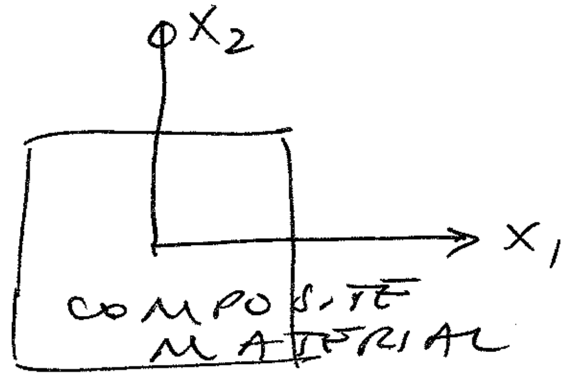
→ For the orthotropic case, the elasticity matrix is:

$$\underset{\sim}{E} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ E_{1122} & E_{2222} & E_{2233} & 0 & 0 & 0 \\ E_{1133} & E_{2233} & E_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2E_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2E_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2E_{1212} \end{bmatrix}$$

(eq. 2)

→ Now take the inverse of the matrix expression of (1) and equate it to the matrix expression of (2). This will give relations between the components of the elasticity tensor and the engineering constants.

M24 (M13.2)




Experiment A: $\sigma_{22} = 50$ ksi
$\epsilon_{11} = -1100$ $\mu$strain
$\epsilon_{22} = +6300$ $\mu$strain

Experiment B: $\sigma_{12} = 30$ ksi
$\epsilon_{12} = +4400$ $\mu$strain

Experiment C: $\sigma_{11} = 30$ ksi
$\sigma_{22} = 15$ ksi
$\epsilon_{11} = +1800$ $\mu$strain

stresses and (in-plane) strains not specified are zero
(except $\epsilon_{22}$ for Experiment C as the gage broke)

(a) Experiments A and C show that extensional stresses cause only extensional strains. Experiment B shows that shear stress causes only shear strain. Thus, there is no shear-extensional coupling and this material behaves "at most" as an orthotropic material.

Since the out-of-plane stresses ($\sigma_{3j}$) are also zero in all three experiments, the cases are plane stress and the in-plane form of the stress-strain equations are sufficient to describe the behavior. This gives:

$$\epsilon_{11} = \frac{1}{E_1}\sigma_{11} - \frac{\nu_{21}}{E_2}\sigma_{22} \qquad (1)$$

$$\epsilon_{22} = -\frac{\nu_{12}}{E_1}\sigma_{11} + \frac{1}{E_2}\sigma_{22} \qquad (2)$$

$$2\epsilon_{12} = \frac{1}{G_{12}}\sigma_{12} \qquad (3)$$

$\longrightarrow$ Use the results of Experiment B in (3):

$$2(4400 \times 10^{-6}) = \frac{1}{G_{12}}(30 \times 10^{3}\ \text{psi})$$

$$\Rightarrow \boxed{G_{12} = 3.4 \times 10^{6}\ \text{psi} = 3.4\ \text{Msi}}$$

→ Use the results of Experiment A (since there is only one stress applied) in (2):

$$6300 \times 10^{-6} = \frac{1}{E_2}\,(50 \times 10^3\ \text{psi})$$

$$\Rightarrow \boxed{E_2 = 7.9 \times 10^6\ \text{psi} = 7.9\ \text{Msi}}$$

and then use the results in (1) along with the value determined for $E_2$:

$$-1100 \times 10^{-6} = \frac{-\nu_{21}}{(7.9 \times 10^6\ \text{psi})}\,(50 \times 10^3\ \text{psi})$$

$$\Rightarrow \boxed{\nu_{21} = 0.174}$$

→ Use the results of Experiment C in (1) along with the values for $\nu_{21}$ and $E_2$ as determined via the results of Experiment A:

$$1800 \times 10^{-6} = \frac{1}{E_1}\,(30 \times 10^3\ \text{psi}) - \frac{0.174}{(7.9 \times 10^6\ \text{psi})}\,(15 \times 10^3\ \text{psi})$$

$$\Rightarrow 1800 \times 10^{-6} = \frac{1}{E_1}(30 \times 10^3 \text{ psi}) - 330 \times 10^{-6}$$

$$\Rightarrow 2130 \times 10^{-6} = \frac{1}{E_1}(30 \times 10^3 \text{ psi})$$

$$\Rightarrow \boxed{E_1 = 14.1 \times 10^6 \text{ psi} = 14.1 \text{ Msi}}$$

→ One can use the <u>reciprocity relation</u>:

$$\nu_{21} E_1 = \nu_{12} E_2$$

and the values determined to get $\nu_{12}$:

$$(0.174)(14.1 \text{ Msi}) = \nu_{12} (7.9 \text{ Msi})$$

$$\Rightarrow \boxed{\nu_{12} = 0.311}$$

This gives the in-plane engineering constants for this orthotropic material.

