PAL
10/16/08

Unified Engineering
Problem Set #7 Week 8
Fall, 2008

SOLUTIONS

M10 (M8.1)

for all cases, recall rules for tensorial/indicial notation:

- Latin subscripts take on values 1, 2, 3
- Greek subscripts take on values 1, 2
- when subscript is repeated in one term, it is a "dummy index" and is summed on
- when subscript appears only once on ~~left~~ side on the equation in one term, it is a "free index" and represents separate equations.

So....

(a) $E = \frac{1}{2} \sigma_{\alpha\beta} \epsilon_{\alpha\beta}$

- α and β are dummy indices and are summed on from 1 to 2 (they are greek subscripts)

$\Rightarrow$ same as: $\epsilon = \frac{1}{2} \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} \sigma_{\alpha\beta} \epsilon_{\alpha\beta}$

Thus:

$$\boxed{\epsilon = \frac{1}{2}\left\{\sigma_{11}\epsilon_{11} + \sigma_{12}\epsilon_{12} + \sigma_{21}\epsilon_{21} + \sigma_{22}\epsilon_{22}\right\}}$$

(b) $D_{ab}(\partial h/\partial x_b) = g_a$

- b is repeated in the first term and is thus a <u>dummy index</u> and is summed on from 1 to 3 (if it is a latin subscript)
- a is a <u>free index</u> (not repeated in terms) and indicates there are 3 equations (latin subscripts $\Rightarrow$ 1, 2, 3)

$\Rightarrow$ same as: $\sum_{b=1}^{3} D_{ab}(\partial h/\partial x_b) = g_a$

Thus:

$$\boxed{\begin{aligned} (a=1)&: \quad D_{11}\,\partial h/\partial x_1 + D_{12}\,\partial h/\partial x_2 + D_{13}\,\partial h/\partial x_3 = g_1 \\ (a=2)&: \quad D_{21}\,\partial h/\partial x_1 + D_{22}\,\partial h/\partial x_2 + D_{23}\,\partial h/\partial x_3 = g_2 \\ (a=3)&: \quad D_{31}\,\partial h/\partial x_1 + D_{32}\,\partial h/\partial x_2 + D_{33}\,\partial h/\partial x_3 = g_3 \end{aligned}}$$

(c) $\sigma_{23} = l_{2i'} l_{3j'} \sigma'_{i'j'}$

- i' and j' are repeated in one term and are <u>dummy indices</u> and are summed on from 1 to 3 (they are latin subscripts)

$$\Rightarrow \text{same as: } \sigma_{23} = \sum_{i'=1}^{3} \sum_{j'=1}^{3} l_{2i'} l_{3j'} \sigma'_{i'j'}$$

Thus:

$$\sigma_{23} = l_{21'} (l_{31'} \sigma'_{1'1'} + l_{32'} \sigma'_{1'2'} \; l_{33'} \sigma'_{1'3'})$$
$$+ l_{22'} (l_{31'} \sigma'_{2'1'} + l_{32'} \sigma'_{2'2'} + l_{33'} \sigma_{2'3'})$$
$$+ l_{23'} (l_{31'} \sigma_{3'1'} + l_{32'} \sigma_{3'2'} + l_{33'} \sigma_{3'3'})$$

(d) $C_n = d_{\sigma\gamma} S_{\sigma\gamma} p_n$

- σ and γ are repeated in one term and are summed thus are <u>dummy indices</u> and are summed on from 1 to 2 (they are greek subscripts)
- n is a <u>free index</u> (not repeated in terms) and indicates separate equations (3 as it is a latin subscript)

$$\Rightarrow \text{same as: } C_n = \sum_{\sigma=1}^{2} \sum_{\gamma=1}^{2} d_{\sigma\gamma} S_{\sigma\gamma} p_n$$

