State Variable Description of LTI systems

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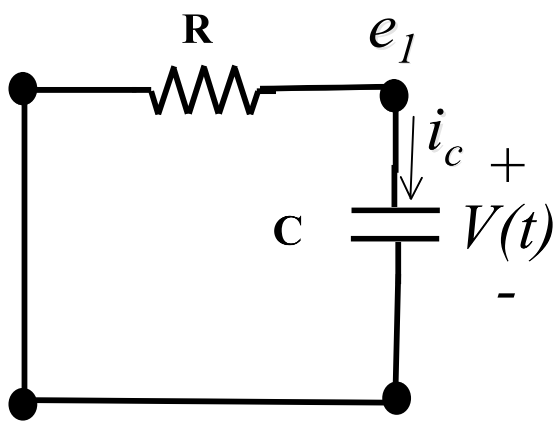
Learning Objectives

- Understand concept of a state
- Develop state-space model for simple LTI systems
  - RLC circuits
  - Simple 1st or 2nd order mechanical systems
  - Input output relationship
- Develop block diagram representation of LTI systems
- Understand the concept of state transformation
  - Given a state transformation matrix, develop model for the transformed system

The State of a System

- **The "*state*" of a system is the minimum information needed about the system in order to determine its future behavior**
  - **Given the state at time t_0, and input up to time $t > t_0$; can determine the output for time t.**
- ***State Variables***
  - **Set of variables of smallest possible size that together with any input to the system is sufficient to determine the future behavior (I.e., output) of the system.**
  - **Each state variable has "*memory*"**
    E.g., voltage in capacitor
  - **Each state variable has an "*initial condition*"**
    E.g., its state at time t_0
  - **State variables are typically associated with energy storage**
- ***State vector*: vector of state variables**

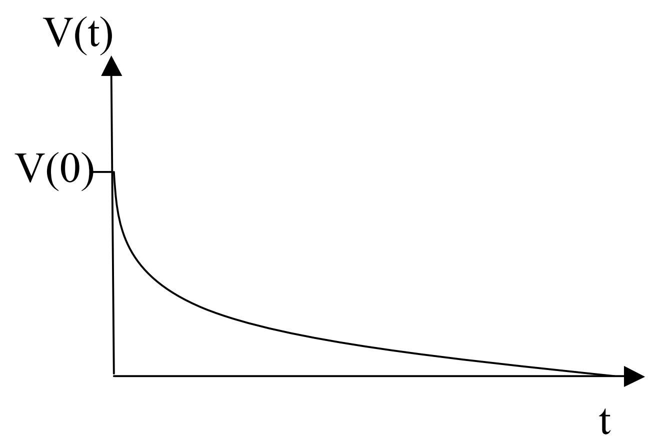
Example: RLC circuit



$$\left.\begin{aligned} i_c &= C\frac{dv}{dt} \\ i_c &= -V/R \end{aligned}\right\} \Rightarrow \frac{dv}{dt} = -\frac{V(t)}{RC}$$

$$V(t) = V(0)e^{-t/RC}$$

- **V(t) is the state of the system at time t**
  - **Initial condition - V(0) is the voltage across the capacitor at time 0**
- **If we know v(t) at any time t, we know it for all future time**
  - **No input in this case**



State Variables

- **In electric circuits, the energy storage devices are the capacitors and inductors**
  - **They contain all of the state information or “memory” in the system**
  - **State variables:**
    - **Voltage across capacitors**
    - **Current through inductors**

- **In mechanical systems, energy is stored in *springs* and *masses***
  - **State variables**
    - **Spring displacement**
    - **Mass position and velocity**

- **Example” Single mass M, moving in one dimension (x), under force F**
  - **State variables (x_1, x_2)**
    - **x_1 = mass position**
    - **x_2 = velocity**

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \ddot{x} = F / M \end{aligned}$$

$$F = M\ddot{x}$$

State Equations

- **State equations in matrix form:**

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad input: u = F/M$$

$$State\ \ equations: \dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F/M$$