Summarizing:

$$\boxed{\begin{array}{ll} E_1 = 14.1 \text{ Msi} & \nu_{12} = 0.311 \\ E_2 = 7.9 \text{ Msi} & \nu_{21} = 0.174 \\ & \\ \multicolumn{2}{c}{G_{12} = 3.4 \text{ Msi}} \end{array}}$$

With these 5 constants (4 independent) and the equations with which the analysis started, the in-plane stress-strain behavior of the material is characterized.

(b) Use the results from Experiment C and the results from part (a) to determine $\epsilon_{22}$ for the Experiment, if possible.

<u>It is possible</u> and the key is in the use of the reciprocity relation to determine a value for $\nu_{12}$. Thus, all the terms in equation (2) are determined, and for Experiment C one can determine:

$$\epsilon_{22} = -\frac{\nu_{12}}{E_1}\sigma_{11} + \frac{1}{E_2}\sigma_{22}$$

$$\Rightarrow \epsilon_{22} = -\frac{0.311}{(14.1\times10^6\,psi)}(30\times10^3\,psi) + \frac{1}{(7.9\times10^6\,psi)}(15\times10^3\,psi)$$

fiver: $\epsilon_{22} = -662 \times 10^{-6} + 1899 \times 10^{-6}$

$$\Rightarrow \boxed{\epsilon_{22} = 1237 \times 10^{-6} = 1237 \mu\text{strain}}$$

what broken strain gage in 2-direction should have read

(c) Begin by using the stress-strain equations in matrix form:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/2G_{12} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

→ Now consider the compliance tensor relations also in matrix form and with the zero stresses ($\sigma_{13}, \sigma_{23}, \sigma_{33}$) eliminated and ignoring the out-of-plane strains as there is no information for those terms. Thus, consider <u>only</u> the <u>in-plane</u> components:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & 2S_{1112} \\ S_{2211} & S_{2222} & 2S_{2212} \\ S_{1211} & S_{1222} & 2S_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

In this case, the <u>COUPLING TERMS</u> are zero based on the experiments showing this to be orthotropic in the testing axes. This leaves 5 terms:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & 0 \\ S_{2211} & S_{2222} & 0 \\ 0 & 0 & 2S_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$$

Compare the two expressions (compliance matrix form and engineering constant matrix form) and this gives:

$$S_{1111} = 1/E_1 = \frac{1}{14.1 \times 10^6 \text{psi}} = 70.9 \times 10^{-9} \frac{1}{\text{psi}}$$

$$S_{1122} = S_{2211} = \frac{-\nu_{21}}{E_2} = \frac{-\nu_{12}}{E_1} =$$

$$\frac{-0.311}{14.1 \times 10^6 \text{psi}} = \frac{-0.174}{7.9 \times 10^6 \text{psi}} = -22.1 \times 10^{-9} \frac{1}{\text{psi}}$$

$$S_{2222} = \frac{1}{E_2} = \frac{1}{7.9\times10^6\,psi} = 127\times10^{-9}\,\frac{1}{psi}$$

$$2S_{1212} = \frac{1}{2G_{12}} = \frac{1}{2(3.4\times10^6\,psi)} = 147\times10^{-9}\,\frac{1}{psi}$$

$$\Rightarrow S_{1212} = 73.5\times10^{-9}\,\frac{1}{psi}$$

Summarizing:

$$\boxed{\begin{array}{ll} S_{1111} = 70.9\times10^{-9}\,\frac{1}{psi} & S_{1122} = S_{2211} = -22.1\times10^{-9}\,\frac{1}{psi} \\ S_{2222} = 127\times10^{-9}\,\frac{1}{psi} & S_{1212} = 73.5\times10^{-9}\,\frac{1}{psi} \\ \multicolumn{2}{c}{S_{1112} = S_{2212} = S_{1211} = S_{1222} = 0} \end{array}}$$

as shown in experiment

And in matrix form:

$$\boxed{\underset{\sim}{S} = \begin{bmatrix} 70.9 & -22.1 & 0 \\ -22.1 & 127 & 0 \\ 0 & 0 & 2(73.5) \end{bmatrix} \times \left[10^{-9}\,\frac{1}{psi}\right]}$$

for: $\underset{\sim}{\epsilon} = \underset{\sim}{S}\,\underset{\sim}{\sigma}$

## Solutions F8

(a) Find the velocity vector.