Thus:

$$\boxed{\begin{aligned}(i=1):\quad & C_1 = (d_{11}S_{11} + d_{12}S_{12} + d_{21}S_{21} + d_{22}S_{22})p_1\\ (i=2):\quad & C_2 = (d_{11}S_{11} + d_{12}S_{12} + d_{21}S_{21} + d_{22}S_{22})p_2\\ (i=3):\quad & C_3 = (d_{11}S_{11} + d_{12}S_{12} + d_{21}S_{21} + d_{22}S_{22})p_3\end{aligned}}$$

(e) $A_{rs} = Q_{rstv}\,\alpha_t\,Y_v$ (for $r=3$, $s=2$)

$\Rightarrow A_{32} = Q_{32tv}\,\alpha_t\,Y_v$

- t and v are <u>dummy</u> <u>indices</u> and are summed on from 1 to 3 (they are latin subscripts)

$\Rightarrow$ same as: $A_{32} = \sum_{t=1}^{3}\sum_{v=1}^{3} Q_{32tv}\,\alpha_t\,Y_v$

Thus:

$$\boxed{\begin{aligned}A_{32} = {} & Q_{3211}\alpha_1 Y_1 + Q_{3212}\alpha_1 Y_2 + Q_{3213}\alpha_1 Y_3\\ & + Q_{3221}\alpha_2 Y_1 + Q_{3222}\alpha_2 Y_2 + Q_{3223}\alpha_2 Y_3\\ & + Q_{3231}\alpha_3 Y_1 + Q_{3232}\alpha_3 Y_2 + Q_{3233}\alpha_3 Y_3\end{aligned}}$$

M 11 (M 8. 2)

$$\begin{bmatrix} P_{111} & 2P_{112} & P_{122} \\ P_{211} & 2P_{212} & P_{222} \\ P_{311} & 2P_{312} & P_{322} \end{bmatrix} \begin{Bmatrix} S_{11} \\ S_{12} \\ S_{22} \end{Bmatrix} = \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix}$$

First write out in full (as it may help):

$$P_{111} S_{11} + 2 P_{112} S_{12} + P_{122} S_{22} = \beta_1$$
$$P_{211} S_{11} + 2 P_{212} S_{12} + P_{222} S_{22} = \beta_2$$
$$P_{311} S_{11} + 2 P_{312} S_{12} + P_{322} S_{22} = \beta_3$$

Look at/consider this piece by piece:

① The subscript on β must be a free index because it changes with the equation and represents separate equations. It must be latin since it takes on the values 1, 2, 3 .... $(= \beta_i)$

② The subscript on S take on the values 1 and 2 and therefore must be greek. They change independently and thus must be different .... $(S_{\alpha\gamma})$

(3) The first subscript on P matches the subscript on β . . . .

$$(P_{i??} \, S_{\alpha\gamma} = \beta_i)$$

(4) The second and third subscripts on P match those on S. By making them the same they are also summed on (as occurs in the equations)

$$\Rightarrow \boxed{P_{i\alpha\gamma} \, S_{\alpha\gamma} = \beta_i}$$

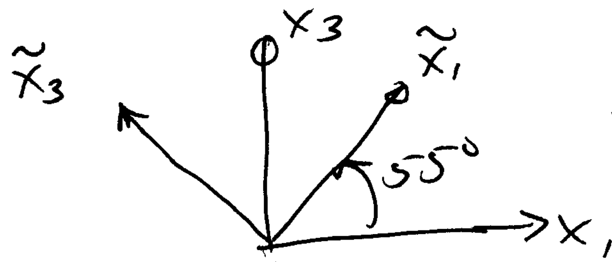
<u>BUT</u>, one must also make the assumption that $S_{\alpha\gamma}$ is symmetric ($S_{\alpha\gamma} = S_{\gamma\alpha}$) and $P_{i\alpha\gamma}$ is symmetric in the last two indices ($P_{i\alpha\gamma} = P_{i\gamma\alpha}$) to put the factor of 2 in the final equations on the S_{12} terms with P_{i12} as multipliers.

