PAL
10/23/08

Unified Engineering

Problem Set 8 - Week 9
Fall, 2008

SOLUTIONS

M14 (M9.1) Given:

3-D body with

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = A$$

$$\sigma_{12} = +Bx_1 - Bx_2 + C_{12} + f_{12}(x_3)$$

(since σ_{12} has a + linear variation of B in x_1 and a - linear variation of B in x_2. Can add a constant to that [use C_{12}] as well as a possible variation in x_3 [use $f_{12}(x_3)$])

NOTE: In most general case, all stresses are functions of all three directions:

$$\sigma_{mn}(x_1, x_2, x_3)$$

Use this information with the stress equations of equilibrium to determine information about the other stresses. First write the 3-D stress equilibrium equations -- there are no body forces:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \qquad (1)$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = 0 \qquad (2)$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} = 0 \qquad (3)$$

→ Take partial derivatives of what is known about stresses to give information

$$\frac{\partial \sigma_{11}}{\partial x_1} = 0 \qquad \frac{\partial \sigma_{22}}{\partial x_2} = 0 \qquad \frac{\partial \sigma_{33}}{\partial x_3} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} = +B \qquad \frac{\partial \sigma_{12}}{\partial x_2} = -B$$

→ Use this information in (1):

$$0 \; - B + \frac{\partial \sigma_{13}}{\partial x_3} = 0$$

$$\Rightarrow \frac{\partial \sigma_{13}}{\partial x_3} = +B$$

→ This implies (using multi-variable calculus and getting functions of integration:

$$\sigma_{13} = Bx_3 + f_{13}(x_1, x_2) + C_{13}$$

→ Use differential results in (2):

$$+B + 0 + \frac{\partial \sigma_{23}}{\partial x_3} = 0$$

$$\Rightarrow \frac{\partial \sigma_{23}}{\partial x_3} = -B$$

Using same procedure gives:

$$\sigma_{23} = -Bx_3 + f_{23}(x_1, x_2) + C_{23}$$

→ Take appropriate partials of these two stresses for use in equation (3):

$$\frac{\partial \sigma_{13}}{\partial x_1} = \frac{\partial f_{13}(x_1, x_2)}{\partial x_1}$$

and
$$\frac{\partial \sigma_{23}}{\partial x_2} = \frac{\partial f_{23}(x_1, x_2)}{\partial x_2}$$

→ Placing in (3):

$$\frac{\partial f_{13}(x_1, x_2)}{\partial x_1} + \frac{\partial f_{23}(x_1, x_2)}{\partial x_2} + 0 = 0$$

→ This is all the information that can be obtained. Thus, this is what can be said about the stress field:

- The extensional stresses are constant with:

$$\boxed{\sigma_{11} = A, \quad \sigma_{22} = A, \quad \sigma_{33} = A}$$

- The shear stress in the $x_1 - x_2$ plane has linear variation in x_1 and x_2 with a value of $+B$ in x_1 and of $-B$ in x_2, plus some function related to x_3 including a constant:

$$\boxed{\sigma_{12} = Bx_1 - Bx_2 + f_{12}(x_3) + C_{12}}$$

- The shear stresses involving the x_3-direction have a linear variation in x_3 (σ_{13} of $+B$, σ_{23} of $-B$) and have functional variations in x_1 and x_2 linked by partial derivatives. In addition, constants must be added:

$$\boxed{\begin{aligned} \sigma_{13} &= Bx_3 + f_{13}(x_1, x_2) + C_{13} \\ \sigma_{23} &= -Bx_3 + f_{23}(x_1, x_2) + C_{23} \\ \text{where: } & \frac{\partial f_{13}(x_1, x_2)}{\partial x_1} = -\frac{\partial f_{23}(x_1, x_2)}{\partial x_2} \end{aligned}}$$

M15 (M9.2)

There is a two-dimensional field of displacement with no displacement in the third (x_3) direction $\Rightarrow u_3 = 0$. Thus, all out-of-plane strains are zero as either u_3 is zero or $\frac{\partial}{\partial x_3} = 0$:

$$\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$$

→ Now define the in-plane strain-displacement relations:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

with the displacement field defined as:

$$\underline{u} = u_1 \underline{i}_1 + u_2 \underline{i}_2$$

Apply this to each case to get the in-plane strains.

