WAVES AND SMALL DISTURBANCES

INTRODUCTION

OUR GOAL IS TO DESCRIBE THE RELATION/INTERACTION BETWEEN WAVE MOTION AND THE RESULTING PRESSURE ON A MOVING BODY. IT IS KNOWN THAT DISTURBANCES IN A FLUID ARE TRANSMITTED TO OTHER PARTS OF THE FLUID AND TO A MOVING BODY IN THE FLUID BY THE PROPAGATION OF WAVES (SOUND WAVES). THE WAVE MOTION THAT IS ESTABLISHED IN A FLUID MUST SATISFY THE CONSERVATION PRINCIPLES AND THE BOUNDARY CONDITIONS ON THE MOVING BODY. THE REQUIREMENT TO SATISFY THE BOUNDARY CONDITIONS ON THE MOVING BODY DETERMINES THE PRESSURE ON THE BODY. ONCE THE PRESSURE DISTRIBUTION ON THE BODY IS KNOWN, THE AERODYNAMIC FORCES, MOMENTS, AND COEFFICIENTS MAY BE COMPUTED.

VON KARMAN'S RULES OF SUPERSONIC FLOW

- THE EFFECT OF PRESSURE CHANGES GENERATED BY A MOVING BODY AT A SPEED FASTER THAN THE SPEED OF SOUND CANNOT REACH POINTS AHEAD OF THE BODY. (RULE OF FORBIDDEN SIGNALS)

- A STATIONARY POINT SOURCE IN SUPERSONIC STREAM PRODUCES EFFECTS ONLY ON POINTS THAT LIE ON OR INSIDE THE MACH CONE EXTENDING DOWNSTREAM FROM THE POINT SOURCE.

- CONVERSELY, THE PRESSURE AND VELOCITY AT AN ARBITRARY POINT OF THE STREAM CAN BE INFLUENCED ONLY BY DISTURBANCES ACTING AT POINTS THAT LIE ON OR INSIDE A CONE EXTENDING UPSTREAM FROM THE POINT CONSIDERED AND HAVING THE SAME VERTEX ANGLE AS THE MACH CONE. (ZONE OF ACTION AND ZONE OF SILENCE)

$$\mu = \alpha = \sin^{-1}\left(\frac{1}{M}\right)$$

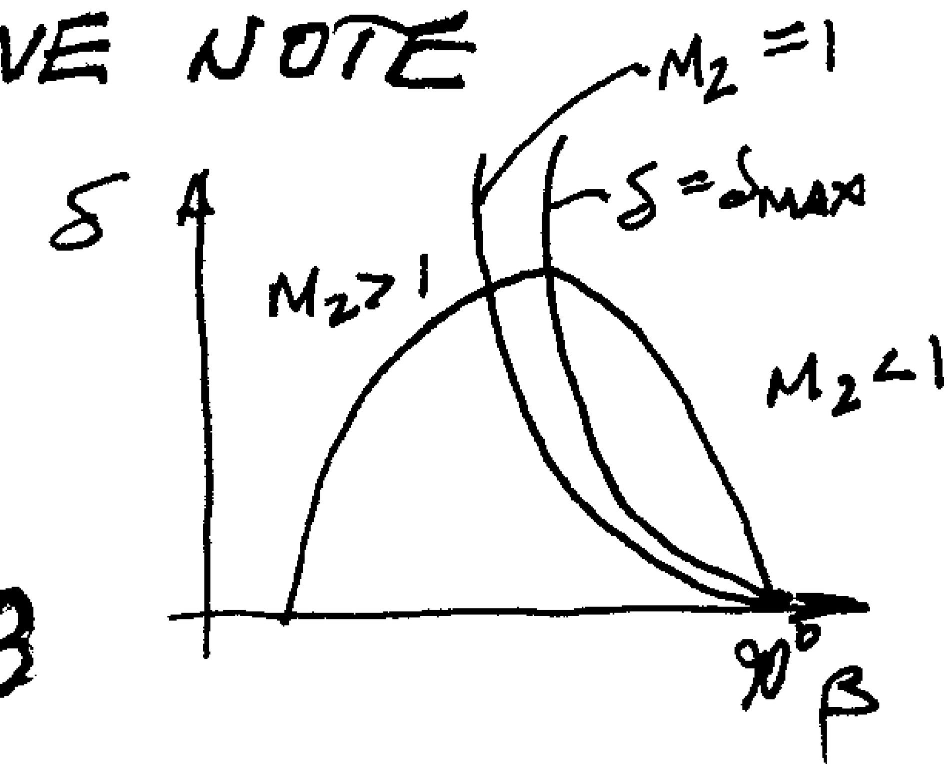
- FOR THE STATIONARY SOURCE, THE INTENSITY OF THE DISTURBANCE IS SYMMETRICAL ABOUT THE SOURCE. FOR A SUBSONIC SOURCE, THE INTENSITY IS UNSYMMETRICAL. FOR A SUPERSONIC SOURCE, THE PRESSURE DISTURBANCE IS LARGELY CONCENTRATED IN THE NEIGHBORHOOD OF THE MACH CONE THAT FORMS THE OUTER LIMITS OF THE ZONE OF ACTION. (RULE OF CONCENTRATED ACTION)