M12 (M8.3)

$$\underline{F} = -6N\,\underline{i}_1 + 3N\,\underline{i}_2 + 2N\,\underline{i}_3$$

(a) Rotation is in $x_1 - x_3$ plane about x_2-axis by angle of 55°. ($\theta = +55°$)

Draw this



+CCW

with x_2 out of the paper

~ represents the rotated system.

$$\tilde{x}_2 = x_2$$

Know that the rotation can be represented via the direction cosines:

$$\tilde{F}_{\tilde{i}} = l_{\tilde{i}j} F_j$$

Determine the direction cosines:

$$l_{\tilde{1}1} = \cos(+\theta) = \cos(+55°) = 0.574$$

$$l_{\tilde{1}2} = \cos(-90°) = 0$$

$$l_{\tilde{1}3} = \cos(-90+\theta) = \cos(90-\theta) = \sin\theta$$
$$= \sin(+55°) = 0.819$$

$$l_{\tilde{2}1} = \cos(+90^\circ) = 0$$
$$l_{\tilde{2}2} = \cos(0^\circ) = 1$$
$$l_{\tilde{2}3} = \cos(+90^\circ) = 0$$
$$l_{\tilde{3}1} = \cos(90+\theta) = -\sin\theta = -\sin(+55^\circ)$$
$$= -0.819$$
$$l_{\tilde{3}2} = \cos(-90^\circ) = 0$$
$$l_{\tilde{3}3} = \cos(\theta) = \cos(+55^\circ) = 0.574$$

writing out the rotational equation:

$$\tilde{F}_{\tilde{i}} = l_{\tilde{i}j} F_j$$

$$\Rightarrow \tilde{F}_{\tilde{1}} = l_{\tilde{1}1} F_1 + l_{\tilde{1}2} F_2 + l_{\tilde{1}3} F_3$$
$$= (0.574)(-6N) + (0.819)(2N)$$
$$= -1.806 N$$

$$\tilde{F}_{\tilde{2}} = l_{\tilde{2}1} F_1 + l_{\tilde{2}2} F_2 + l_{\tilde{2}3} F_3$$
$$= (1)(3N) = 3N$$

$$\tilde{F}_{\tilde{3}} = l_{\tilde{3}1} F_1 + l_{\tilde{3}2} F_2 + l_{\tilde{3}3} F_3$$
$$= (-0.819)(-6N) + (0.574)(2N)$$
$$= 6.062 N$$

$$\Rightarrow \boxed{\underset{\sim}{\tilde{F}} = (-1.806N)\underline{i}_1 + (3N)\underline{i}_2 + (6.062N)\underline{i}_3}$$

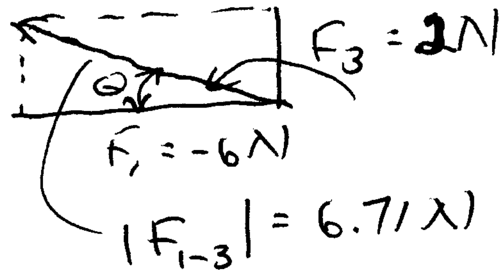
(b) to prove that these expressions for $\underline{F}$ (and $\tilde{\underline{F}}$) are equivalent one can look at the overall magnitude of the vector (along with directions)

<u>Step 1</u> Note that the magnitude in the 2-direction is the same since there is no rotation there (that axis is rotated about)

<u>Step 2</u> The 2-D vector in the $x_1 - x_3$ plane creates a right triangle with the x_3-magnitude as one side and the x_1-magnitude as the other.