$\rightarrow$ Select 2-D coordinate system; for a given $z \Rightarrow (r, \theta)$

then $\bar{V} = V_r \hat{i_r} + V_\theta \hat{i_\theta}$ for a given $z$ ( ↳ polar coordinates )

From $\phi = A\theta$ and $\psi = -A \ln r$

$$\Rightarrow \bar{V} = \nabla\phi = \frac{\partial\phi}{\partial r}\hat{i_r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{i_\theta} = \frac{A}{r}\hat{i_\theta} \quad \text{so } V_r = 0$$

$\Rightarrow$ In polar coordinates, line element is $d\bar{l} = r\,d\theta\,\hat{i_\theta} + dr\,\hat{i_r}$

the streamline equation is given by $\bar{V} \times d\bar{l} = 0$, or $\frac{dr}{d\theta} = \frac{rV_r}{V_\theta}$

And since $d\psi = \frac{\partial\psi}{\partial r}dr + \frac{\partial\psi}{\partial\theta}d\theta \Rightarrow$ for $\psi_1 \equiv$ const $\frac{dr}{d\theta} = -\frac{(\partial\psi/\partial\theta)}{(\partial\psi/\partial r)}$ compare

We have $V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$ $\quad V_\theta = -\frac{\partial\psi}{\partial r}$

with $\psi = -A\ln r$ $\quad V_r = 0$, $V_\theta = \frac{A}{r}$ $\quad$ In both cases $\boxed{\bar{V} = \frac{A}{r}\hat{i_\theta}}$

(b) for irrotational flow, we require $\nabla \times \bar{V} = 0$

In polar coordinates $\nabla\times\bar{V} = \left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta) - \frac{1}{r}\frac{\partial V_r}{\partial\theta}\right)\hat{i_z}$

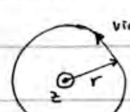
since $V_r = 0$ $\frac{\partial V_r}{\partial\theta} = 0$ and $\frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r}\frac{A}{\cancel{r}}\right) = 0$ since $A \equiv$ constant.

Flow is irrotational.

(c) The streamlines at any given $z$ are circles $\Rightarrow \frac{dr}{d\theta} = \frac{rV_r}{V_\theta} = 0 \Rightarrow \boxed{r = \text{const}}$

This means, Bernoulli's equation can be applied for any circular streamline:

$$\boxed{\frac{1}{2}\rho V_\theta^2 + p + \rho g z = \text{const}}$$



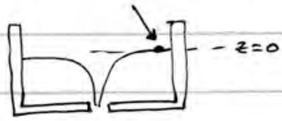

In addition, since the flow is irrotational, all streamlines share the same constant.

To find the constant, we can apply Bernoulli's at some point where the conditions are known.

Solutions F8 contd'

We select this point far away from the axis, where $r$ is large.

This means $V_\theta = \frac{A}{r} \to$ small and $V_\theta^2 \to 0$ approximately.




we also reference the $z$ axis to this point so that the constant in Bernoulli's eqn. is:

$$\text{const} = \frac{1}{2}\rho \cancel{V_\theta^2}^{\,0} + P_{z=0} + \rho g \cancel{z}^{\,0} \quad z=0$$

At any other point on the surface $\boxed{P_z = P_{z=0}}$ hydrostatic pressures are practically the same.

Therefore $\frac{1}{2}\rho V_\theta^2 + \rho g z = 0$ and since $V_\theta = \frac{A}{r}$

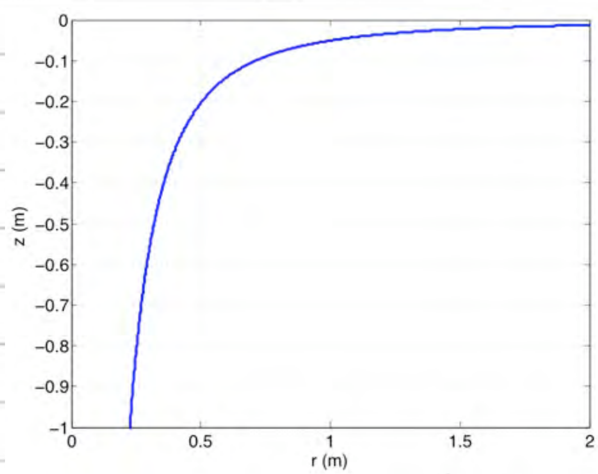
$$\boxed{z = -\frac{A^2}{2gr^2}}$$ independent of $\rho$!