- **Output equations:**
  - **suppose output is $y=x_1$ (position)**

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- **General form of state equations:**

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

- **We will focus (to start) mainly on homogeneous case:** $\dot{x} = Ax$

General form of state equations

- In general a system can have
  - n states, state vector X(t), m inputs, u(t), and l outputs, y(t)

$$\vec{X}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \vec{U}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}, \quad \vec{Y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

- The state and output equations can be written as:

$$\dot{\vec{X}} = A\vec{X} + B\vec{U}$$

$$\vec{Y} = C\vec{X} + D\vec{U}$$

- A,B,C,D are constant matrices
  - A is an nxn "system dynamics" matrix
  - B is an nxm "input matrix"
  - C is an lxn matrix relating states to outputs
  - D is an lxm matrix relating inputs to outputs

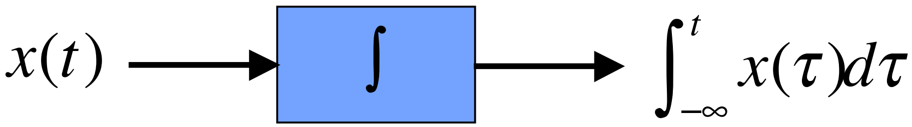
Why the state-space approach?

- **Very general approach to describe Linear time-invariant (LTI) systems**
  - **Rich theory describing the solutions**
  - **Simplifies analysis of complex systems with multiple inputs and outputs**
  - **Approach is central to "modern" control**

- **History of state-space approach**
  - **State-space approach to control system design was introduced in the 1950's**
    - **Up to that point "classical" control used root-locus or frequency response methods (more in 16.060)**
  - **"New" approach was named "modern" control and still have that name**
  - **Related state space approach to describing differential equations is over 100 years old**