With this, determine the overall magnitude of the vector via:

$$|F_{1-3}| = \sqrt{(F_1)^2 + (F_3)^2}$$
$$= \sqrt{(-6N)^2 + (2N)^2} = \sqrt{40N^2}$$
$$= 6.32\,N$$



and the angle relative to $x_1 - x_3$:

$$\tan^{-1}\left(\frac{F_3}{F_1}\right) = \tan^{-1}\left(-\frac{2N}{6N}\right) =$$

$$\tan^{-1}\left(-\frac{1}{3}\right) = \underline{\underline{-18.4^\circ}}$$

$$\Rightarrow \theta = +161.6^\circ \qquad (+ccw)$$

Step 3 Do the same for the rotated $\tilde{x}_1 - \tilde{x}_3$ system

$$|\tilde{F}_{1-3}| = \sqrt{(\tilde{F}_{\tilde{1}})^2 + (\tilde{F}_3)^2}$$

$$= \sqrt{(-1.806N)^2 + (6.062N)^2}$$

$$= \sqrt{3.261N^2 + 36.75N^2}$$

$$\cong \sqrt{40N^2} = \underline{\underline{6.32\ N}}$$ ✓ the same

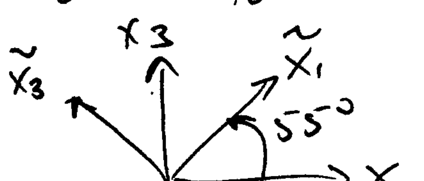
and then the angle relative to $\tilde{x}_1 - \tilde{x}_3$:

$$\tan^{-1}\left(\frac{\tilde{F}_{\tilde{1}}}{\tilde{F}_{\tilde{3}}}\right) = \tan^{-1}\left(-\frac{6.062}{1.806}\right)$$

$$= \tan^{-1}(-3.36) = -73.4^\circ$$

$$\Rightarrow \theta = +106.6 \ (+ccw)$$

and then the angle relative to $x_1 - x_3$ ... must add the rotation angle, 55°:



$$106.6 + 55^\circ = \underline{\underline{161.6^\circ}}$$ ✓ the same

$\boxed{\text{Proven}}$ Q.E.D.

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M13 (M 8.4)

Begin by writing out the stress equilibrium equations as we have them in tensorial notation:

$$\frac{\partial \sigma_{mn}}{\partial x_n} + f_n = 0$$

expand this:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = 0$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0$$

Recall the symmetry of the stress tensor: $\sigma_{mn} = \sigma_{nm}$

(a) To go from tensorial to engineering notation, recall:

$$x_1 \to x$$
$$x_2 \to y$$
$$x_3 \to z$$

and a similar conversion on subscripts on the stresses. So:

$$\boxed{\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x = 0\\
&\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y = 0\\
&\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0
\end{aligned}}$$

Notes:
- expression of stresses with subscripts still remains symmetric
- τ can be used in place of σ for shear stresses (2 different subscripts: $\tau_{xy}, \tau_{xz}, \tau_{yz}$)

(b) A state of plane stress has:
- no out-of-plane components

$$\Rightarrow \sigma_z = \sigma_{yz} = \sigma_{xz} = 0$$

- no out-of-plane gradient

$$\Rightarrow \frac{\partial}{\partial z} = 0$$

And, since there are no forces in the out-of-plane direction the body force in that direction (f_z) must be zero.

This results in two equations:

$$\boxed{\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + f_x = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0 \end{aligned}}$$

This could also be done in tensorial notation:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + f_1 = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 = 0$$

T14

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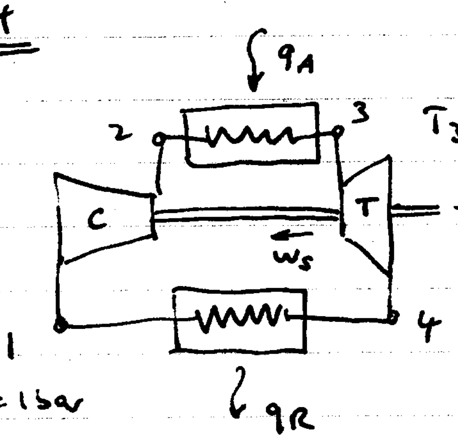
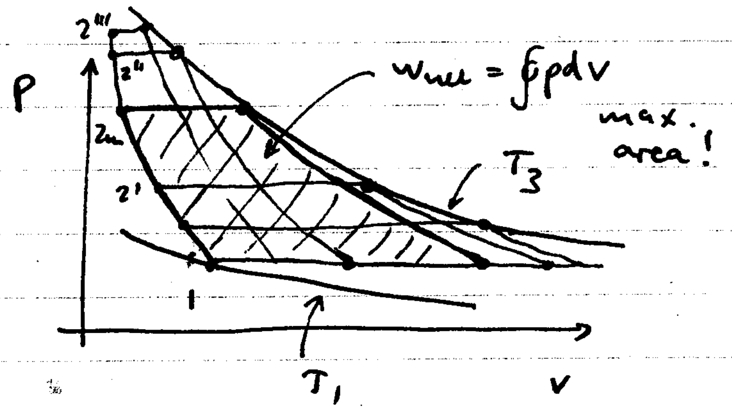


