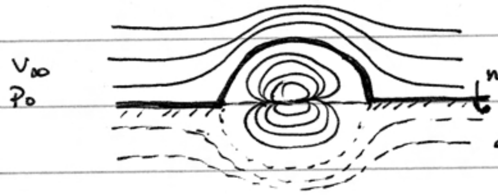
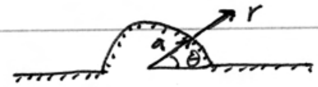
Solutions F10

Flow is a superposition of doublet and uniform (along x)



↳ this reduces to:



the stream function is $\psi = V_\infty r \sin\theta \left(1 - \left(\frac{a}{r}\right)^2\right)$

the velocity components are:

$$V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_\infty \cos\theta \left(1 - \left(\frac{a}{r}\right)^2\right)$$

$$V_\theta = -\frac{\partial\psi}{\partial r} = -V_\infty \sin\theta \left(1 + \left(\frac{a}{r}\right)^2\right)$$

(a) Along the hump: $r = a$ then $V_r = 0$

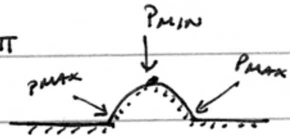
$V_\theta = -2V_\infty \sin\theta$

Bernoulli $\Rightarrow$ $\frac{1}{2}\rho V_\infty^2 + P_0 = \frac{1}{2}\rho V_\theta^2 + P \Rightarrow P = P_0 + \frac{1}{2}\rho V_\infty^2\left(1 - \left(\frac{V_\theta}{V_\infty}\right)^2\right)$

$P = P_0 + \frac{1}{2}\rho V_\infty^2 (1 - 4\sin^2\theta)$

$\boxed{P_{max} = P_0 + \frac{1}{2}\rho V_\infty^2}$ at $\theta = 0$ and $\theta = \pi$

$\boxed{P_{MIN} = P_0 - \frac{3}{2}\rho V_\infty^2}$ at $\theta = \pi/2$



(b) for $\psi = 0$ at $r = a$, then ψ increases for other streamlines from zero.

at $r = 2a$, $\theta = \frac{\pi}{2}$: $\psi = V_\infty 2a \sin\frac{\pi}{2}\left(1 - \left(\frac{a}{2a}\right)^2\right)$

$= 2V_\infty a \left(\frac{3}{4}\right) = \frac{3}{2} V_\infty a$

This means $\psi = \frac{3}{2}V_\infty a$ is constant on the streamline, therefore:

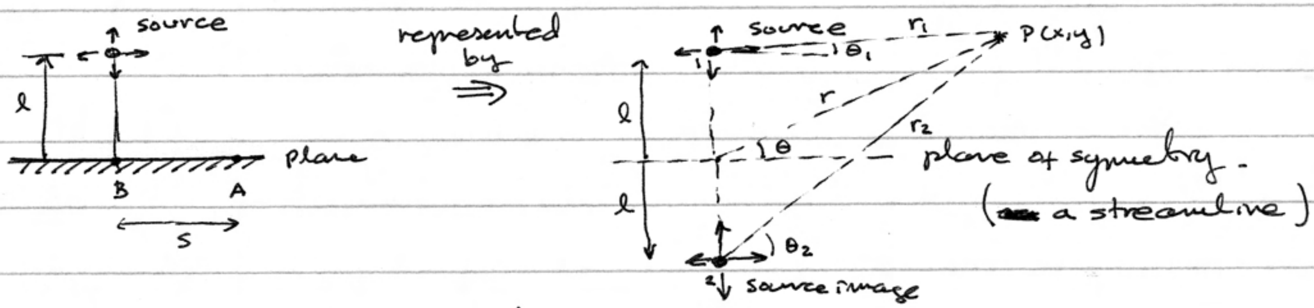
$\frac{3}{2}\cancel{V_\infty} a = \cancel{V_\infty} r \sin\theta \left(1 - \left(\frac{a}{r}\right)^2\right)$ solve for r:

$\Rightarrow r^2 - \frac{3a}{2\sin\theta} r - a^2 = 0$

$$\boxed{r = \frac{3a}{4\sin\theta}\left(1 + \sqrt{1 + \frac{16}{9}\sin^2\theta}\right)}$$

