Analysis of RC circuits

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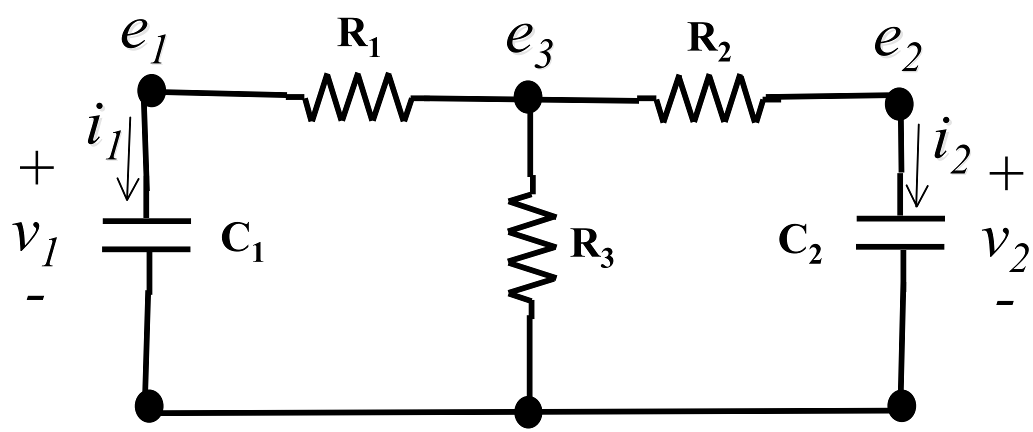
Learning Objectives

- **Analysis of basic circuit with capacitors, no inputs**
  - **Derive the differential equations for the voltage across the capacitors**
- **Solve a system of first order homogeneous differential equations using classical method**
  - **Identify the exponential solution**
  - **Obtain the characteristic equation of the system**
  - **Obtain the natural response of the circuit**
  - **Solve for the complete solution using initial conditions**

Second order RC circuits



$R_1 = R_2 = R_3 = 1\Omega$

$C_1 = C_2 = 1F$

$$i_1 = C_1 \frac{dv_1}{dt} \qquad i_2 = C_2 \frac{dv_2}{dt}$$

Node equations :

$$e_1: \quad i_1 + (e_1 - e_3)/R_1 = 0$$

$$e_2: \quad (e_2 - e_3)/R_2 + i_2 = 0$$

$$e_3: \quad (e_3 - e_1)/R_1 + (e_3 - e_2)/R_2 + e_3/R_3 = 0$$

The differential equations

$$Note: \quad v_1 = e_1 \;; v_2 = e_2$$

Differential equations:

$$C_1 \frac{de_1}{dt} + (e_1 - e_3)/R_1 = 0$$

$$C_2 \frac{de_2}{dt} + (e_2 - e_3)/R_2 = 0$$

$$(e_3 - e_1)/R_1 + (e_3 - e_2)/R_2 + e_3/R_3 = 0$$

Plugging in values for resistors capacitors:

$$(1)\ \dot{e}_1 + e_1 - e_3 = 0$$

$$(2)\ \dot{e}_2 + e_2 - e_3 = 0$$

$$(3)\ 3e_3 - e_2 - e_1 = 0$$

Guessing a solution

We "guess" that the solution is an exponential of the form:

$$Guess\ \ e_i = E_i e^{st} \Rightarrow \dot{e}_i = E_i s e^{st}$$

$$\left.\begin{aligned} E_1 s e^{st} + E_1 e^{st} - E_3 e^{st} &= 0 \\ E_2 s e^{st} + E_2 e^{st} - E_3 e^{st} &= 0 \\ 3E_3 e^{st} - E_2 e^{st} - E_1 e^{st} &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} E_1 s + E_1 - E_3 &= 0 \\ E_2 s + E_2 - E_3 &= 0 \\ 3E_3 - E_2 - E_1 &= 0 \end{aligned}$$

$$\begin{array}{cccc} E_1(1+s) & & -E_3 & = 0 \\ & E_2(1+s) & -E_3 & = 0 \\ -E_1 & -E_2 & +3E_3 & = 0 \end{array}$$

Solution to homogeneous equations

- **If equations are linearly independent we have one unique solution:**
  - **$E_1=E_2=E_3=0$**
  - **“trivial” solution**
- **In order to obtain non-trivial solution, the equations cannot be linearly independent**
  - **For what values of s are the equations not linearly independent?**
- **Rewrite equations in the form: AE = 0**

$$\begin{bmatrix} (1+s) & 0 & -1 \\ 0 & 1+s & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = 0$$

