General Response of Second Order System

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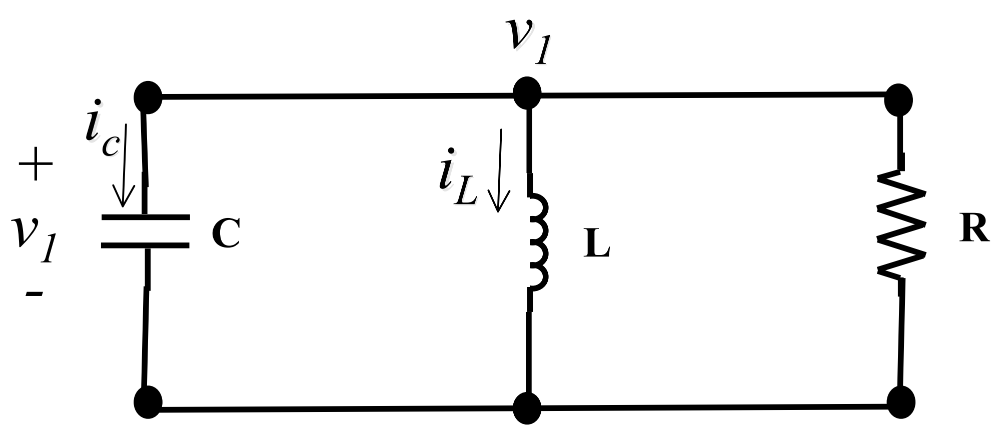
Learning Objectives

- **Learn to analyze a general second order system and to obtain the general solution**
  - **Identify the over-damped, under-damped, and critically damped solutions**
  - **Convert complex solution to real solution**
  - **Suspended "mass-spring-damper" equivalent system**

Second order RC circuit

- **System with 2 state variables**
  - **Described by two coupled first-order differential equations**



- **States**
  - **Voltage across the capacitor - V_1**
  - **Current through the inductor - i_L**

- **What to obtain state equations of the form: x' = Ax**
  - **Need to obtain expression for dv_1/dt in terms of V_1 and i_L**
  - **Need to obtain expression for di_l/dt in terms of V_1 and i_L**

State Differential Equations

- **For the capacitor and inductor we have,**

$$\frac{dv_1}{dt} = \frac{i_c}{C}, \quad \frac{di_L}{dt} = \frac{v_L}{L} = \frac{v_1}{L}$$

- **Above immediately gives us an expression for di_l/dt in terms of V_1**
- **Need to obtain expression for dv_1/dt in terms of V_1 and i_L**

- **Write node equation at v_1 to obtain:**

$$i_c + i_L + v_1 / R = 0$$

- **Hence the state equations are given by:**

$$(1) \quad \frac{dv_1}{dt} = \frac{-i_L}{C} - \frac{v_1}{RC}$$

$$(2) \quad \frac{di_L}{dt} = \frac{v_1}{L}$$

Solving the state differential equations

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ i_L \end{bmatrix} = \underbrace{\begin{bmatrix} -1/RC & -1/C \\ 1/L & 0 \end{bmatrix}}_{A}\begin{bmatrix} v_1 \\ i_L \end{bmatrix}$$

$$[sI - A] = \begin{bmatrix} s + 1/RC & 1/C \\ -1/L & s \end{bmatrix}$$

$$\phi(s) = characteristic\ eqn = \det(sI - A) = 0$$

$$\det(sI - A) = s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

$$solve\ quadratic\ equation\ to\ obtain:$$

$$s_1 = \frac{-\frac{1}{RC} + \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}{2}, \quad s_2 = \frac{-\frac{1}{RC} - \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}{2}$$

The natural response of RLC circuits

- **Three cases**
  - **Over-damped response:**
    - Characteristic equation has two (negative) real roots
    - Response is a decaying exponential
    - No oscillation (hence the name over-damped, because the resistor damps out the frequency of oscillation)
  - **Under-damped response:**
    - Characteristic equation has two distinct complex roots
    - Response is a decaying exponential that oscillates
  - **Critically-damped response:**
    - Characteristic equation has two read, distinct roots
    - Solution no longer a pure exponential
    - Response is on the verge of oscillation
- **Analogy to oscillating suspended spring-mass-damper system; where energy is stored in the spring and mass**