Note that the velocity field contains a singularity at $r=0$ $(V_\theta \to \infty)$, this means the circulation can be non-zero, even if the flow is irrotational.

$\Gamma = -\oint \bar{V} \cdot d\bar{\ell}$ in our case $\bar{V} = \frac{A}{r}\hat{i}_\theta$ $\quad d\bar{\ell} = r\,d\theta\,\hat{i}_\theta$

Then $\Gamma = -\int_0^{2\pi} \left(\frac{A}{r}\right) r\,d\theta = -2\pi A$ $\quad$ Then: $\boxed{z = \frac{-\Gamma^2}{8\pi^2 g r^2}}$

(d) if $A=1 \Rightarrow \boxed{\Gamma = -2\pi}$ then $z = -\frac{1}{2gr^2}$ assume $g = 9.8\,\frac{m}{s^2}$




The sketch corresponds to a tank of radius 2 m.

The strength of the vortex is given by the magnitude of the circulation.

The model breaks up fast as $r \to 0$: the dimensions will exceed physical Boundaries and assumptions.

Solutions F9

(a) Velocity vector : $\vec{V} = V_r \hat{i_r} + V_\theta \hat{i_\theta} = \nabla\phi = \frac{\partial\phi}{\partial r}\hat{i_r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{i_\theta}$
(cylindrical-polar)

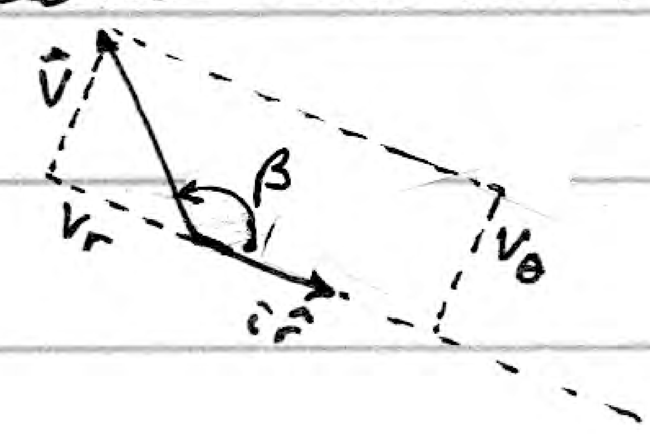
with $\phi = A\theta - B\ln r$

$$\frac{\partial\phi}{\partial r} = -\frac{B}{r}\ ; \quad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{A}{r}$$

$$\boxed{\vec{V} = -\frac{B}{r}\hat{i_r} + \frac{A}{r}\hat{i_\theta}}$$

(b) $\beta \equiv$ angle between the velocity vector and radial direction.

then $\vec{v}\cdot\hat{i_r} = |\vec{V}||\hat{i_r}|\cos\beta$ also $\hat{v}\cdot\hat{i_r} = -V_r$
($|\vec{V}| = V$, $|\hat{i_r}| = 1$)



then $V_r = -V\cos\beta$ or $\cos\beta = -\frac{V_r}{V}$

$V = \sqrt{V_r^2 + V_\theta^2}$ then $\cos\beta = \frac{-V_r}{\sqrt{V_r^2+V_\theta^2}} = \frac{-1}{\sqrt{1+\left(\frac{V_\theta}{V_r}\right)^2}}$

But $\frac{V_\theta}{V_r} = \frac{A/r}{-B/r} = -\frac{A}{B} \equiv$ constant, therefore $\boxed{\cos\beta \equiv \text{constant}}$

(c) Vorticity $\xi = \nabla\times\vec{V} = \left[\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta) - \frac{1}{r}\frac{\partial V_r}{\partial\theta}\right]\hat{i_z}$

$$= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r}\frac{A}{\cancel{r}}\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{-B}{r}\right)\right]\hat{i_z} = 0$$

Vorticity $=0$
FLOW is irrotational.

Again, the circulation might be finite because of the singularities, but the vorticity is still zero.

(d) Calculate $\nabla\cdot\vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta}$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{rB}{r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{A}{r}\right) = 0$$

flow is incompressible.

