Trajectory Calculation
Lab 2 Lecture Notes

Nomenclature

t	time	ρ	air density
h	altitude	g	gravitational acceleration
V	velocity, positive upwards	m	mass
F	total force, positive upwards	C_D	drag coefficient
D	aerodynamic drag	A	drag reference area
T	propulsive thrust	$\dot{m}_{\text{fuel}}$	fuel mass flow rate
Δt	time step	u_e	exhaust velocity
$\dot{()}$	time derivative $(= d()/dt)$	i	time index

Trajectory equations

The vertical trajectory of a rocket is described by the altitude, velocity, and total mass, $h(t)$, $V(t)$, $m(t)$, which are functions of time. These are called *state variables* of the rocket. Figure 1 shows plots of these functions for a typical trajectory. In this case, the *initial values* for the three state variables h_0, V_0, m_0 are prescribed.

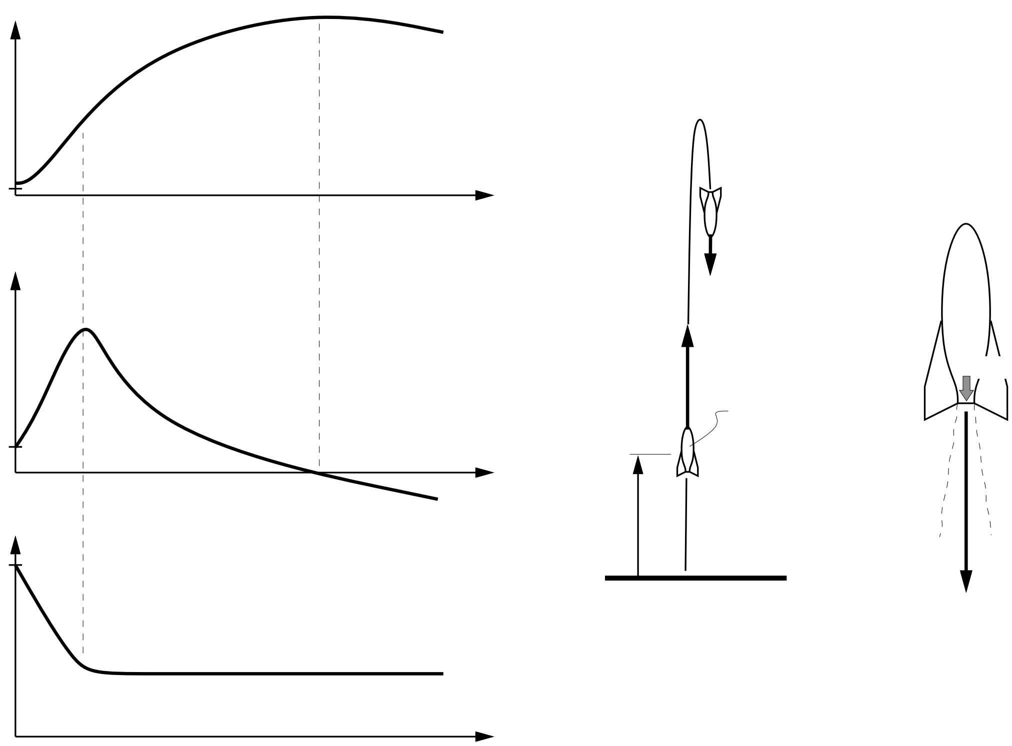

Figure 1: Time traces of altitude, velocity, and mass for a rocket trajectory.

The trajectories are governed by *Ordinary Differential Equations* (ODEs) which give the time rate of change of each state variable. These are obtained from the definition of velocity, from Newton's 2nd Law, and from mass conservation.

$$\dot{h} = V \tag{1}$$
$$\dot{V} = F/m \tag{2}$$
$$\dot{m} = -\dot{m}_{\text{fuel}} \tag{3}$$

The total force F on the rocket has three contributions: the gravity force, the aerodynamic drag force, and the thrust.

$$F = \begin{cases} -mg - D + T & , \text{ if } V > 0 \\ -mg + D - T & , \text{ if } V < 0 \end{cases} \tag{4}$$

The two sign cases in (4) are required because F is defined positive up, so the drag D and thrust T can subtract or add to F depending in the sign of V. In contrast, the gravity force contribution $-mg$ is always negative.

