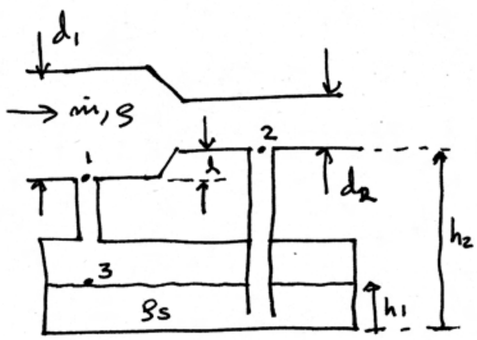
Solutions Quiz #3 (fluids) $\boxed{\text{problem 1}}$



Incipient injection when the liquid column climbs to point 2.

$$P_3 - P_2 = \rho_s g (h_2 - h_1)$$

and also:

$$P_3 - P_1 = \rho g (h_2 - l - h_1)$$

Therefore:

$$(*) \quad P_1 - P_2 = \rho_s g \Delta h - \rho g (\Delta h - l) \qquad \text{with } \Delta h = h_2 - h_1$$

This same pressure difference needs to be supplied by the flow, then through Bernoulli:

$$P_1 + \tfrac{1}{2}\rho V_1^2 + \rho g y_1 = P_2 + \tfrac{1}{2}\rho V_2^2 + \rho g y_2 \qquad \text{select } \begin{matrix} y_1 = 0 \\ y_2 \end{matrix}$$

from here:

$$P_1 - P_2 = \tfrac{1}{2}\rho V_1^2 \left[\left(\frac{V_2}{V_1}\right)^2 - 1\right] + \rho g (y_2 - y_1)$$

and since: $V_1 A_1 = V_2 A_2$ and $y_2 - y_1 = l$

$$P_1 - P_2 = \tfrac{1}{2}\rho V_1^2 \left[\left(\frac{A_1}{A_2}\right)^2 - 1\right] + \rho g l$$

then, from $(*)$:

$$\rho_s g \Delta h - \rho g \Delta h + \cancel{\rho g l} = \tfrac{1}{2}\rho V_1^2 \left[\left(\frac{A_1}{A_2}\right)^2 - 1\right] + \cancel{\rho g l}$$

solve for V_1:

$$V_1 = \sqrt{\frac{2(\rho_s - \rho)\Delta h g}{\rho\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]}} \qquad \text{and since } \dot{m} = \rho V_1 A_1$$

then:

$$\dot{m}_{MIN} = A_1 \sqrt{\frac{2\rho(\rho_s - \rho)\Delta h g}{\left(\frac{A_1}{A_2}\right)^2 - 1}} \quad \text{or} \quad \boxed{\dot{m}_{MIN} = \frac{\pi d_1^2}{4}\sqrt{\frac{2\rho(\rho_s - \rho)\Delta h g}{\left(\frac{d_1}{d_2}\right)^4 - 1}}}$$

If ρ is a gas, then $\rho_s \gg \rho$ and:

$$\dot{m}_{MIN} = \frac{\pi d_1^2}{4}\sqrt{\frac{2\rho\rho_s \Delta h g}{\left(\frac{d_1}{d_2}\right)^4 - 1}}$$

Solutions Quiz #3 (fluids) $\boxed{\text{problem 2}}$

$$\phi = -\frac{1}{\tau}(x^2 - y^2) \qquad \rho = \rho_0\left(1 + \frac{xy}{L^2}\right)$$

a) $\bar{V} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} = \boxed{-\frac{2x}{\tau}\hat{\imath} + \frac{2y}{\tau}\hat{\jmath}}$

b) Yes. from $\rho u = \frac{\partial\Psi}{\partial y}$ and $\rho v = -\frac{\partial\Psi}{\partial x}$ (compressible flow)

$$\Psi = \int \rho u\, dy = \int \rho_0\left(1 + \frac{xy}{L^2}\right)\left(-\frac{2x}{\tau}\right)dy$$
$$= \rho_0\left(y + \frac{xy^2}{2L^2}\right)\left(-\frac{2x}{\tau}\right) + f(x) + C$$
$$= -\frac{\rho_0}{\tau}\left(2xy + \frac{x^2y^2}{L^2}\right) + f(x) + C$$

$$\Psi = \int -\rho v\, dx = -\int \rho_0\left(1 + \frac{xy}{L^2}\right)\left(\frac{2y}{\tau}\right)dx$$
$$= -\rho_0\left(x + \frac{x^2y}{2L^2}\right)\left(\frac{2y}{\tau}\right) + f(y) + C$$
$$= -\frac{\rho_0}{\tau}\left(2xy + \frac{x^2y^2}{L^2}\right) + f(y) + C$$

since $\Psi = \Psi$

then: $f(x) = f(y) = 0$ or const.

finally $\boxed{\psi = -\frac{\rho_0 xy}{\tau}\left(2 + \frac{xy}{L^2}\right) + \text{const}}$

c) It should since Ψ obeys $\nabla\cdot\rho\bar{V} = 0$. (compressible flow!)

Verify: $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = \frac{\partial}{\partial x}\left(\rho_0\left(1 + \frac{xy}{L^2}\right)\left(-\frac{2x}{\tau}\right)\right) + \frac{\partial}{\partial y}\left(\rho_0\left(1 + \frac{xy}{L^2}\right)\left(+\frac{2y}{\tau}\right)\right)$

$$= \rho_0\left[-\frac{2}{\tau} - \frac{2xy}{L^2\tau} + \frac{2}{\tau} + \frac{2xy}{L^2\tau}\right] = 0$$

d) Should be irrotational since it comes from a potential $\vec{V} = \nabla\phi$ and $\nabla\times\nabla\phi = 0$

But verify: $\xi_z = \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

e) Cannot use Bernoulli $\Rightarrow$ compressible flow. Use $\rho\frac{D\bar{V}}{Dt} = -\nabla p$

and since $\frac{\partial}{\partial t} = 0 \Rightarrow \rho(\bar{V}\cdot\nabla)\bar{V} = -\nabla p$ $\qquad \frac{\partial p}{\partial x} = -\rho\bar{V}\cdot\nabla u \qquad \frac{\partial p}{\partial y} = -\rho\bar{V}\cdot\nabla v$

$$\left.\begin{aligned}\frac{\partial p}{\partial x} &= -\rho_0\left(1 + \frac{xy}{L^2}\right)\left(-\frac{2x}{\tau}\right)\left(-\frac{2}{\tau}\right) = -\rho_0\left(1 + \frac{xy}{L^2}\right)\frac{4x}{\tau^2}\\ \frac{\partial p}{\partial y} &= -\rho_0\left(1 + \frac{xy}{L^2}\right)\left(\frac{2y}{\tau}\right)\left(\frac{2}{\tau}\right) = -\rho_0\left(1 + \frac{xy}{L^2}\right)\frac{4y}{\tau^2}\end{aligned}\right\} \quad \nabla p = \frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath}$$