(e) Streamlines in polar coordinates $\Rightarrow \frac{dr}{d\theta} = \frac{rV_r}{V_\theta}$

and since $V_r = \frac{-B}{r}$ and $V_\theta = \frac{A}{r} \Rightarrow \frac{dr}{d\theta} = -r\frac{B}{A}$

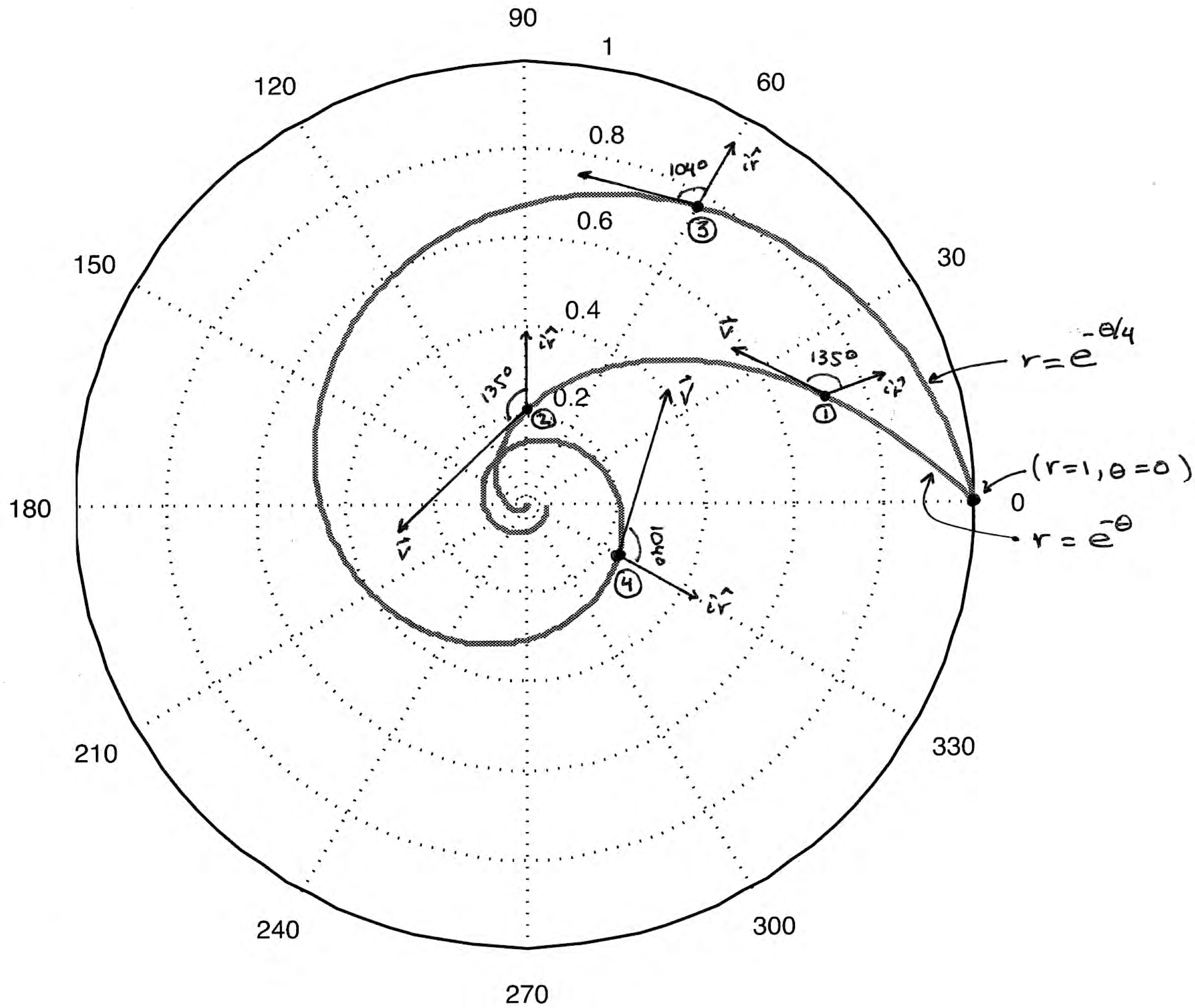
$$\frac{dr}{r} = -\frac{B}{A}d\theta \Rightarrow \ln\frac{r}{r_0} = -\frac{B}{A}(\theta-\theta_0) \Rightarrow \boxed{r = r_0 e^{-\frac{B}{A}(\theta-\theta_0)}}$$