ASSUMPTIONS

- STEADY FLOWS
- TWO-DIMENSIONAL / PLANAR FLOWS
- TWO-DIMENSIONAL / PLANAR SUPERSONIC FLOWS
- STEADY WAVE SYSTEM
- INDIRECT APPROACH
  - PROPOSE A SIMPLE STATIONARY WAVE SYSTEM
  - FIND COMPATIBLE BOUNDARY CONDITIONS

MACH LINES

CONSIDER THE δ-β-M CHART. SOLUTIONS FOR WHICH $M_2 > 1$ (WEAK SHOCK CASE) WE NOTE



$$\frac{d\delta}{d\beta} > 0$$

HENCE, AS δ DECREASES SO DOES β

WE DENOTE THE LIMITING VALUE OF β AT $\underline{\delta = 0}$ BY μ:

$$\beta = \mu \quad \text{AT} \quad \delta = 0$$

THE δ-β-M PLOT IS BASED ON THE FOLLOWING EQUATION

$$M_1^2 \sin^2\beta - 1 = \frac{\gamma+1}{2} M_1^2 \frac{\sin\beta \sin\delta}{\cos(\beta - \delta)}$$

SUBSTITUTING $\beta = \mu$ AND $\delta = 0$:

$$M_1^2 \sin^2\mu - 1 = \frac{\gamma+1}{2} M_1^2 \frac{\sin\mu \sin 0}{\cos(\mu - 0)}$$

$$= 0$$

$$\therefore \quad \mu = \sin^{-1}\left(\frac{1}{M_1}\right)$$

AT THIS LIMIT, $\beta = \mu$, $\delta = 0$, WE NOTE THAT ALL THE JUMP QUANTITIES ARE ZERO, i.e.

$$\frac{\rho_2}{\rho_1} = 1 \qquad \frac{p_2 - p_1}{p_1} = 0 \qquad \frac{T_2}{T_1} = 1 \qquad \frac{s_2 - s_1}{R} = 0$$

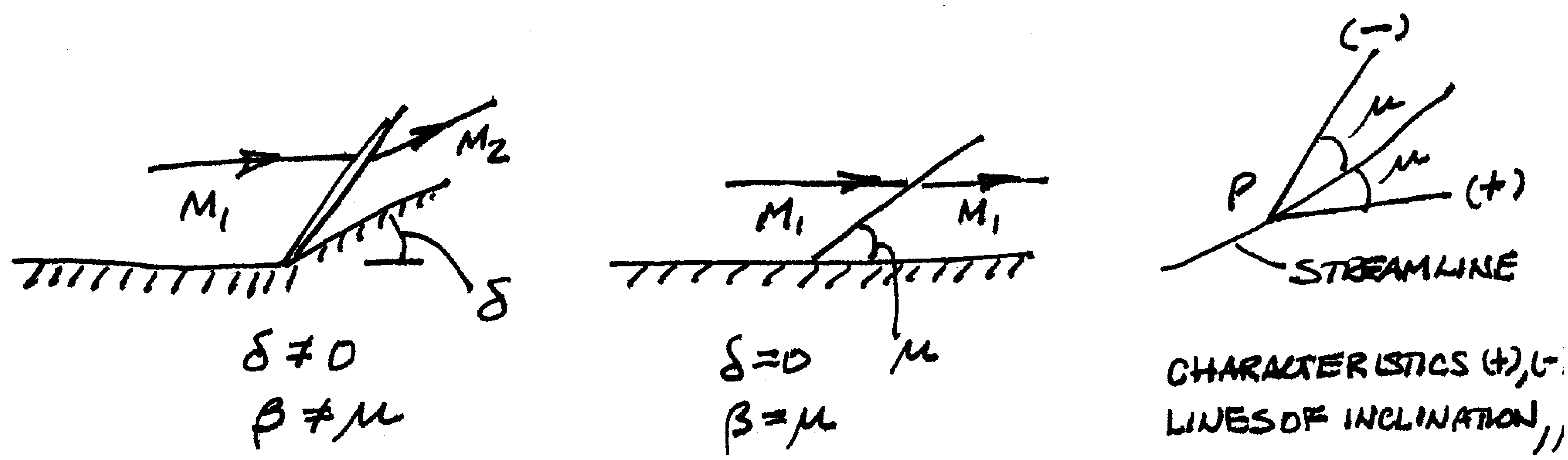
HENCE WE HAVE NO DISTURBANCES IN THE FLOW. AT NO POINT IS THE FLOW UNIQUE. μ IS CALLED THE MACH ANGLE OR CHARACTERISTIC ANGLE. IF

$$M = M(\bar{x}, t)$$

THEN

$$\mu = \sin^{-1}\left(\frac{1}{M}\right) = \mu(M(\bar{x}), t)$$

AND THE MACH LINES ARE CURVED.