Over-damped response

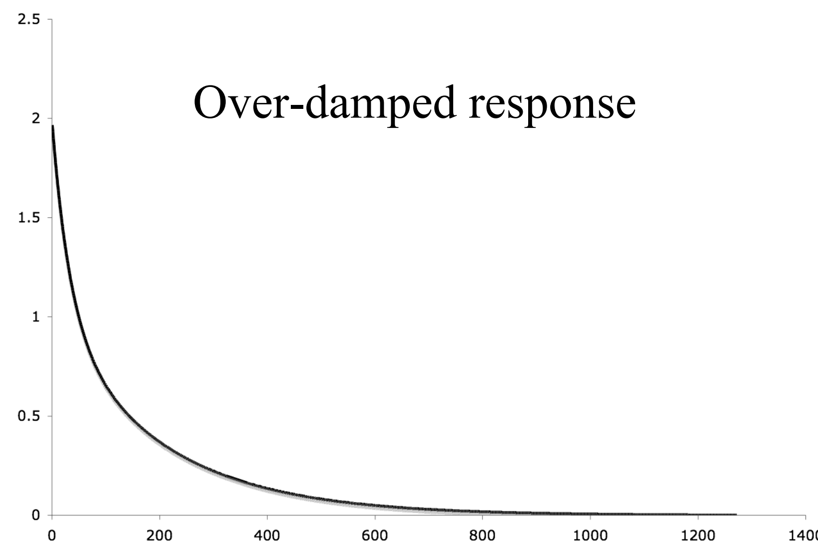
$$\frac{1}{(RC)^2} > \frac{4}{LC}$$

- **Characteristic equation has two distinct real roots; s_1, s_2**
  - **Both $s_1 > 0$, $s_2 > 0$ (why?)**

- **Solution of the form:**

$$v_1 = aE_1^{s_1} e^{s_1 t} + bE_1^{s_2} e^{s_2 t}$$
$$i_l = aE_2^{s_1} e^{s_1 t} + bE_2^{s_2} e^{s_2 t}$$

- **Decaying exponential response**



Critically-damped response

$$\frac{1}{(RC)^2} = \frac{4}{LC}$$

- **Characteristic equation has two real repeated roots; s_1, s_2**
  - **Both $s_1 = s_2 = -1/2RC$**

- **Solution no longer a pure exponential**
  - **"defective eigen-values" $\Rightarrow$ only one independent eigen-vector**
    Cannot solve for (two) initial conditions on inductor and capacity

- **However, solution can still be found and is of the form:**

$$v_1 = a_1 e^{st} + b_1 t e^{st}$$

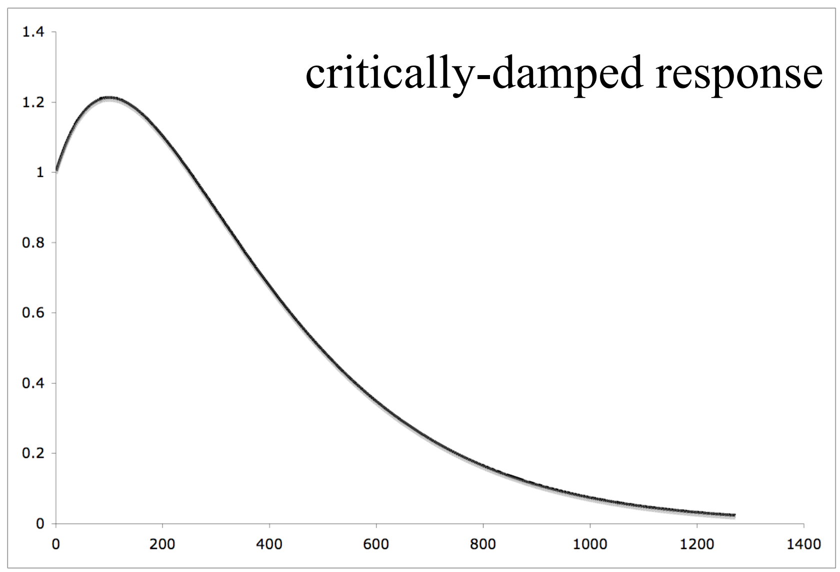
$$i_l = a_2 e^{s_1 t} + b_2 t e^{s_2 t}$$

- **See chapter 5 of Edwards and Penny; or 18.03 notes**

Critically-damped response, cont.

$$v_1 = ae^{-t/2RC} + bte^{-t/2RC}$$

- **Response is on the verge of oscillation**
- **Known as "critically damped"**



Under-damped response

$$\frac{1}{(RC)^2} < \frac{4}{LC}$$

- **Characteristic equation has two distinct complex roots; s_1, s_2**

$$s_1 = -\alpha + j\beta, \;\; s_2 = s_1^* = -\alpha - j\beta$$

$$\alpha = 1/2RC, \; \beta = \frac{\sqrt{1/(RC)^2 - 4/LC}}{2}$$

- **Two distinct eigenvectors, V_1 and V_1^***
  - **Complex eigenvalues and eigenvectors**
- **Solution of the form:**

$$v_1 = a_1 e^{s_1 t} + a_1^* e^{s_2 t}$$

$$i_l = a_2 e^{s_1 t} + a_2^* e^{s_2 t}$$

Under-damped response, cont.