$T_3 = 1200K$

$p_1 = 1 bar$
$T_1 = 300K$

a)



Concepts: - Brayton cycle, 1st law, thermal eff.

b) $w_{net} = q_A - q_R$ (from 1st law cycle $\Delta U = 0$) ; ideal gas $dh = c_p\,dT$

$q_A = c_p(T_3 - T_2)$, $q_R = c_p(T_4 - T_1)$ from 1st law CV around heat exchangers

$w_{net} = c_p(T_3 - T_2 - T_4 + T_1)$; choice of T_2 (equivalently p_2) sets cycle work for fixed T_1, T_3

so $\frac{dw_{net}}{dT_2} = 0$ for max work

yields $0 = -1 - \frac{dT_4}{dT_2} \rightarrow \frac{dT_4}{dT_2} = -1$ but $\frac{T_3}{T_4} = \frac{T_2}{T_1}$ since same PR in compr. and turbine (ad. rev. proc.)

$T_4 = \frac{T_3 T_1}{T_2} \rightarrow \frac{dT_4}{dT_2} = -\frac{T_3 T_1}{T_2^2} = -1$

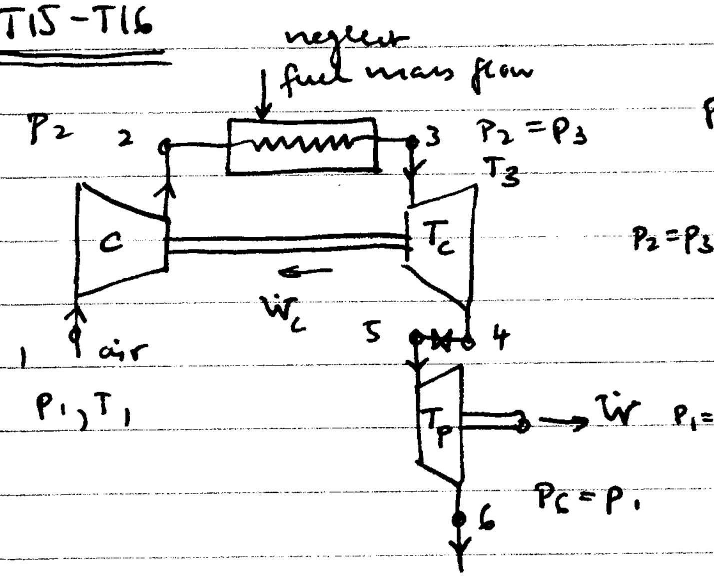
$T_2\big|_{\text{max work}} = \sqrt{T_1 T_3}$ and $p_2 = p_1\left(\frac{T_2}{T_1}\right)^{\gamma/\gamma-1} \rightarrow \underline{p_2\big|_{\text{max work}} = p_1\left(\frac{T_3}{T_1}\right)^{\frac{\gamma}{2(\gamma-1)}}}$