Block Diagram Representation

Integrator block diagram

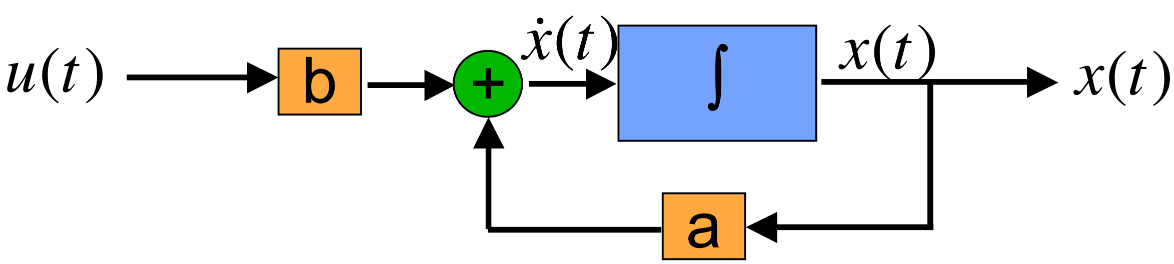


$$\frac{dx(t)}{dt} = ax(t) + bu(t)$$

$$x(t) = \int_{-\infty}^{t} ax(\tau) + bu(\tau)d\tau$$

- **x(t) - state variables**
- **u(t) - inputs**

System block diagram



$x(t)$ – integrator output

$\dot{x}(t)$ – integrator input

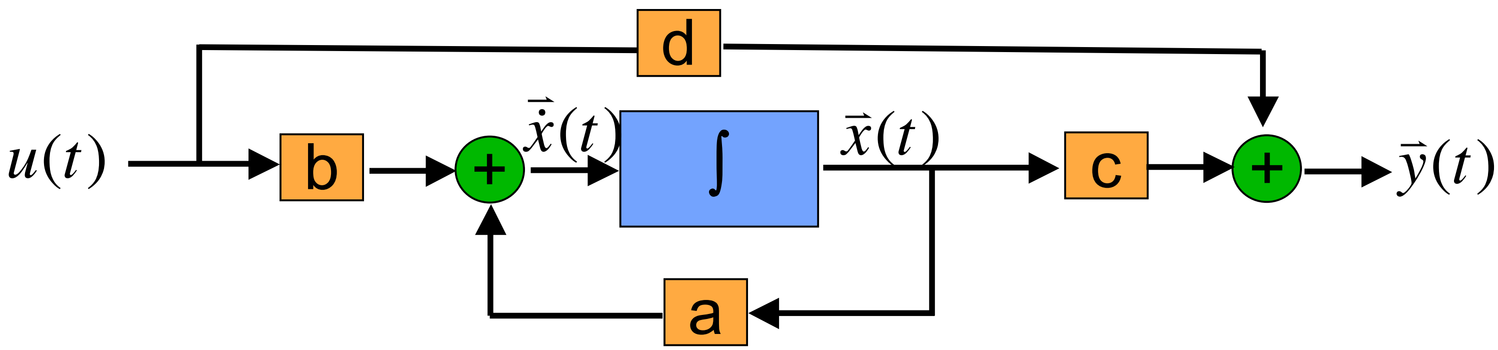
$$\dot{x}(t) = ax(t) + bu(t)$$

State is "fed-back" into the system

General system block diagram

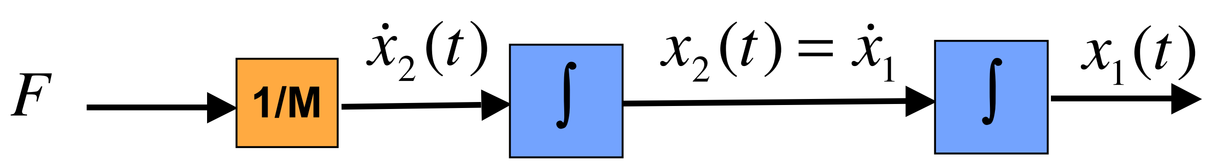
$$\dot{\vec{X}} = A\vec{X} + B\vec{U}$$

$$\vec{Y} = C\vec{X} + D\vec{U}$$



Force-mass example:

$$\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \Rightarrow \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{x} = \dot{x}_1 = F/M \end{array}$$



State Transformation

- The state variable description of a system is not unique
- Different state variable descriptions are obtained by "state transformation"
  - New state variables are weighted sum of original state variables
  - Changes the form of the system equations, but not the behavior of the system
- Some examples: original system ~ $x_1(t)$, $x_2(t)$
  - Transformed systems ~ $z_1(t)$, $z_2(t)$

  (1) $z_1(t) = x_2(t)$, $z_2(t)=x_1(t)$
  (2) $z_1(t) = x_1(t)+x_2(t)$, $z_2(t)=x_2(t)$
  (3) $z_1(t) = 2x_1(t) - x_2(t)$, $z_2(t) = x_1(t)+2x_2(t)$

- State transformation can simplify system description and analysis

State Transformation, continued

- **In general, we can transform $\vec{x}$ to a new state vector $\vec{\tilde{x}}$ by,**

$$\vec{\tilde{x}} = T\vec{x}$$

- **Where T is the state transformation matrix**

- **Relationship between $\vec{x}$ and $\vec{\tilde{x}}$ must be one-to-one (i.e., the mapping must be invertible)**
  - $\Rightarrow$ **T must be non-singular**
  - $\Rightarrow$ **T^{-1} must exist**

$$original\ \ system: \quad \dot{x} = Ax + Bu$$

$$Transformed\ \ system: \tilde{x} = Tx \Rightarrow \dot{\tilde{x}} = T\dot{x} = TAx + TBu$$

$$\Rightarrow \dot{\tilde{x}} = TAT^{-1}x + TBu \quad (recall: \ \ x = T^{-1}\tilde{x})$$

$$\Rightarrow \tilde{A} = TAT^{-1}, \quad \tilde{B} = TB$$

$$\Rightarrow \dot{\tilde{x}} = \tilde{A}x + \tilde{B}u$$

State Transformation (example)

- **Try to write down the state equations explicitly**
- **Notice that the state transformation is a linear combination of the original system states**
- **Notice that the new transformed system has a much simpler (to understand) structure**
  - **Two Decoupled first order differential equations**
  - **The system is still the same - just the description is simpler**