Solutions F9 contd'

(f) the angle $\beta$ is defined as $\beta = \cos^{-1} \dfrac{1}{\sqrt{1 + \left(\frac{V_\theta}{V_r}\right)^2}}$ with $\dfrac{V_\theta}{V_r} = -\dfrac{A}{B}$

Select 2 points on each streamline below:

- points ① and ② on streamline $r = e^{-\theta}$
- points ③ and ④ on streamline $r = e^{-\theta/4}$

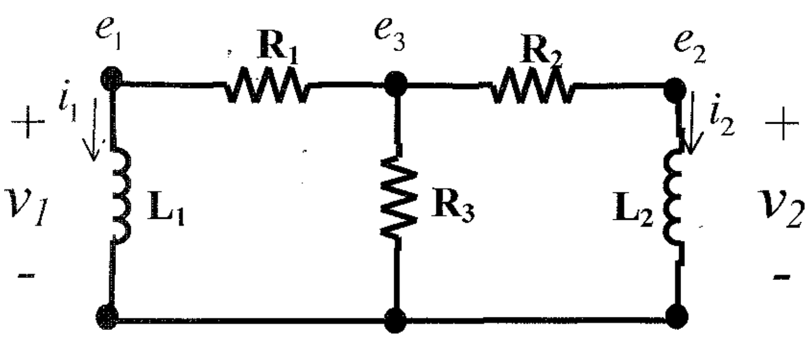



On the streamline $r = e^{-\theta}$ : $\dfrac{V_\theta}{V_r} = -\dfrac{A}{B} = -1 \Rightarrow \beta = \cos^{-1} \dfrac{-1}{\sqrt{2}} = 135°$

On the streamline $r = e^{-\theta/4}$ : $\dfrac{V_\theta}{V_r} = -4 \Rightarrow \beta = \cos^{-1} \dfrac{-1}{\sqrt{17}} = 104°$

Note: velocity vectors are <u>not</u> scaled.

Problem S10 (Signals and Systems)




Consider the circuit above with

$R_1 = R_2 = R_3 = 1\ \Omega$ and $L_1 = L_2 = 1H$

The initial conditions on the inductor currents are,

$i_1(0) = 5A$, $i_2(0) = -5A$

Find $i_1(t)$ and $i_2(t)$ using the node voltage method

$$V_1 = L_1 \frac{di_1}{dt}, \quad V_2 = L_2 di_2/dt$$

$$\left.\begin{aligned} e_1 &: \quad i_1 + \frac{e_1 - e_3}{R_1} = 0 \\ e_2 &: \quad i_2 + \frac{e_2 - e_3}{R_2} = 0 \\ e_3 &: \quad \frac{e_3 - e_1}{R_1} + \frac{e_3}{R_3} + \frac{e_3 - e_2}{R_2} = 0 \end{aligned}\right\} \text{Node equations}$$

Note: $e_1 = V_1 = L_1 di_1/dt, \quad e_2 = V_2 = L_2 di_2/dt$

Also, equation at $e_3 \Rightarrow \boxed{e_3 = \frac{e_1 + e_2}{3}} \quad (R_1 = R_2 = R_3 = 1)$

Substituting for $e_3$ we get:

① $i_1 + \frac{2e_1}{3} - \frac{e_2}{3} = 0$

② $i_2 + \frac{2e_2}{3} - \frac{e_1}{3} = 0$

① $\Rightarrow$ $i_1 + \frac{2}{3}\frac{di_1}{dt} - \frac{1}{3}\frac{di_2}{dt} = 0$

② $\Rightarrow$ $i_2 + \frac{2}{3}\frac{di_2}{dt} - \frac{1}{3}\frac{di_1}{dt} = 0$

Guess solution: $i_1 = I_1 e^{st}$ , $i_2 = I_2 e^{st}$

We now have:

$$I_1 e^{st} + \frac{2}{3} I_1 s e^{st} - \frac{1}{3} I_2 s e^{st} = 0$$

$$I_2 e^{st} + \frac{2}{3} I_2 s e^{st} - \frac{1}{3} I_1 s e^{st} = 0$$

$$\Rightarrow \quad I_1 \left(1 + \frac{2}{3}s\right) - I_2\, s/3 = 0$$

$$I_1 \left(-\frac{1}{3}s\right) + I_2 \left(1 + \frac{2}{3}s\right) = 0$$

$$A = \begin{bmatrix} \left(1 + \frac{2}{3}s\right) & -s/3 \\ -s/3 & \left(1 + \frac{2}{3}s\right) \end{bmatrix}$$

$$\det(A) = \left(1+\tfrac{2}{3}s\right)^2 - \frac{s^2}{9} = 0$$

$$1 + \frac{4}{3}s + \frac{4}{9}s^2 - \frac{s^2}{9} = 0$$

$$s^2 + 4s + 3 = 0$$

$$(s+1)(s+3) = 0 \Rightarrow s=-1,\ s=-3$$

- To solve for $I_1, I_2$ plug back onto A.