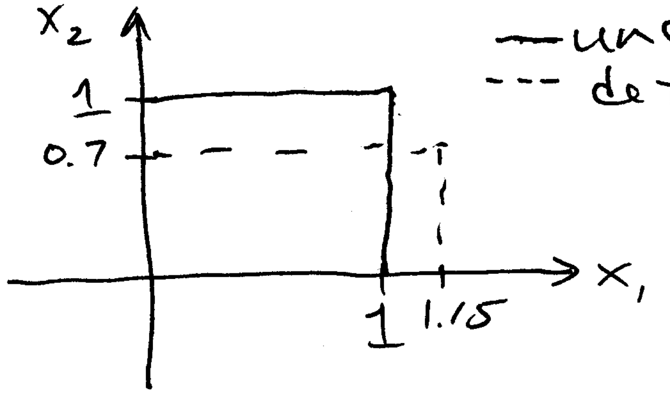
(a) $\underline{u} = (0.015 x_2)\,\underline{i}_1 - (0.030 x_2)\,\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0.015$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = -0.030$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0$$

— undeformed
--- deformed



$$\boxed{\begin{aligned} \epsilon_{11} &= 0.015 \\ \epsilon_{22} &= -0.030 \\ \epsilon_{12} &= 0 \end{aligned}}$$

This is pure elongation in two directions (one positive, one negative)

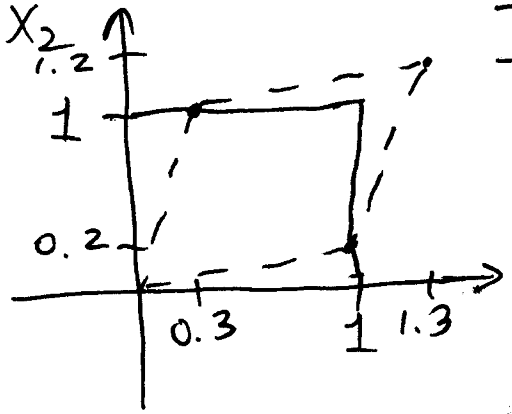
(b) $\underline{u} = (0.030\, x_2)\, \underline{i}_1 + (0.020\, x_1)\, \underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) = \frac{1}{2}(0.030 + 0.020)$$

$$= 0.025$$



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$$\boxed{\begin{aligned} \epsilon_{11} &= 0 \\ \epsilon_{22} &= 0 \\ \epsilon_{12} &= 0.025 \end{aligned}}$$

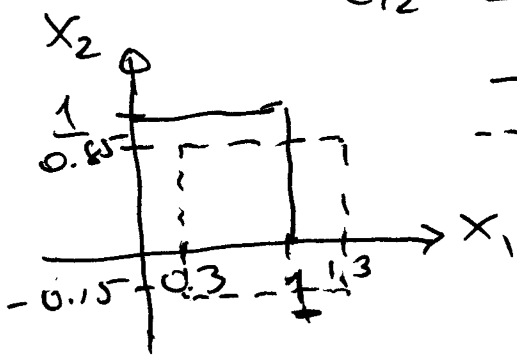
This is <u>pure</u> <u>shear</u>

(c) $\underline{u} = (0.030)\,\underline{i}_1 - (0.015)\,\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0$$



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$$\boxed{\begin{aligned} \epsilon_{11} &= 0 \\ \epsilon_{22} &= 0 \\ \epsilon_{12} &= 0 \end{aligned}}$$

This is <u>pure</u> <u>translation</u> in both x_1 and x_2

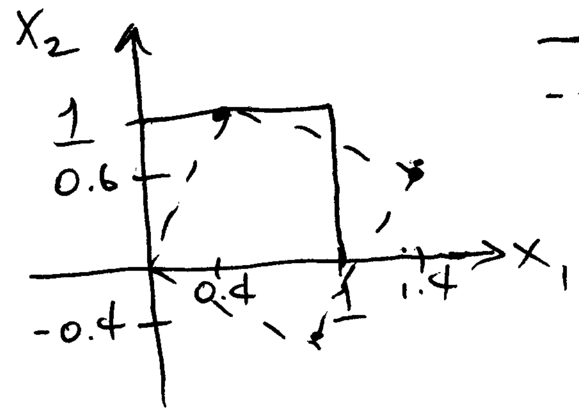
(d) $\underline{u} = (0.040 x_2)\underline{i}_1 - (0.040 x_1)\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_1} = 0$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$= \frac{1}{2}(0.040 - 0.040) = 0$$