In general, F will be some function of time, and may also depend on the characteristics of the particular rocket. For example, the T component of F will become zero after all the fuel is expended, after which point the rocket will be *ballistic*, with only the gravity force and the aerodynamic drag force being present.

A convenient way to express the drag is

$$D = \frac{1}{2}\rho V^2 C_D A \tag{5}$$

The reference area A used to define the drag coefficient C_D is arbitrary, but a good choice is the rocket's frontal area. Although C_D in general depends on the Reynolds number, it can be often assumed to be constant throughout the ballistic flight. Typical values of C_D vary from 0.1 for a well streamlined body, to 1.0 or more for an unstreamlined or bluff body.

A convenient way to relate the rocket's thrust to the propellant mass flow rate $\dot{m}_{\text{fuel}}$ is via the exhaust velocity u_e.

$$T = \dot{m}_{\text{fuel}}\, u_e \tag{6}$$

Both $\dot{m}_{\text{fuel}}$ and u_e will depend on the rocket motor characteristics, and the motor throttle setting.

With the above force component expressions, the governing ODEs are written as follows.

$$\dot{h} = V \tag{7}$$
$$\dot{V} = -g - \frac{1}{2}\rho V|V| \frac{C_D A}{m} + \frac{V}{|V|} \frac{\dot{m}_{\text{fuel}}\, u_e}{m} \tag{8}$$
$$\dot{m} = -\dot{m}_{\text{fuel}} \tag{9}$$

By replacing V^2 with $V|V|$, and using the $V/|V|$ factor, the drag and thrust contributions now have the correct sign for both the $V > 0$ and $V < 0$ cases.

Numerical Integration

In the presence of drag, or $C_D > 0$, the equation system (7), (8), (9) cannot be integrated analytically. We must therefore resort to numerical integration.

Discretization

Before numerically integrating equations (7), (8), (9), we must first *discretize* them. We replace the continuous time variable t with a *time index* indicated by the subscript i, so that the state variables h,V,m are defined only at discrete times $t_0, t_1, t_2 \ldots t_i \ldots$

$$\begin{array}{ccc} t & \rightarrow & t_i \\ h(t) & \rightarrow & h_i \\ V(t) & \rightarrow & V_i \\ m(t) & \rightarrow & m_i \end{array}$$

