Water Rocket Calculations
Lab 4 Lecture Notes

Nomenclature

$\mathcal{V}$	air volume inside rocket	ℓ	length of launch rod
p	air pressure insider rocket	A_e	area of launch rod and nozzle exit
$p_{\rm atm}$	atmospheric pressure	$\rho_{\rm w}$	water density
$F_{\rm rod}$	force on launch rod	$\dot{m}_{\rm w}$	water mass flow rate
$(\,)_{\rm c}$	charge condition, start of Phase 1	u_e	water exhaust velocity
$(\,)_0$	final condition, end of Phase 1	T	thrust

Liftoff Initial Condition Calculation

Phase 1 of the water rocket flight consists of the rocket sliding up on the launcher rod, via the action of the pressure of the internal compressed air.

Pressure-force work

The launcher rod of cross-sectional area A_e feels the compressed-air pressure p on the inside, and the atmospheric pressure $p_{\rm atm}$ on the outside. The net axial force on the launcher rod is

$$F_{\rm rod} \;=\; (p - p_{\rm atm})A_e \tag{1}$$

which will in general decrease as the launch rod is expelled and the air partially expands as a result. The mechanical work performed by this force over the length ℓ of the rod must therefore be determined via a work integral over the axial distance z.

$$W \;=\; \int_0^\ell F_{\rm rod}\,dz \;=\; \int_0^\ell (p - p_{\rm atm})A_e\,dz \tag{2}$$

Using the relation $A_e\,dz = d\mathcal{V}$, the integration is more conveniently performed over the volume change.

$$W \;=\; \int_{\mathcal{V}_{\rm c}}^{\mathcal{V}_0} (p - p_{\rm atm})\,d\mathcal{V} \;=\; \int_{\mathcal{V}_{\rm c}}^{\mathcal{V}_0} p\,d\mathcal{V} \;-\; p_{\rm atm}(\mathcal{V}_0 - \mathcal{V}_{\rm c}) \tag{3}$$

Evaluation of the remaining $p\,d\mathcal{V}$ integral requires knowing how the pressure p of the air in the rocket varies with its volume $\mathcal{V}$. Assuming that the expansion is isentropic (i.e. both loss-free and adiabatic), the pressure and volume will then be related by the isentropic relation

$$p\mathcal{V}^\gamma \;=\; \text{constant} \;=\; p_{\rm c}{\mathcal{V}_{\rm c}}^\gamma \tag{4}$$

$$\text{or} \qquad p \;=\; p_{\rm c}{\mathcal{V}_{\rm c}}^\gamma\,\mathcal{V}^{-\gamma} \tag{5}$$

The $p(\mathcal{V})$ relation (5) can then be used to evaluate the integral in the work expression (3).

$$\int_{\mathcal{V}_{\rm c}}^{\mathcal{V}_0} p\,d\mathcal{V} \;=\; p_{\rm c}{\mathcal{V}_{\rm c}}^\gamma \int_{\mathcal{V}_{\rm c}}^{\mathcal{V}_0} \mathcal{V}^{-\gamma}\,d\mathcal{V} \tag{6}$$

$$=\; \frac{1}{\gamma-1}\,p_{\rm c}{\mathcal{V}_{\rm c}}^\gamma \left[{\mathcal{V}_{\rm c}}^{1-\gamma} \;-\; {\mathcal{V}_0}^{1-\gamma}\right] \tag{7}$$

$$=\; \frac{1}{\gamma-1}\,p_{\rm c}\left[\mathcal{V}_{\rm c} \;-\; \mathcal{V}_0\left(\frac{\mathcal{V}_{\rm c}}{\mathcal{V}_0}\right)^\gamma\right] \tag{8}$$

Again using the isentropic relation we further replace the volume ratio in (8) with the pressure ratio,

$$\frac{p_0}{p_c} = \left(\frac{\mathcal{V}_c}{\mathcal{V}_0}\right)^{\gamma} \tag{9}$$

so that the pressure integral takes on a fairly simple form.

$$\int_{\mathcal{V}_c}^{\mathcal{V}_0} p \, d\mathcal{V} = \frac{1}{\gamma - 1} \left[p_c \mathcal{V}_c - p_0 \mathcal{V}_0 \right] \tag{10}$$

The overall pressure-work integral (3) is then explicitly given as follows.

$$W = \frac{1}{\gamma - 1} \left[p_c \mathcal{V}_c - p_0 \mathcal{V}_0 \right] - p_{\mathrm{atm}} (\mathcal{V}_0 - \mathcal{V}_c) \tag{11}$$

Energy balance

The net work W on the launch rod shows up as the kinetic and potential energy change of the rocket during the expansion.

$$W = \Delta(KE) + \Delta(PE) \tag{12}$$

Note: You are to use this energy balance to determine the initial velocity V_0 of the rocket at the moment it leaves the launcher rod. You may assume that all the quantities in (11) needed to compute the work W are known or calculated. The initial mass m_0 is also known. Neglect the air drag and the rod friction during this phase.

Rocket Thrust Calculation

The thrust T of the rocket is given by the following momentum balance relation.

$$T = \dot{m}_{\mathrm{w}} \, u_e \tag{13}$$

Both the water mass flow $\dot{m}_{\mathrm{w}}$ and the exit velocity u_e will depend on the instantaneous air pressure p. Hence, both $\dot{m}_{\mathrm{w}}$ and u_e will decrease during the flight as the water is expelled and p decreases from the resulting air expansion.

Note: You are to use the Bernoulli equation to determine u_e as a function of the air pressure p and the nozzle exit pressure $p_e = p_{\mathrm{atm}}$.

With u_e determined, the water mass flow then follows from the simple channel mass flow relation.

$$\dot{m}_{\mathrm{w}} = \rho_{\mathrm{w}} u_e A_e \tag{14}$$