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$$\boxed{\begin{aligned} \epsilon_{11} &= 0 \\ \epsilon_{22} &= 0 \\ \epsilon_{12} &= 0 \end{aligned}}$$

This is <u>pure</u> <u>rotation</u>

(e) $\underline{u} = (0.050 x_1 - 0.050 x_2)\underline{i}_1 + (-0.050 x_1 - 0.020 x_2)\underline{i}_2$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = 0.050$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2} = -0.020$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$= \frac{1}{2}(-0.050 - 0.050) = -0.050$$

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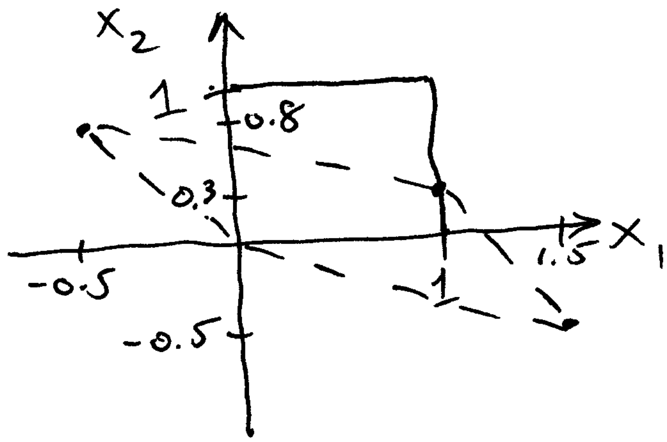


$$\boxed{\begin{aligned} \epsilon_{11} &= 0.050 \\ \epsilon_{22} &= -0.020 \\ \epsilon_{12} &= -0.050 \end{aligned}}$$

This is <u>combined</u> <u>elongation</u> <u>and</u> <u>shear</u>

M16 (M9.3)

$$\epsilon_{11} = a x_1 + b x_2^2 + C_{11}$$

$$\epsilon_{22} = -(b/2) x_1^2 + a x_2 + C_{22}$$

$$\epsilon_{33} = 0$$

$$\epsilon_{13} = \epsilon_{23} = 0$$

(a) To determine the three-dimensional field of displacements, use the strain-displacement relations:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

$$\epsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\epsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

$$\epsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

→ use the expression for ϵ_{11}, integrate and use multi-variable calculus and get functions (and constants) of integration for u_1:

$$u_1 = \int \epsilon_{11}\, dx_1 = \frac{a}{2} x_1^2 + b x_1 x_2^2 + C_{11} x_1 + f_1(x_3) + d_1$$

→ similarly for ϵ_{22} to get expression for u_2:

$$u_2 = \int \epsilon_{22}\, dx_2 = -\frac{b}{2} x_1^2 x_2 + \frac{a}{2} x_2^2 + C_{22} x_2 + f_2(x_3) + d_2$$

→ and for ϵ_{33} to get expression for u_3:

$$u_3 = \int \epsilon_{33}\, dx_3 = f_3(x_1, x_2) + d_3$$

→ Now use each of the ϵ_{13} and ϵ_{23} equations since these strains are equal to zero.

First take derivatives:

$$\frac{\partial u_1}{\partial x_3} = \frac{\partial f_1(x_3)}{\partial x_3}$$

$$\frac{\partial u_2}{\partial x_3} = \frac{\partial f_2(x_3)}{\partial x_3}$$

$$\frac{\partial u_3}{\partial x_1} = \frac{\partial f_3(x_1, x_2)}{\partial x_1}$$

$$\frac{\partial u_3}{\partial x_2} = \frac{\partial f_3(x_1, x_2)}{\partial x_2}$$

and use these to determine:

$$\text{via } \epsilon_{13} = 0 \Rightarrow \frac{\partial f_1(x_3)}{\partial x_3} + \frac{\partial f_3(x_1, x_2)}{\partial x_1} = 0$$

$$\text{via } \epsilon_{23} = 0 \Rightarrow \frac{\partial f_2(x_3)}{\partial x_3} + \frac{\partial f_3(x_1, x_2)}{\partial x_2} = 0$$

→ NOTE: Cannot use the equation for ϵ_{12} since no information is given concerning that shear strain.