Solutions F11

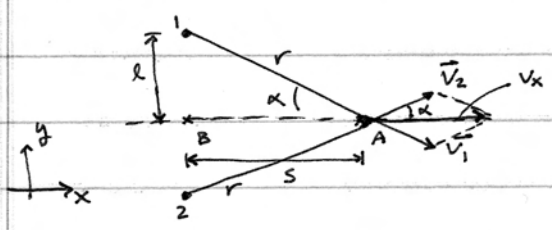
To solve this problem, we will use the method of images and place an identical source (pipe) at a symmetric point "inside" the plane surface in order to force the mid-plane be a streamline :



the stream function of the ^dual source is $\psi = \frac{\Lambda}{2\pi}(\theta_1 + \theta_2)$

and in the most general case we would find ψ at the point (x,y) by finding expressions for $\theta_1 = \theta_1(r,\theta)$ and $\theta_2 = \theta_2(r,\theta)$

In our case, this is not required as we look only for the velocity distribution along the surface. We only need information on the contribution of each source to the velocity : $V_r = \frac{\Lambda}{2\pi r}$



y - components cancel out.

x - component is $V_x = 2V_r \cos\alpha$

and $\cos\alpha = \frac{s}{r}$ $\Rightarrow$ $V_x = 2V_r \frac{s}{r} = \frac{\Lambda}{\pi}\frac{s}{r^2}$

or $V_x = \frac{\Lambda}{\pi}\frac{s}{s^2+l^2}$. From Bernoulli's : $\frac{1}{2}\rho V_A^2 + P_A = \frac{1}{2}\rho V_B^2 + P_B$

and $V_A = V_x$, while $V_B = 0$ (symmetric point).

then $\boxed{P_B - P_A = \frac{1}{2}\rho V_x^2 = \frac{1}{2}\rho\left(\frac{\Lambda}{\pi}\frac{s}{s^2+l^2}\right)^2}$

Values: $\rho = 62\ lb/ft^3$

$\Lambda = \frac{1}{2}\pi$ $(= \dot{V}/\text{depth})$

$s = 4\ ft$

$l = 3\ ft$

$\boxed{P_B - P_A = 0.1984\ \frac{lb}{ft^2}}$

Signals & Systems

S14

A) From S12 we have

$$\begin{bmatrix} \dot{i}_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

B) $$(sI-A) = \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}$$

$$|sI-A| = (s+2)^2-1 = 0 \Rightarrow s^2+4s+3 = 0$$

$$s_1 = -1,\ s_2 = -3$$

c) eigen-vectors

$$s=-1 \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} I_1^{s1} \\ I_2^{s1} \end{bmatrix} = 0 \qquad I_1^{s_1} = -I_2^{s1}$$

$$I^{s1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$s=-3 \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow I^{s2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$i_1(t) = -ae^{-t} + be^{-3t}$$
$$i_2(t) = ae^{-t} + be^{-3t}$$

$$\left.\begin{aligned} i_1(0) &= -a+b = 10\,A \Rightarrow \\ i_2(0) &= a+b = 0 \Rightarrow a=-b \end{aligned}\right\} \begin{aligned} 2b&=10 \\ b&=5 \\ a&=-5 \end{aligned}$$

$$i_1(t) = 5e^{-t} + 5e^{-3t} \qquad i_2(t) = -5e^{-t} + 5e^{-3t}$$

a) $\tilde{x} = \begin{bmatrix} i_1(t) \\ i_3(t) \end{bmatrix}$

Note: $i_1 + i_2 + i_3 = 0 \Rightarrow i_3 = -i_1 - i_2$

State transformation matrix $T = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$, $\tilde{x} = Tx$.

b) $T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$

$\tilde{A} = TAT^{-1}$ where A is $\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$

$\tilde{A} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}$

$sI - \tilde{A} = \begin{bmatrix} s+1 & -1 \\ 0 & (s+3) \end{bmatrix}$ $\quad |sI - \tilde{A}| = (s+1)(s+3) = 0$

$s_1 = -1, \; s_2 = -3$

eigen-vectors $I^{s1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $I^{s2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Hence, $i_1(t) = ae^{-t} - be^{-3t}$

$i_2(t) = 2be^{-3t}$

Note that $i_3 = -i_1 - i_2$

Hence, $i_3(0) = -i_1(0) - i_2(0) = -10A$.