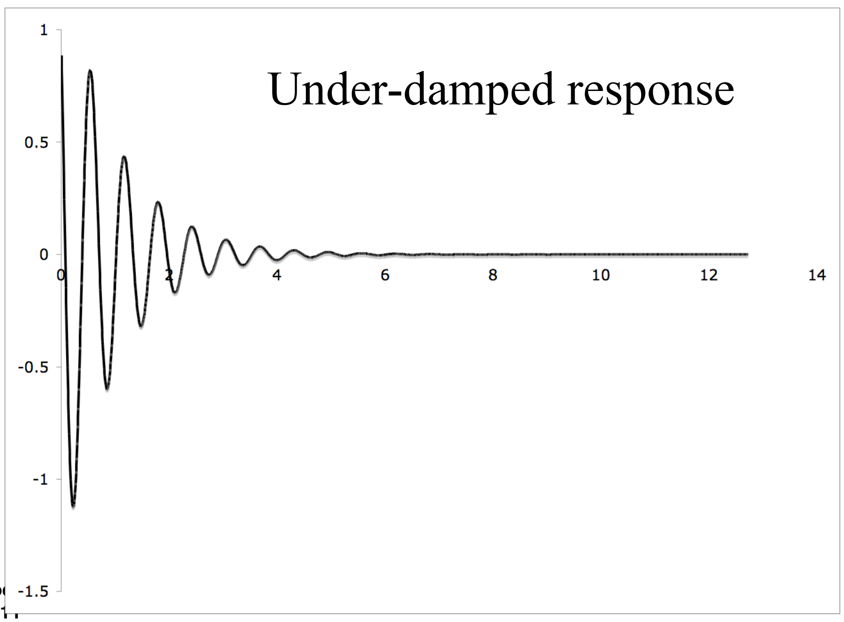
$$v_1 = a_1 e^{(-\alpha + j\beta)t} + a_1^* e^{(-\alpha - j\beta)t}$$

$$Euler's\ formula: \ \ e^{\alpha + j\beta} = e^{\alpha}(\cos(\beta) + j\sin(\beta))$$

$$\Rightarrow v_1 = a_1 e^{-\alpha t}(\cos(\beta t) + j\sin(\beta t)) + a_1^* e^{-\alpha t}(\cos(-\beta t) + j\sin(-\beta t))$$

$$\cos(-\beta) = \cos(\beta), \ \ \sin(-\beta) = -\sin(\beta)$$

$$\Rightarrow v_1 = 2e^{-\alpha t}\left[\mathrm{Re}[a_1]\cos(\beta t) - \mathrm{Im}[a_1]\sin(\beta t)\right]$$



Decaying oscillation between -Re(a_1) and + Re(a_1) or - Im(a_1) and + Im(a_1)

Example of over-damped response (complex eigen-values and eigen-vectors)

- Consider the same circuit with the following values
  - C=1/2 F, L=1/5 H, R=1
  - Initial conditions, $V_1(0)$=1v, $i_c(0)$=1A

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ i_L \end{bmatrix} = \underbrace{\begin{bmatrix} -1/RC & -1/C \\ 1/L & 0 \end{bmatrix}}_{A}\begin{bmatrix} v_1 \\ i_L \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ i_L \end{bmatrix}$$

$$\left[sI - A\right] = \begin{bmatrix} s+2 & 2 \\ -5 & s \end{bmatrix}$$

$$\left|\begin{bmatrix} s+2 & 2 \\ -5 & s \end{bmatrix}\right| = s^2 + 2s + 10 = 0$$

$$s_1 = -1 + 3j, \quad s_2 = -1 - 3j$$

Example, continued: Eigen-vectors

- **Now find the eigen-vectors**

$$s_1 = -1 + 3j \Rightarrow sI - A = \begin{bmatrix} 1+3j & 2 \\ -5 & -1+3j \end{bmatrix} \begin{bmatrix} V_1^{s1} \\ V_2^{s1} \end{bmatrix} = 0$$

$$\Rightarrow -5V_1^{s1} + (-1+3j)V_2^{s1} = 0$$

$$\Rightarrow V_1^{s1} = \left[-1/5 + 3/5\,j\right] V_2^{s1}$$

$$\Rightarrow V^{s1} = \begin{bmatrix} V_1^{s1} \\ V_2^{s1} \end{bmatrix} = \begin{bmatrix} -1/5 + 3/5\,j \\ 1 \end{bmatrix}$$

$$s_2 = s_1^* = -1 - 3j$$

$$V^{s2} = \begin{bmatrix} V_1^{s1} \\ V_2^{s1} \end{bmatrix}^* = \begin{bmatrix} -1/5 + 3/5\,j \\ 1 \end{bmatrix}^* = \begin{bmatrix} -1/5 - 3/5\,j \\ 1 \end{bmatrix}$$