This gives all the information that can be obtained. Thus, this is what can be said about the displacement field:

- The displacement in x_1 (u_1) has linear and quadratic relations to x_1 and x_3, has a functional relation to x_3 and a constant:

$$\boxed{u_1 = \frac{a}{2} x_1^2 + b_1 x_1 x_2^2 + C_{11} x_1 + f_1(x_3) + d_1}$$

- The displacement in x_2 (u_2) has similar relations as for u_1:

$$\boxed{u_2 = -\frac{b}{2} x_1^2 x_2 + \frac{a}{2} x_2^2 + C_{22} x_2 + f_2(x_3) + d_3}$$

- The displacement in x_3 (u_3) has a functional relation to (x_1, x_2) and a constant:

$$\boxed{u_3 = f_3(x_1, x_2) + d_3}$$

- The three functional forms are related via the following derivatives:

$$\boxed{\begin{aligned} \frac{\partial f_1(x_3)}{\partial x_3} &= -\frac{\partial f_3(x_1, x_2)}{\partial x_1} \\ \frac{\partial f_2(x_3)}{\partial x_3} &= -\frac{\partial f_3(x_1, x_2)}{\partial x_2} \end{aligned}}$$

(b) Use the expression:

$$\epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

→ Take derivatives of the expressions for the displacements:

$$\frac{\partial u_1}{\partial x_2} = 2b x_1 x_2$$

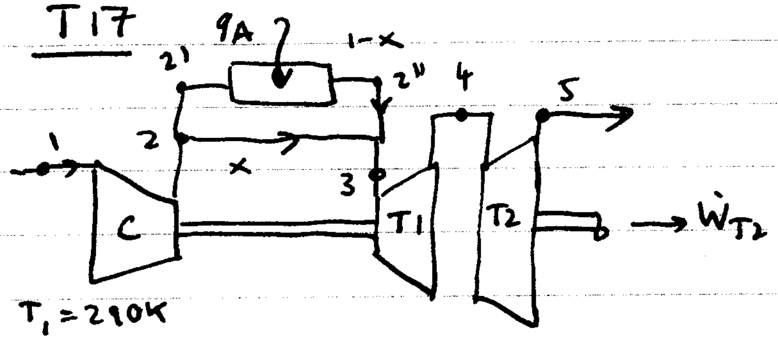
$$\frac{\partial u_2}{\partial x_1} = -b x_1 x_2$$

→ Use these in the ϵ_{12} equation:

$$\Rightarrow \epsilon_{12} = \frac{1}{2}(2b x_1 x_2 - b x_1 x_2)$$

giving: $\boxed{\epsilon_{12} = \frac{b}{2} x_1 x_2}$

T17 | 16. Unified Fall 2008 | 25



$T_1 = 290K$

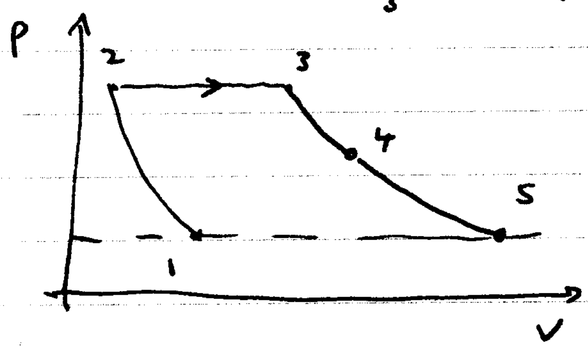
$p_1 = 1bar$ $\quad p_2 = 4.5bar$ $\quad p_5 = 1bar$