- **Rows of A are linearly dependent if determinant of A = 0**
  - **Non-trivial solution**

The characteristic equation

$\det A = 0$ (The characteristic equation)

see linear algebra primer on taking determinants

$$|\mathrm{A}| = 3(1+s)(1+s) - (1+s) - (1+s) = 0$$

$$3s^2 + 4s + 1 = 0 \Rightarrow s = \frac{-4 \pm \sqrt{16-12}}{6} = \frac{-4 \pm 2}{6}$$

$$\Rightarrow s_1 = -1,\ s_2 = -1/3$$

$Two\ \ solutions:$

$$s = -1 \Rightarrow e_i = E_i e^{-t}$$

$$s = -1/3 \Rightarrow e_i = E_i e^{-t/3}$$

Non-trivial solution

- Now we must solve for the values of E_i's
- Recall
$$\begin{aligned} E_1(1+s) \quad\quad\quad\quad\quad &- E_3 = 0 \\ E_2(1+s) \; &- E_3 = 0 \\ -E_1 \quad\quad -E_2 \quad\quad\quad &+3E_3 = 0 \end{aligned}$$
- For s = -1 we have,

$$\begin{aligned} E_1(0) \quad\quad\quad\quad\quad &- E_3 = 0 \Rightarrow E_3 = 0 \\ E_2(0) \; &- E_3 = 0 \Rightarrow E_3 = 0 \\ -E_1 \quad\quad -E_2 \quad\quad\quad &+3E_3 = 0 \Rightarrow E_1 = -E_2 \end{aligned}$$

$$Solution\ \ not\ \ unique,\ choose:\ \ E_2 = 1,\ E_1 = -1$$

$$\Rightarrow e_1 = -1e^{-t},\ \ e_2 = 1e^{-t},\ e_3 = 0$$

Non-trivial solution, continued

- **Similarly, for s = -1/3**

$$
\begin{aligned}
&E_1(2/3) \qquad\qquad\qquad\qquad - E_3 = 0 \Rightarrow E_1 = 3/2E_3 \\
&\qquad\qquad\qquad E_2(2/3) - E_3 = 0 \Rightarrow E_2 = 3/2E_3 \\
&-E_1 \qquad -E_2 \qquad\qquad +3E_3 = 0
\end{aligned}
$$

$$
\begin{aligned}
&Solution\ \ not\ \ unique,\ choose:\ \ E_3 = 1,\ E_1 = E_2 = 3/2 \\
&\Rightarrow e_1 = 3/2e^{-t/3},\ \ e_2 = 3/2e^{-t/3},\ e_3 = e^{-t/3}
\end{aligned}
$$

- **Note that in general we can solve for the E's using row reduction, etc. (see linear algebra notes)**

The complete solution

- **We now have two possible solutions:**

$$(1): \quad e_1 = 3/2e^{-t/3}, \;\; e_2 = 3/2e^{-t/3}, \; e_3 = e^{-t/3}$$

$$(2): \;\; e_1 = -1e^{-t}, \;\; e_2 = 1e^{-t}, \; e_3 = 0$$

- **Since the system is linear and homogeneous, any linear combination of these solution is also a solution. Hence,**

$$\vec{e}(t) = a\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + b\begin{bmatrix} 3/2 \\ 3/2 \\ 1 \end{bmatrix} e^{-t/3}$$

- **Where a, and b are constants that depend on initial conditions**
  - **Given initial values for $e_1 = v_1$ and $e_2 = v_2$ we can solve for a and b**

The complete solution, continued

- Suppose we are given the initial voltage values across the capacitors: $e_1(0) = v_1(0)$, $e_2(0) = v_2(0)$

- Plug these into the solution to obtain:

$$\left.\begin{aligned} e_1(0) &= -ae^0 + 3/2be^0 = -a + 3/2b \\ e_2(0) &= ae^0 + 3/2be^0 = \;a + 3/2b \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow a = \frac{e_2(0) - e_1(0)}{2}, \quad b = \frac{e_1(0) + e_2(0)}{3}$$

$Finally,$

$$e_1(t) = \frac{e_1(0) - e_2(0)}{2} e^{-t} + \frac{e_1(0) + e_2(0)}{2} e^{-t/3}$$

$$e_2(t) = \frac{e_2(0) - e_1(0)}{2} e^{-t} + \frac{e_1(0) + e_2(0)}{2} e^{-t/3}$$

$$e_3(t) = \frac{e_1(0) + e_2(0)}{3} e^{-t/3}$$