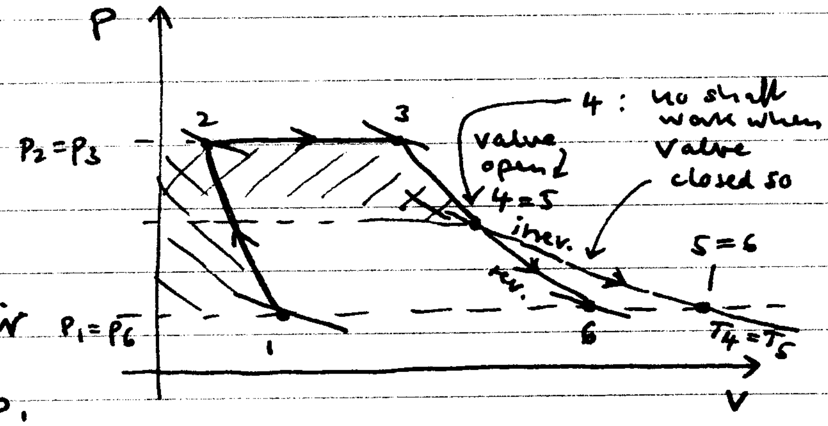
$\underline{p_{2\,\text{max work}} = 11.3 \text{ bar}}$, $T_{2\,\text{max work}} = 600K$

c) $\eta_{th}^B = 1 - \frac{T_1}{T_2}$ so $\eta_{th}^B\big|_{\text{max work}} = 1 - \sqrt{\frac{T_1}{T_3}}$

$\underline{\eta_{th}^B\big|_{\text{max work}} = 0.5}$

T15 - T16 16. Unified Fall 08





Concepts: 1st law, enthalpy, Brayton cycle, rev / irrev process

a) valve open: $w_c = w_{T_c}$ shaft power balance

$$c_p(T_2 - T_1) = c_p(T_3 - T_4) \quad \text{and} \quad T_2 = T_1\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

so $T_4 = T_3 - T_1\left[\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$, $T_4 = 949.8\,K$, $p_4 = p_2\left(\frac{T_4}{T_3}\right)^{\gamma/\gamma-1} = 2.76\,\text{bar}$

find $\boxed{T_5 = T_4 = 949.8\,K}$ and $\boxed{p_5 = p_4 = 2.76\,\text{bar}}$

find $T_6 = T_5\left(\frac{p_6}{p_5}\right)^{\frac{\gamma-1}{\gamma}} \rightarrow \boxed{T_6 = 710.2\,K}$, $\boxed{p_6 = 1\,\text{bar}}$

b) $w_c = c_p(T_2 - T_1) = c_p(T_3 - T_4)$ (1st law) $\rightarrow \boxed{w_c = 178\,kJ/kg}$

c) $w = c_p(T_5 - T_6)$ (1st law) $\rightarrow \boxed{w = 241\,kJ/kg}$

d) $q_A = c_p(T_3 - T_2) = c_p\left(T_3 - T_1\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right) \rightarrow \boxed{q_A = 653\,kJ/kg}$

e) $\dot{W} = \dot{m}w \rightarrow \dot{m} = \frac{\dot{W}}{w}$ find $\boxed{\dot{m} = 1.04\,kg/s}$

f) Heating value $\Delta h_f = 44\,MJ/kg$, $\dot{m}q_A = \dot{m}_f \Delta h_f \rightarrow \boxed{\dot{m}_f = \frac{q_A \dot{m}}{\Delta h_f} = 0.015\,\frac{kg}{s}}$

$\boxed{SFC = \frac{\dot{m}_f}{\dot{W}} = 0.216\,kg/kWh}$

g) $\eta_{th} = \frac{w}{q_A} \rightarrow \boxed{\eta_{th} = 0.37}$

valve closed (slightly open)
→ free expansion! → only lost work and no turbine work!

h) valve closed: 4 ⋈ 5

1st law: $0 = \dot{m}(h_4 - h_5)$ no work no heat transfer

ideal gas: $dh = c_p dT$ → $h_4 = h_5$ → $\boxed{T_5 = T_4 = 949.8\,K}$

j) $\boxed{w = 0}$ no expansion, no work $p_5 = p_6$

pressure drops: $\boxed{p_5 = p_6 = 1\,\text{bar}}$

$\boxed{T_6 = T_5 = 949.8\,K}$