$T_3 = 1000K$

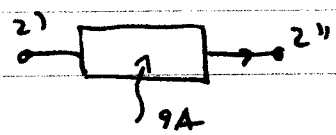
concepts: - 1st law
- cycles
- ad. rev. processes

$q_A = 1200\ kJ/kg$

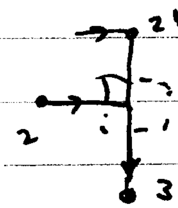
assume: - ideal gas with constant spec. heats
- ideal turbomachines
- neglect $\Delta KE, \Delta PE$



a) ad rev. process 1→2: $T_2 = T_1 \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$, $T_2 = 445.7\ K$

1st law combustor:  $c_p(T_2'' - T_2') = q_A$

and $T_2' = T_2$

so $T_2'' = \frac{q_A}{c_p} + T_2$; 1st law mixing 

$0 = \dot{H}_2 + \dot{H}_{2''} - \dot{H}_3$, $0 = x h_2 + (1-x) h_2'' - h_3$

$x = \frac{h_3 - h_2''}{h_2 - h_2''}$ $\quad \underline{x = 0.536}$ $\quad T_2'' = 1640.3K$

b) shaft power balance : $w_c = w_{T1}$ (note: same mass flow

$c_p(T_2 - T_1) = c_p(T_3 - T_4) \longrightarrow T_4 = T_3 - T_2 + T_1$, $\underline{T_4 = 844.3K}$

ad rev. exp: $p_4 = p_3 \left(\frac{T_4}{T_3}\right)^{\gamma/\gamma-1}$, $p_3 = p_2$ so $\underline{p_4 = 2.49 bar}$

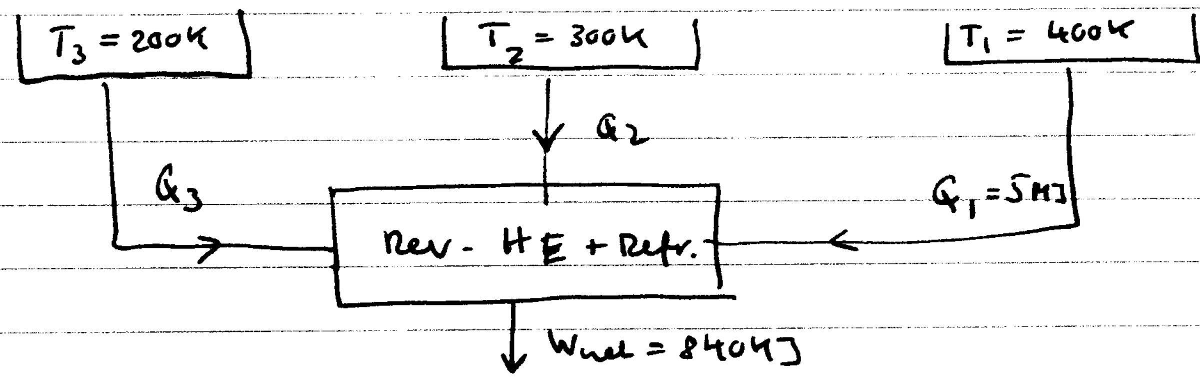
c) 1st law: $0 = h_4 - h_5 - w_{T2}$

ad. rev. exp: $T_5 = T_4 \left(\frac{p_5}{p_4}\right)^{\frac{\gamma-1}{\gamma}}$ $\quad w_{T2} = c_p T_4 \left(1 - \left(\frac{p_5}{p_4}\right)^{\frac{\gamma-1}{\gamma}}\right)$, $\underline{w_{T2} = 194.5\ kJ/kg}$

T18

16. Unified Fall 08

ZS



Concepts: - 1st law
- cycles
- 2nd law
- reversible process

Assume: all processes and cycles are reversible
$\rightarrow \Delta S_{total} = 0$

1st law : cycles $\quad 0 = Q_3 + Q_2 + Q_1 - W_{net} \quad$ (i)

2nd law : $\quad \Delta S_{total} = \Delta S_{cycles} + \Delta S_3 + \Delta S_2 + \Delta S_1 = 0$

$\Delta S_{cycles} = 0$ (entropy is state variable)