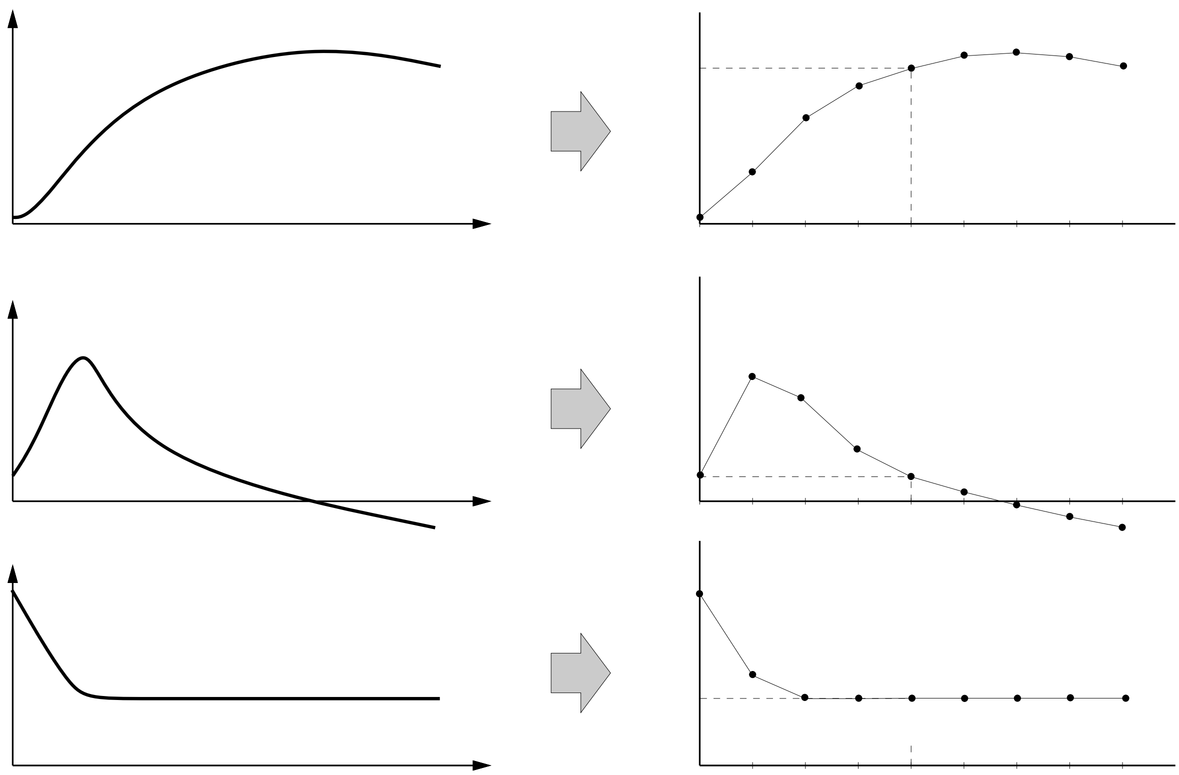

Figure 2: Continuous time traces approximated by discrete time traces.

The governing ODEs (7), (8), (9) can then be used to determine the discrete rates at each time level.

$$\dot{h}_i = V_i \tag{10}$$

$$\dot{V}_i = -g - \frac{1}{2}\rho V_i |V_i| \frac{C_D\,A}{m_i} + \frac{V_i}{|V_i|}\frac{\dot{m}_{\mathrm{fuel}_i}\, u_{e_i}}{m_i} \tag{11}$$

$$\dot{m}_i = -\dot{m}_{\mathrm{fuel}_i} \tag{12}$$

As shown in Figure 3, the rates can also be approximately related to the changes between two successive times.

$$\dot{h}_i = \frac{dh}{dt} \simeq \frac{\Delta h}{\Delta t} = \frac{h_{i+1} - h_i}{t_{i+1} - t_i} \tag{13}$$

$$\dot{V}_i = \frac{dV}{dt} \simeq \frac{\Delta V}{\Delta t} = \frac{V_{i+1} - V_i}{t_{i+1} - t_i} \tag{14}$$

$$\dot{m}_i = \frac{dm}{dt} \simeq \frac{\Delta m}{\Delta t} = \frac{m_{i+1} - m_i}{t_{i+1} - t_i} \tag{15}$$

Equating (10) with (13), (11) with (14), and (12) with (15), gives the following *difference equations* governing the discrete state variables.

$$\frac{h_{i+1} - h_i}{t_{i+1} - t_i} = V_i \tag{16}$$

$$\frac{V_{i+1} - V_i}{t_{i+1} - t_i} = -g - \frac{1}{2}\rho V_i |V_i| \frac{C_D\, A}{m_i} + \frac{V_i}{|V_i|} \frac{\dot{m}_{\text{fuel}_i}\, u_{e_i}}{m_i} \tag{17}$$

$$\frac{m_{i+1} - m_i}{t_{i+1} - t_i} = -\dot{m}_{\text{fuel}_i} \tag{18}$$

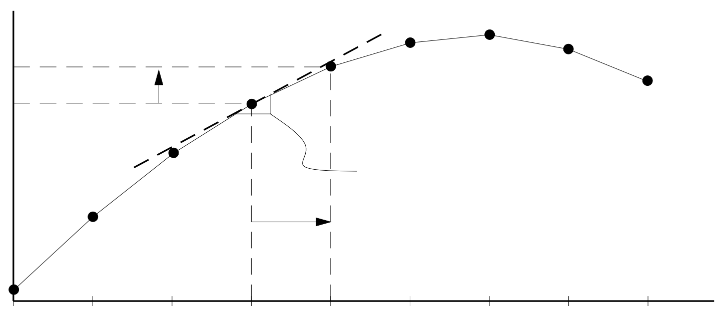

Figure 3: Time rate $\dot{h}$ approximated with finite difference $\Delta h/\Delta t$.