The total solution

$$V_1(t) = a_1(-1/5 + 3/5j)e^{(-1+3j)t} + a_2(-1/513/5j)e^{(-1-3j)t}$$

$$i_c(t) = a_1(1)e^{(-1+3j)t} + a_2(1)e^{(-1-3j)t}$$

$$Use\ V_1(0) = 1,\ \ i_c(0) = 1\ \ to\ obtain:\ \ a_1 = 1/2 - j,\ \ a_2 = 1/2 + j$$

$$V_1(t) = (1/2 + 1/2j)e^{(-1+3j)t} + (1/2 - 1/2j)e^{(-1-3j)t}$$

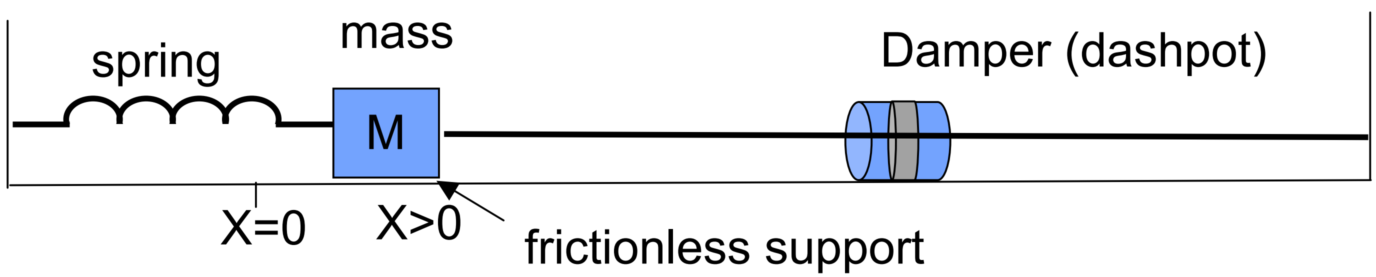
$$i_c(t) = (1/2 - j)e^{(-1+3j)t} + (1/2 + j)e^{(-1-3j)t}$$

to express as real values use,

Euler's formula: $\mathrm{e}^{\alpha+\mathrm{j}\beta} = e^{\alpha}(\cos(\beta) + j\sin(\beta))$

some algebra $\Rightarrow \begin{cases} V_1(t) = \left[\cos(3t) - \sin(3t)\right]e^{-t} \\ i_c(t) = \left[\cos(3t) + 2\sin(3t)\right]e^{-t} \end{cases}$

2nd order mechanical systems mass-spring-damper



- Force exerted by spring is proportional to the displacement (x) of the mass from its equilibrium position and acts in the opposite direction of the displacement
  - $F_s = -kx$
  - $F_s < 0$ if $x > 0$ (I.e., spring stretched)
  - $F_s > 0$ if $x < 0$ (spring compressed)

- Damper force is proportional to velocity ($v = x'$) and acts in opposite direction to motion
  - $F_r = -bv = -bx'$

Spring-mass-damper system

- **If Fr and Fs are the only forces acting on the mass m, then**
  - **F = ma where a = dv/dt = x''**
  - **F = m x'' = Fs + Fr = -kx - bx'**
- **So the differential equation for the mass position, x, is given by:**

mx'' + bx' + kx = 0

  - **This is a second order system**
  - **Guessing $x = Ae^{st}$ for the solution we get,**

$$mAs^2e^{st} + bAse^{st} + kAe^{st} = 0$$

  - **The characteristic equation is thus**

$$s^2 + (b/m)s + k/m = 0$$

$$s_1 = \frac{(-b/m) + \sqrt{(b/m)^2 - 4(k/m)}}{2}, \quad s_2 = \frac{(-b/m) - \sqrt{(b/m)^2 - 4(k/m)}}{2}$$

Response of Spring-mass-damper system

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

where A_1 and A_2 are chosen to satisfy initial conditions

- **Note that for this system the state can be described by**
  - **Position, x(t), Velocity, x'(t)**
  - **Hence, the initial conditions would be x(0) and x'(0)**

- **Note similarity to RLC circuit response:**

$$s_1 = \frac{-1/RC + \sqrt{\left(1/RC\right)^2 - 4/LC}}{2}, \quad s_2 = \frac{-1/RC - \sqrt{\left(1/RC\right)^2 - 4/LC}}{2}$$

- **Notice relationship between 1/R in RLC circuit and damping factor (b) in spring-mass-damper system**
  - **B ~ 0 ⇒ un-damped system ⇒ oscillation**
  - **This is the basis for the terminology, over-damped, under-damped, etc.**
  - **Over-damped system ⇒ damping factor is large and system does not oscillate (just exponential decay)**