$\Delta S_i = \dfrac{Q_i}{T_i}$ for $i = 1, 2, 3$ (as heat reservoirs @ constant temp.)

so $\quad 0 = \dfrac{Q_3}{T_3} + \dfrac{Q_2}{T_2} + \dfrac{Q_1}{T_1} \quad$ (ii)

combine (i) and (ii): $Q_3 = W_{net} - Q_2 - Q_1$

$$Q_2 = \frac{-Q_1/T_1 + Q_1/T_3 - W_{net}/T_3}{1/T_2 - 1/T_3},$$

$\rightarrow \underline{Q_2 = -4.98 \text{ MJ}}$ ← [note: heat rejected from cycles since sign is negative but arrow chosen to point into cycles]

$\rightarrow \underline{Q_3 = 820 \text{ KJ}}$ ← heat absorbed

16.001 | October 20
Unified Engineering I | MIT, Fall 2008

Problem S1 (Signals and Systems)

1. Consider the system of equations

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 1 \\ x & - & 3y & + & 2z & = & -2 \\ -2x & + & 3y & + & z & = & 3. \end{array}$$

   Solve for x, y, and z, in three separate ways. The goal of part (1) is to practice solving systems of equations, so that when you get to part (2), you will have a fair basis of comparison.

   (a) Determine x, y, and z using (symbolic) elimination of variables.

   (b) Determine x, y, and z by Gaussian reduction.

   (c) Determine x, y, and z using Cramer's rule.

2. Consider the system of equations

$$\begin{array}{rcrcrcr} x & + & 6y & - & 6z & = & 2 \\ 3x & - & 2y & + & 3z & = & 3 \\ -4x & - & 2y & + & 3z & = & -4. \end{array}$$

   Again, solve for x, y, and z, in three separate ways. This time, please time each part (a), (b), (c) below.

   (a) Determine x, y, and z using (symbolic) elimination of variables.

   (b) Determine x, y, and z by Gaussian reduction.

   (c) Determine x, y, and z using Cramer's rule.

   (d) How much time did each method take?

   (e) Which method do you prefer?

   (f) When answering this question, think about how much time might be required for a larger system, say, one that is 5×5.

16.001 October 20
Unified Engineering I MIT, Fall 2008

Solution for Problem S1 (Signals and Systems)

1. As we will show below, parts (a)–(c) give an identical solution: $x = -\frac{1}{4}$, $y = \frac{3}{4}$, $z = \frac{1}{4}$.

   (a) We start with the original system of equations

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 1 \\ x & - & 3y & + & 2z & = & -2 \\ -2x & + & 3y & + & z & = & 3 \end{array} \tag{1}$$

   First, we eliminate variable x from the second and third rows by setting Row 2 $\leftarrow$ Row 2 $-$ Row 3 and setting Row 3 $\leftarrow$ Row 3 $+$ $2 \times$ Row 3. These operations result in a new system of equations below

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 1 \\ & - & 5y & + & 3z & = & -3 \\ & & 7y & - & z & = & 5 \end{array} \tag{2}$$

   Next, we eliminate variable y from the last rows of (2) by setting Row 3 $\leftarrow$ Row 3 $+ \frac{7}{5} \times$ Row 2. This operation results in a new system of equations below:

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 1 \\ & - & 5y & + & 3z & = & -3 \\ & & & & \frac{16}{5}z & = & \frac{4}{5} \end{array} \tag{3}$$

   The solution for equations in (3) can be obtained as follows. The third row implies that $z = \frac{1}{4}$. Substituting $z = \frac{1}{4}$ into the second row gives

$$y = \frac{-3 - 3z}{-5} = \frac{3}{4}.$$

Substituting $z = \frac{1}{4}$ and $y = \frac{3}{4}$ into the first row gives

$$x = 1 - 2y + z = -\frac{1}{4}.$$

The solution of the system of equations in (3) is identical to the solution of the original system of equations in (1), since we arrive at (3) from (1) by a series of row operations. Therefore, the solution of (1) is also $(x, y, z) = (-\frac{1}{4}, \frac{3}{4}, \frac{1}{4})$.