Now $i_1(0) = a - b = 10A \Rightarrow a = 10 - b$

$i_3(0) = 2b = -10A \Rightarrow b = -5$

$\Rightarrow a = 5$

Complete solution:

$$i_1(t) = 5e^{-t} + 5e^{-3t}$$

$$i_3(t) = -10e^{-3t}$$

d) Compare to S14.

$i_1(t)$ is the same ✓

In S14 we obtained i_1 & i_2

$$i_3(t) = -i_1(t) - i_2(t)$$

$$= -5e^{-t} - 5e^{-3t} + 5e^{-t} - 5e^{-3t}$$

$$= -10e^{-3t}$$ ✓

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a) state variables: $V_c(t)$, $i_L(t)$

· Node equations

$$① \quad \left.\begin{aligned} & i_c + \frac{V_c - V_L}{R} = 0 \\ & i_c = C\,dv_c/dt \end{aligned}\right\} \; C\,dv_c/dt + \frac{V_c}{R} - \frac{V_L}{R} = 0 \quad (*)$$

$$② \quad \left.\begin{aligned} & i_L + \frac{V_L - V_c}{R} = 0 \\ & V_L = L\,di_L/dt \end{aligned}\right\} \; i_L + \frac{L}{R}\frac{di_L}{dt} - \frac{V_c}{R} = 0 \quad (**)$$

Note: $V_L = V_c - i_L \cdot R$

$$(*) \Rightarrow C\frac{dv_c}{dt} + \frac{V_c}{R} - \frac{V_c}{R} + i_L = 0 \Rightarrow \frac{dv_c}{dt} = -\frac{i_L}{C}.$$

$$(**) \Rightarrow \frac{di_L}{dt} = \frac{V_c}{L} - i_L \cdot \frac{R}{L}$$

$$\begin{bmatrix} \dot{V_c} \\ \dot{i_L} \end{bmatrix} = \begin{bmatrix} 0 & -1/C \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} V_c \\ i_L \end{bmatrix}$$

$$|sI-A| = \begin{vmatrix} s & +1/C \\ -1/L & s+R/L \end{vmatrix} = s(s+R/L) + \frac{1}{LC} = 0$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s_1 = \frac{-R/L + \sqrt{(R/L)^2 - 4/LC}}{2}$$

$$s_2 = \frac{-R/L - \sqrt{(R/L)^2 - 4/LC}}{2}$$

b) $R = 0\,\Omega$, $L = 2H$, $C = 2F$

$s_1 = j/2$, $s_2 = -j/2$ $\qquad j = \sqrt{-1}$

possible solutions: $e^{j/2\, t}$, $e^{-j/2\, t}$.

$$s_1 = j/2 \Rightarrow sI - A = \begin{bmatrix} j/2 & 1/2 \\ -1/2 & j/2 \end{bmatrix} \begin{bmatrix} E_1^{s_1} \\ E_2^{s_1} \end{bmatrix}$$

$$j/2\, E_1 + \tfrac{1}{2} E_2 = 0 \Rightarrow E_1 = -E_2/j = j \cdot E_2$$

$$E^{s_1} = \begin{bmatrix} j \\ 1 \end{bmatrix}$$

Similarly $\quad E^{s_2} = \begin{bmatrix} -j \\ 1 \end{bmatrix}$

S16 - Cont. (3).

$$V_c(t) = aj\,e^{j/2\,t} - jb\,e^{-j/2\,t}.$$

$$i_L(t) = a e^{j/2\,t} + b e^{-j/2\,t}$$

$$\left.\begin{aligned} V_c(0) &= 10v \\ i_L(0) &= 0A. \end{aligned}\right\} \Rightarrow \quad \begin{aligned} &aj - bj = 10 \\ &a + b = 0 \Rightarrow a = -b \end{aligned}$$

$$\Rightarrow -2bj = 10 \Rightarrow b = -5/j = 5j$$

$$a = -5j$$

~~Tot~~ Solution:

$$V_c(t) = 5e^{j/2\,t} + 5e^{-j/2\,t}.$$

$$i_L(t) = -5je^{j/2\,t} + 5j\,e^{-j/2\,t}.$$

c) $e^{j\beta} = \cos(\beta) + j\sin(\beta)$

$$\cos(-\beta) = \cos(\beta), \quad \sin(-\beta) = -\sin(\beta)$$

$$e^{j/2\,t} = \cos(t/2) + j\sin(t/2)$$

$$e^{-j/2\,t} = \cos(t/2) - j\sin(t/2)$$

$$V_c(t) = 5\left(\cos(t/2) + j\sin(t/2)\right) + 5\left(\cos(\tfrac{t}{2}) - j\sin(\tfrac{t}{2})\right)$$

$$= 10\cos(t/2)$$

$$i_L(t) = -5j\left[\cos(\tfrac{t}{2}) + j\sin(\tfrac{t}{2})\right] + 5j\left[\cos(\tfrac{t}{2}) - j\sin(\tfrac{t}{2})\right]$$

$$= 10\sin(\tfrac{t}{2})$$

you can now check this answer

$$\frac{dV_c}{dt} = \frac{-i_L}{C} \Rightarrow i_L = -C\frac{dV_c}{dt} \qquad C=2$$

$$\frac{d}{dt}\, 10\cos\left(\frac{t}{2}\right) = -5\sin(t/2)$$

$$\Rightarrow i_L = 10\sin(t/2)$$

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d) without a resistor in the circuit $(R=0)$ energy does not dissipate in this circuit. As a result energy moves back-and-forth between the inductor and the capacitor, leading to the oscillation in the current and voltage.

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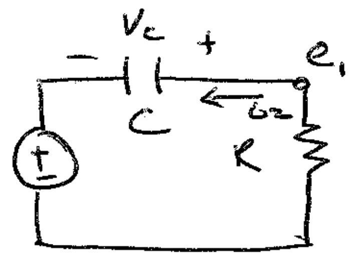
$$i_z = C\frac{dv_c}{dt} \Rightarrow \frac{dv_c}{dt} = \frac{i_z}{C}.$$

$u(t)$



$$e_1 = u(t) + v_c(t)$$

Node equation at e_1: $\quad i_z + \frac{e_1}{R} = 0$

$$\Rightarrow e_1/R = -i_z$$

$$\Rightarrow \frac{u(t) + v_c(t)}{R} = -i_z = -C\frac{dv_c}{dt}$$

$$\frac{dv_c}{dt} = \frac{-u(t) - v_c(t)}{RC}$$

<u>Homogeneous solution:</u> $\cancel{v_c = ae^{st}}$ $\quad u(t) = 0$

$$\Rightarrow \frac{dv_c}{dt} = \frac{-v_c(t)}{RC}$$

$$v_c = ae^{st}$$

$$\Rightarrow ase^{st} = \frac{-ae^{st}}{RC}$$

$$as + \frac{a}{RC} = 0 \quad a\left(s + \frac{1}{RC}\right) = 0 \Rightarrow s = -\frac{1}{RC}.$$

S17 Cont.

$$V_H = a e^{-t/RC}$$

with $u(t) = 1v$ we guess a constant forced response.

$$V_F = B \quad , \quad \dot{V}_F = 0$$

$V_c = V_F = B$

⊛ $$\frac{dV_c}{dt} = \frac{-u(t) - V_c(t)}{RC} = \frac{-1-B}{RC} = 0$$

with $V_F = B$

$\dot{V}_F = 0.$

⊛ Note we use $V_c = V_F = B$

$$\Rightarrow B = -1$$

$$V_F = -1$$

$$V_c(t) = V_H + V_F = a e^{-t/RC} - 1$$

$$V_c(0) = 0 \Rightarrow a - 1 = 0$$
$$\Rightarrow a = 1$$

$$\boxed{V_c(t) = \left(e^{-t/RC} - 1\right)V}$$

Note that as $t \to \infty$ $V_c \to -1v$

this makes sense because as $t \to \infty$ the capacitor becomes an open circuit, current stops flowing,

$\Rightarrow e_1 = 0 \Rightarrow V_c = -u(t) = -1v.$