Original system: $\dot{x} = Ax + Bu$

$$A = \begin{bmatrix} -1 & 4 \\ 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

State transformation:

$$\tilde{x}_1 = \frac{-x_1 + x_2}{2}, \; \tilde{x}_2 = \frac{x_1 + x_2}{2}$$

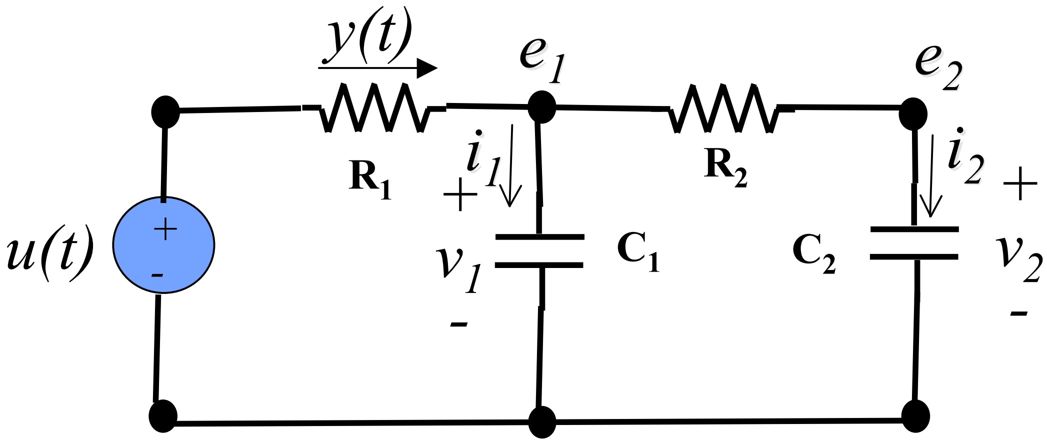
$$T = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \; T^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} -5 & 0 \\ 0 & 3 \end{bmatrix}, \; \tilde{B} = TB = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\dot{\tilde{x}}_1 = -5\tilde{x}_1 + u$$

$$\dot{\tilde{x}}_2 = 3\tilde{x}_2 + 3u$$

State variable description of RLC circuits



$$i_1(t) = C_1 \frac{d}{dt} v_1(t)$$

$$i_2(t) = C_2 \frac{d}{dt} v_2(t)$$

- Input: voltage source ~ u(t)
- System state: capacitor voltages ~ $v_1(t)$, $v_2(t)$
- Output: current through resistor R_1 ~ y(t)
- Node equations:

$$e_1: \quad \frac{v_1 - u(t)}{R_1} + i_1 + \frac{v_1 - v_2}{R_2} = 0 \Rightarrow \frac{dv_1}{dt} = \frac{v_2 - v_1}{C_1 R_2} - \frac{u(t) - v_1}{C_1 R_1}$$

$$e_2: \quad \frac{v_2 - v_1}{R_2} + i_1 = 0 \Rightarrow \frac{dv_2}{dt} = -\frac{v_2 - v_1}{C_2 R_2} = \frac{v_1 - v_2}{C_2 R_2}$$

Example, continued

$$\dot{v}_1 = -v_1\left(\frac{1}{C_1R_1} + \frac{1}{C_1R_2}\right) + v_2\frac{1}{C_1R_2} + u(t)\frac{1}{C_1R_1}$$

$$\dot{v}_2 = -v_2\frac{1}{C_2R_2} + v_1\frac{1}{C_2R_2}$$

$$\dot{V} = \begin{bmatrix} -\frac{1}{C_1R_1} - \frac{1}{C_1R_2} & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & -\frac{1}{C_2R_2} \end{bmatrix} V + \begin{bmatrix} \frac{1}{C_1R_1} \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \frac{u(t) - v_1}{R_1} \Rightarrow Y = \begin{bmatrix} \frac{-1}{R_1} & 0 \end{bmatrix} V + \begin{bmatrix} \frac{1}{R_1} \end{bmatrix} u(t)$$