(b) Gaussian reduction is a representation of (1)–(3) in matrix forms:

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 1 & -3 & 2 & -2 \\ -2 & 3 & 1 & 3 \end{array}\right]$$

$\Longrightarrow$

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & -3 \\ 0 & 7 & -1 & 5 \end{array}\right]$$

$\Longrightarrow$

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -5 & 3 & -3 \\ 0 & 0 & \frac{16}{5} & -\frac{4}{5} \end{array}\right]. \tag{4}$$

If we stop now, we can use a back substitution as we did in (a) to arrive at the solution $(x, y, z) = (-\frac{1}{4}, \frac{3}{4}, \frac{1}{4})$.

Alternatively, we can continue the reduction until the 3×3 sub-matrix becomes the identity matrix.

We turn the diagonal elements of the 3×3 sub-matrix into 1's by setting Row 2 $\leftarrow -\frac{1}{5}$Row 2 and setting Row 3 $\leftarrow -\frac{5}{16}$Row 3. These operations result in the matrix representation of linear equations below

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & -\frac{3}{5} & -\frac{3}{5} \\ 0 & 0 & 1 & \frac{1}{4} \end{array}\right].$$

Next, we eliminate variable z from the first and second rows by setting Row 1 $\leftarrow$ Row 1 + Row 3 and setting Row 2 $\leftarrow$ Row 2 + $\frac{3}{5}$Row 3. These operations result in the matrix representation

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & \frac{5}{4} \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{4} \end{array}\right].$$

Finally, we eliminate variable y from the first row by setting Row 1 $\leftarrow$ Row 1 − 2Row 2. This operation results in the matrix representation

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{4} \end{array}\right].$$

The left-most column is the solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \\ \frac{1}{4} \end{bmatrix},$$

which is consistent with the solution obtained from back substitution.

(c) Let $\mathbf{A}$ denote the coefficient matrix in the given system of equations:

$$\mathbf{A} \triangleq \begin{bmatrix} 1 & 2 & -1 \\ 1 & -3 & 2 \\ -2 & 3 & 1 \end{bmatrix}.$$

Cramer's rule implies that the solution is given by

$$x = \frac{\det \begin{bmatrix} 1 & 2 & -1 \\ -2 & -3 & 2 \\ 3 & 3 & 1 \end{bmatrix}}{\det \mathbf{A}} = \frac{4}{-16} = -\frac{1}{4}$$

$$y = \frac{\det \begin{bmatrix} 1 & 1 & -1 \\ 1 & -2 & 2 \\ -2 & 3 & 1 \end{bmatrix}}{\det \mathbf{A}} = \frac{-12}{-16} = \frac{3}{4}$$

$$z = \frac{\det \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & -2 \\ -2 & 3 & 3 \end{bmatrix}}{\det \mathbf{A}} = \frac{-4}{-16} = \frac{1}{4},$$

where

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + cdh + bfg - ahf - dbi - gef.$$

The determinant of $\mathbf{A}$ can be obtained from the above identity or from the product of the diagonal elements of the sub-matrix in (5), which results from the row operations:

$$\det \mathbf{A} = 1 \times (-5) \times \frac{16}{5} = -16.$$

2. The solution is $x = 1$, $y = \frac{1}{2}$, and $z = \frac{1}{3}$. The approach to obtain the solution is similar to that of part (1). Details of the approach will be omitted for brevity.

   (a) The solution is obtained from the following chain of reductions:

$$\begin{array}{rcrcrcr} x & + & 6y & - & 6z & = & 2 \\ 3x & - & 2y & + & 3z & = & 3 \\ -4x & - & 2y & + & 3z & = & -4 \end{array}$$

$\Longrightarrow$ (Row 2 $\leftarrow$ Row $2 - 3 \times$ Row 1; Row 4 $\leftarrow$ Row $4 + 4 \times$ Row 1)

$$\begin{array}{rcrcrcr} x & + & 6y & - & 6z & = & 2 \\ & - & 20y & + & 21z & = & -3 \\ & & 22y & - & 21z & = & 4 \end{array}$$