Time stepping (time integration)

Time stepping is the successive application of the difference equations (16), (17), (18) to generate the sequence of state variables h_i, V_i, m_i. To start the process, it is necessary to first specify *initial conditions*, just like in the continuous case. These initial conditions are simply the state variable values h_0, V_0, m_0 corresponding to the first time index $i\!=\!0$. Then, given the values at any i, we can compute values at $i\!+\!1$ by rearranging equations (16), (17), (18).

$$h_{i+1} = h_i + \Big(V_i \Big) \, (t_{i+1} - t_i) \tag{19}$$

$$V_{i+1} = V_i + \left(-g - \frac{1}{2}\rho V_i |V_i| \frac{C_D\, A}{m_i} + \frac{V_i}{|V_i|} \frac{\dot{m}_{\text{fuel}_i}\, u_{e_i}}{m_i} \right) (t_{i+1} - t_i) \tag{20}$$

$$m_{i+1} = m_i + (-\dot{m}_{\text{fuel}_i}) \, (t_{i+1} - t_i) \tag{21}$$

The resulting equations (19), (20), (21) are an example of *Forward Euler Integration*. These will always have the form

$$y_{i+1} \;=\; y_i \;+\; \Big(y\text{-rate at } t_i\Big)\,(t_{i+1} - t_i)$$

where $y(t)$ is the state variable being integrated. There are other, more accurate discrete equation forms. For example, *Trapezoidal Integration* has the form

$$y_{i+1} \;=\; y_i \;+\; \left(\frac{y\text{-rate at } t_i \;+\; y\text{-rate at } t_{i+1}}{2}\right)(t_{i+1} - t_i)$$

But such alternative methods bring more complexity, and will not be considered in this introductory treatment.

Numerical implementation

A spreadsheet provides a fairly simple means to implement the time stepping equations (19), (20), (21). Such a spreadsheet program is illustrated in Figure 4. The time t_{i+1} in equations

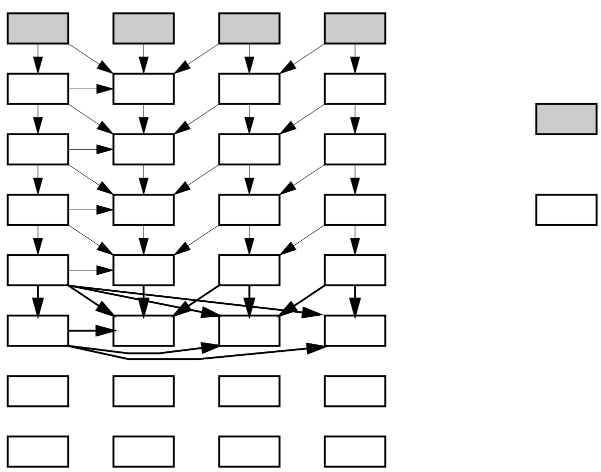

Figure 4: Spreadsheet for time stepping. Arrows show functional dependencies.

(19), (20), (21) is most conveniently defined from t_i and a specified *time step*, denoted by Δt.

$$t_{i+1} \;=\; t_i \;+\; \Delta t \tag{22}$$

It is most convenient to make this Δt to have the same value for all time indices i, so that equation (22) can be coded into the spreadsheet to compute each time value t_{i+1}, as indicated in Figure 4. This is much easier than typing in each t_i value by hand.

More spreadsheet rows can be added to advance the calculation in time for as long as needed. Typically there will be some *termination criteria*, which will depend on the case at hand. For the rocket, suitable termination criteria might be any of the following.

$h_{i+1} < h_0$	rocket fell back to earth
$h_{i+1} < h_i$	rocket has started to descend
$V_{i+1} < 0$	rocket has started to descend

Accuracy

The discrete sequences h_i, V_i, m_i are only approximations to the true analytic solutions $h(t)$, $V(t)$, $m(t)$ of the governing ODEs. We can define *discretization errors* as

$$\mathcal{E}_{h_i} = h_i - h(t_i) \quad (23)$$

$$\mathcal{E}_{V_i} = V_i - V(t_i) \quad (24)$$

$$\mathcal{E}_{m_i} = m_i - m(t_i) \quad (25)$$

although $h(t)$, $V(t)$, $m(t)$ may or may not be available. A discretization method which is *consistent* with the continuous ODEs has the property that

$$|\mathcal{E}| \to 0 \qquad \text{as} \qquad \Delta t \to 0$$

The method described above is in fact consistent, so that we can make the errors arbitrarily small just by making Δt sufficiently small.