<u>$s_1 = -1$</u> $\quad \frac{1}{3} I_1^{s_1} + \frac{1}{3} I_2^{s_1} = 0 \Rightarrow I_1^{s_1} = -I_2^{s_1}$

$\Rightarrow I_1^{s_1} = 1,\ I_2^{s_1} = -1$ (or any multiple)

$s_2 = -3 \quad -I_1^{s_2} + I_2^{s_2} = 0 \Rightarrow I_1^{s_2} = I_2^{s_2}$

$\Rightarrow I_1^{s_2} = 1,\ I_2^{s_2} = 1$ (or any multiple)

---

<u>The total solution</u>

$$i_1 = ae^{-t} + be^{-3t}$$

$$i_2 = -ae^{-t} + be^{-3t}$$

$i_1(0) = 5A, \quad i_2(0) = -5A$

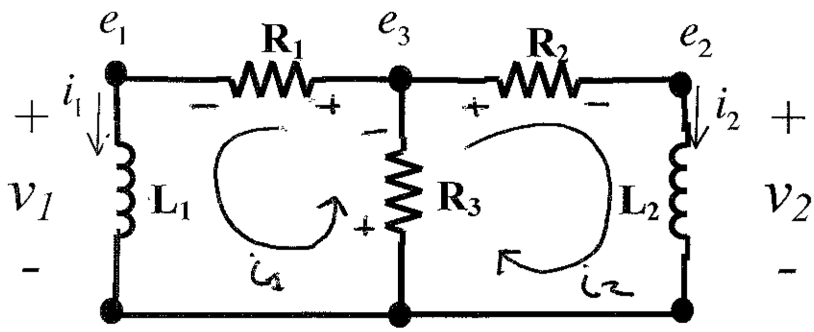
$$\left.\begin{aligned} a+b &= 5 \\ -a+b &= -5 \end{aligned}\right\} \Rightarrow b=0,\ a=5$$

$$i_1 = 5e^{-t} + 0$$

$$i_2 = -5e^{-t} + 0$$

Problem S11 (Signals and Systems)

For the same circuit in S10, with the same initial conditions. Find $i_1(t)$ and $i_2(t)$ using the mesh current method. Which method did you find easier?




Writing the loop equations we get:

$$V_1 + (i_1 + i_2)R_3 + i_1 R_1 = 0$$

$$V_2 + (i_1 + i_2)R_3 + i_2 R_2 = 0$$

$V_1 = di_1/dt$ , $V_2 = di_2/dt$ (Note $L_1 = 1$, $L_2 = 1$)

substitute $R_1 = R_2 = R_3 = 1$

(1) $$di_1/dt + 2i_1 + i_2 = 0$$

(2) $$di_2/dt + 2i_2 + i_1 = 0$$

Guess: $i_1(t) = I_1 e^{st}$ , $i_2(t) = I_2 e^{st}$.

(1) $I_1 s e^{st} + 2I_1 e^{st} + I_2 e^{st} = 0$

(2) $I_2 s e^{st} + 2I_2 e^{st} + I_1 e^{st} = 0$

(1) $\Rightarrow I_1 (2+s) + I_2 = 0$

(2) $\Rightarrow I_1 + I_2 (2+s) = 0$

$$A = \begin{bmatrix} 2+s & 1 \\ 1 & 2+s \end{bmatrix}$$

$\det(A) = 0 \Rightarrow (2+s)^2 - 1 = 0$

$\Rightarrow s^2 + 4s + 4 - 1 = 0$

$s^2 + 4s + 3 = 0$

$(s+1)(s+3) = 0$

$\Rightarrow s_1 = -1, \; s = -3$

The rest clearly follows exactly as in the node method.

Problem S12 (Signals and Systems)

For the same circuit in S10, with the same initial conditions. Let the state of the system be,

$$x(t) = \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$$

a) Find the state-space equations that describe the evolution of the circuit, in the form: $\dot{x} = Ax$

b) Suppose that the output of the system is the voltage at the middle node ($e_3$); write the system output equations, and draw the block diagram representation for the complete system.