$\Longrightarrow$ (Row 3 $\leftarrow$ Row 3 $-\frac{22}{20}\times$ Row 2)

$$\begin{array}{rcrcrcr} x & + & 6y & - & 6z & = & 2 \\ & - & 20y & + & 21z & = & -3 \\ & & & & \frac{21}{10}z & = & \frac{14}{20}. \end{array}$$

The last row implies that

$$z = \frac{14}{20} \times \frac{10}{21} = \frac{1}{3}.$$

Substituting $z = \frac{1}{3}$ into the second row gives

$$y = \frac{-3 - 21z}{-20} = \frac{1}{2}.$$

Substituting $y = \frac{1}{2}$ and $z = \frac{1}{3}$ into the first row gives

$$x = 2 - 6y + 6z = 1.$$

Therefore the solution is $(x, y, z) = (1, \frac{1}{2}, \frac{1}{3})$.

(b) Gaussian reduction is given below:

$$\left[\begin{array}{rrr|r} 1 & 6 & -6 & 2 \\ 3 & -2 & 3 & 3 \\ -4 & 2 & 3 & -4 \end{array}\right]$$

$\Longrightarrow$

$$\left[\begin{array}{rrr|r} 1 & 6 & -6 & 2 \\ 0 & -20 & 21 & -3 \\ 0 & 22 & -21 & 4 \end{array}\right]$$

$\Longrightarrow$

$$\left[\begin{array}{rrr|r} 1 & 6 & -6 & 2 \\ 0 & -20 & 21 & -3 \\ 0 & 0 & \frac{21}{10} & \frac{14}{20} \end{array}\right]. \tag{5}$$

If we stop now, we can use a back substitution as we did in (a) to arrive at the solution $(x, y, z) = (1, \frac{1}{2}, \frac{1}{3})$.

Alternatively, we can continue the reduction until the 3×3 submatrix becomes the identity matrix:

Make the diagonal elements equal 1

$$\left[\begin{array}{ccc|c} 1 & 6 & -6 & 2 \\ 0 & 1 & \frac{21}{20} & \frac{3}{20} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$$

$\Longrightarrow$ (Eliminate z from rows one and two)

$$\left[\begin{array}{ccc|c} 1 & 6 & 0 & 4 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$$

$\Longrightarrow$ (Eliminate z from rows one and two)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right]$$

The solution is given by the last column of the matrix: $(x, y, z) = (1, \frac{1}{2}, \frac{1}{3})$.

(c) Let $\mathbf{A}$ denote the coefficient matrix in the given system of equations:

$$\mathbf{A} \triangleq \begin{bmatrix} 1 & -6 & -6 \\ 3 & -2 & 3 \\ -4 & -2 & 3 \end{bmatrix}.$$

Cramer's rule implies that the solution is given by

$$x = \frac{\det \begin{bmatrix} 2 & 6 & -6 \\ 3 & -2 & 3 \\ -4 & -2 & 3 \end{bmatrix}}{\det \mathbf{A}} = \frac{-42}{-42} = 1$$

$$y = \frac{\det \begin{bmatrix} 1 & 2 & -6 \\ 3 & 3 & 3 \\ -4 & -4 & 3 \end{bmatrix}}{\det \mathbf{A}} = \frac{-21}{-42} = \frac{1}{2}$$

$$z = \frac{\det \begin{bmatrix} 1 & 6 & 2 \\ 3 & -2 & 3 \\ -4 & -2 & -4 \end{bmatrix}}{\det \mathbf{A}} = \frac{-14}{-42} = \frac{1}{3}.$$

(d) Let n denotes the number of variables and the number of linear equations. Let T denote the amount of computational time required to obtain the solution using the elimination of variables. Then, Gaussian reduction requires approximately T time unit as well, since Gaussian reduction and elimination of variables have roughly the same computational complexity.

The Cramer's rule requires the evaluation of $n + 1$ determinants. To obtain a determinant, we can perform Gaussian reduction until we arrive at a triangular matrix. Therefore, an amount of time to obtain the solution using Cramer's rule is approximately $(n+1)T$.

(e) From the reasoning in part (d), elimination of variables and Gaussian reduction are approximately $n+1$ times faster than Cramer's rule. From a standpoint of computational time, elimination of variables or Gaussian reduction is a preferred method.