A Primer on Solving Systems of Linear Equations

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In Signals and Systems, as well as other subjects in Unified, it will often be necessary to solve systems of linear equations, such as

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ 2x & + & 5y & + & 2z & = & 2 \\ x & + & y & + & z & = & 3 \end{array} \tag{1}$$

There are are at least three ways to solve this set of equations: Elimination of variables, Gaussian reduction, and Cramer's rule. These three approaches are discussed below.

1 Elimination of Variables

Elimination of variables is the method you learned in high school. In the example, you first eliminate x from the second two equations, by subtracting twice the first equation from the second, and subtracting the first equation from the third. The three equations then become

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ & & y & - & 4z & = & 0 \\ & & -y & - & 2z & = & 2 \end{array} \tag{2}$$

Next, y is eliminated from the third equation, by adding the (new) second equation to the third, yielding

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ & & y & - & 4z & = & 0 \\ & & & - & 6z & = & 2 \end{array} \tag{3}$$

From the third equation, we conclude that

$$z = -\frac{1}{3} \tag{4}$$

From the second equation, we conclude that

$$y = 4z = -\frac{4}{3} \tag{5}$$

Finally, from the first equation, we find that

$$x = 1 - 2y - 3z = \frac{14}{3} \tag{6}$$

2 Gaussian Reduction

A more organized way of solving the system of equations is *Gaussian reduction.* First, the augmented matrix of the system is formed:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right] \tag{7}$$

Each row of the augmented matrix corresponds to one equation in the system of equations. The first three elements of each row are the coefficients of x, y, and z in the equation. The fourth element in each row is the right-hand side of the corresponding equation.

The goal of the reduction process is to apply row operations to the matrix to get zeros below the first diagonal of the matrix. To do this for the example, subtract twice the first row from the second, and the first row from the third, to obtain

$$\left[\begin{array}{rrr|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & -1 & -2 & 2 \end{array}\right] \tag{8}$$

This produces zeros in the first column below the diagonal. Then add the second row to the third to obtain

$$\left[\begin{array}{rrr|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & -6 & 2 \end{array}\right] \tag{9}$$

Normally, the process is stopped at this point, and "back substitution" is used to solve for the variables. That is, the array above is really the same as the last reduction we did in elimination of variables, so we can proceed as before.

Alternatively, we could continue the process further, by dividing the third row by -6, and then eliminating terms above the diagonal as well, as follows:

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right] \tag{10}$$

Subtract 3 times the third row from the first, and add 4 times the third row to the second:

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right] \tag{11}$$

Finally, subtract 2 times the second row from the first:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{14}{3} \\ 0 & 1 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \end{array}\right] \tag{12}$$

from which we conclude that

$$\begin{aligned} x &= \frac{14}{3} \\ y &= -\frac{4}{3} \\ z &= -\frac{1}{3} \end{aligned}$$

as before.

There is really no need to repeatedly write down the augmented matrix. Instead, it is convenient to write down the matrix with space between the rows, and update a row at a time, crossing out the old row.

Note that the two methods, elimination of variables and Gaussian reduction, are really the same approach. The only real difference is that in Gaussian reduction, we don't bother to write down x, y, and z. Instead, the variables are associated with columns of the augmented matrix.

3 Cramer's Rule

Cramer's Rule is a method that is useful primarily for low-order systems, with two or three unknowns. Cramer's rules states that each unknown can be expressed as the ratio of two matrix determinants. For example, x (the first variable) is given by

$$x = \frac{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{-28}{-6} = \frac{14}{3} \tag{13}$$

The denominator is the determinant of the matrix of coefficients of the equation, *i.e.*,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

The numerator is the determinant of the same matrix, except with the first column replaced by the three numbers on the right-hand side of the equations. Likewise, y is found by a similar ratio, with the top matrix found by replacing the second column of the denominator matrix by the right-hand side column:

$$y = \frac{\begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 2 \\ 1 & 3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{8}{-6} = -\frac{4}{3} \tag{14}$$

Finally,

$$z = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{2}{-6} = -\frac{1}{3} \tag{15}$$

Cramer's rule is very handy for second- and third-order systems. However, it is much less useful for larger systems, because the determinant calculation becomes prohibitive, if done in the conventional way. (See section below.) Determinants can also be found by Gaussian reduction; however, the reduction process does more than determine the determinant, it also solves the equations! So just use Cramer's rule for small "toy" problems.

4 Determinants

Finally, you need to know how to take determinants of matrices. For second-order matrices,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{16}$$

For third-order systems,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg \tag{17}$$

Both of these have an obvious pattern—each of the terms is the product of diagonals, with a + sign for one direction of diagonal, and a − sign for the other direction. Note that for the third-order case, the diagonals "wrap around."

For higher-order systems, there are two approaches to find the determinant. One approach is to do Gaussian reduction to obtain a triangular matrix. The determinant is then the product of the terms on the main diagonal.

The other approach is to decompose the determinant along, say, the first row of the matrix. For each element A_{1j} along the first row of the matrix, find the matrix formed by deleting the first row and jth column. Call the determinant of that matrix d_j. Then

$$|a| = A_{11}d_1 - A_{12}d_2 + \ldots - (-1)^n A_{1n}d_n \tag{18}$$

For example,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \tag{19}$$

Actually, we can expand on any row or column as above. When expanding on an even-numbered row or column, the sign of the determinant is changed.

The only problem with the approach is that the amount of calculation required to calculate the determinant using this approach goes like $n!$, but only goes like n^3 using Gaussian reduction. So it usually isn't used for systems much higher than third order.