We need to express $i_1'$ and $i_2'$ in terms of the state variables $i_1$ & $i_2$.

This comes out immediately out of the mesh current equations.

(1) $di_1/dt + 2i_1 + i_2 = 0$

$$\Rightarrow di_1/dt = -2i_1 - i_2$$

(2) $\frac{di_2}{dt} + 2i_2 + i_1 = 0$

$$\Rightarrow \frac{di_2}{dt} = -i_1 - 2i_2$$

$$\begin{bmatrix} \dot{i}_1 \\ \dot{i}_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Note that you could have obtained the same equations using the Node Voltage method. However, the node voltage equations give $i_1$ & $i_2$ in terms of $e_1$ and $e_2$. You would have to manipulate to obtain $e_1$ in terms of $i_1$ & $i_2$, and $e_2$ in terms of $i_1, i_2$

In particular, the node voltage gives

(1) $i_1 + 2e_1/3 - e_2/3 = 0$

(2) $i_2 + 2e_2/3 - e_1/3 = 0$

manipulating the above equations to eliminate $e_2$ from (1) and $e_1$ from (2) we get:

$2 \times (1) + (2) \Rightarrow i_2 + 2i_1 + e_1 = 0$

$(1) + 2 \times (2) \Rightarrow i_1 + 2i_2 + e_2 = 0$

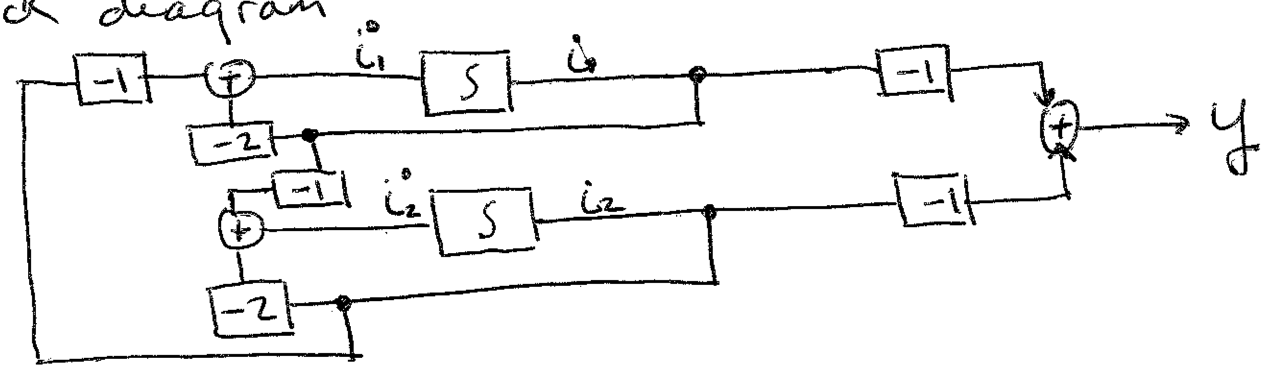
$$\left.\begin{aligned} di_1/dt = e_1 &= -2i_1 - i_2 \\ di_2/dt = e_2 &= -i_1 - 2i_2 \end{aligned}\right\} \begin{bmatrix} \dot{i}_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Part b : $y = e_3$

$e_3 + i_1 + i_2 = 0 \Rightarrow e_3 = -i_1 - i_2$

$$Y = \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Block diagram



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Problem S13 (Signals and Systems)

For the matrix,

$$A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

a) Find the characteristic polynomial of A
b) Find the eigen-values and eigen-vectors of A

A) $SI - A = \begin{bmatrix} S-2 & 2 & 0 \\ 2 & S-4 & 2 \\ 0 & 2 & S-2 \end{bmatrix}$

$\det(SI-A) = (S-2)(S-4)(S-2) - 8(S-2)$

$= S(S-2)(S-6) = 0$ (characteristic polynomial)

$S=0,\ S=2,\ S=6$ are eigenvalues

$S=0 \Rightarrow$ ~~[illegible]~~ $A_1 = A_2 = A_3 \Rightarrow V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$S=2 \Rightarrow A_2 = 0,\ A_1 = -A_3 \Rightarrow V_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$S=6 \Rightarrow A_1 = A_3,\ A_2 = -2A_1$

$V_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

(*eigenvectors: $\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$