MACH LINES AND CHARACTERISTICS ARE THE DIRECTION OF FLOW OR THE DIRECTION OF INCREASING TIME (KARMAN'S RULE).

WEAK OBLIQUE SHOCK WAVES

CONSIDER THE CASE IN WHICH THE DEFLECTION ANGLE IS SMALL:

$$\underline{\delta \ll 1}$$

HENCE,

$$\sin\delta \cong \delta$$

$$\cos\delta \cong 1$$

AND

$$M_1^2 \sin^2\beta - 1 \cong \frac{\gamma+1}{2} M_1^2 \frac{(\sin\beta)\,\delta}{\cos\beta}$$

$$\cong \frac{\gamma+1}{2} M_1^2 (\tan\beta)\,\delta$$

FROM THE δ-β-M CHART, β IS NEAR $\pi/2$ WHEN $M_2 < 1$ OR β IS NEAR μ FOR $M_2 > 1$ FOR SMALL δ. HENCE, WE HAVE:

$$\tan\beta \cong \tan\mu = \frac{1}{\sqrt{M_1^2 - 1}}$$

SUBSTITUTING

$$M_1^2 \sin^2\beta - 1 \cong \frac{\gamma+1}{2} M_1^2 \frac{\delta}{\sqrt{M_1^2 - 1}}$$

$$M_{1n}^2 \cong 1 + \frac{\gamma+1}{2} \frac{M_1^2}{\sqrt{M_1^2 - 1}}\,\delta$$

THIS RESULT LEADS TO:

$$\frac{p_2 - p_1}{p_1} \cong \frac{\gamma M_1^2}{\sqrt{M_1^2 - 1}}\,\delta$$

OR

$$\frac{\Delta p}{p_1} \propto \delta \qquad \frac{\Delta T}{T} \propto \delta \qquad \frac{\Delta \rho}{\rho} \propto \delta \qquad \frac{\Delta s}{R} \propto \delta^3$$

WAVE ANGLE – MACH ANGLE RELATIONS

ASSUME

$$\beta = \mu + \epsilon$$

WHERE

$$\epsilon \ll 1$$

HENCE

$$\sin\beta = \sin(\mu + \epsilon)$$

$$= \cos\epsilon \sin\mu + \sin\epsilon \cos\mu$$

$$\cong \sin\mu + \epsilon \cos\mu$$

RECALL

$$\sin\mu = \frac{1}{M_1} \qquad \cot\mu = \sqrt{M_1^2 - 1}$$

HENCE

$$M_1 \sin\beta \cong M_1 \sin\mu + \epsilon M_1 \cos\mu$$

$$\cong 1 + \epsilon M_1 \frac{\cos\mu}{\sin\mu} \sin\mu$$

$$\cong 1 + \epsilon M_1 \sin\mu \cot\mu$$

$$\cong 1 + \epsilon \cdot 1 \cdot \sqrt{M_1^2 - 1}$$

$$\cong 1 + \epsilon \sqrt{M_1^2 - 1}$$

SQUARING BOTH SIDES OF THE ABOVE EQUATION,

$$M_1^2 \sin^2\beta \cong 1 + 2\epsilon \sqrt{M_1^2 - 1} + \epsilon^2 (M_1^2 - 1)$$

$$\cong 1 + 2\epsilon \sqrt{M_1^2 - 1}, \quad \epsilon \ll 1$$

BY COMPARISON

$$1 + 2\epsilon\sqrt{M_1^2 - 1} = 1 + \frac{\gamma+1}{2}\frac{M_1^2}{\sqrt{M_1^2-1}}\,\delta$$

$$\epsilon = \frac{\gamma+1}{4}\frac{M_1^2}{(M_1^2-1)}\,\delta$$

$$\epsilon = O(\delta)$$

$$\beta - \mu = \epsilon = O(\delta)$$

THIS RESULT LEADS TO

$$\frac{\Delta V}{V_1} \cong -\frac{\delta}{\sqrt{M_1^2-1}} \propto \delta$$

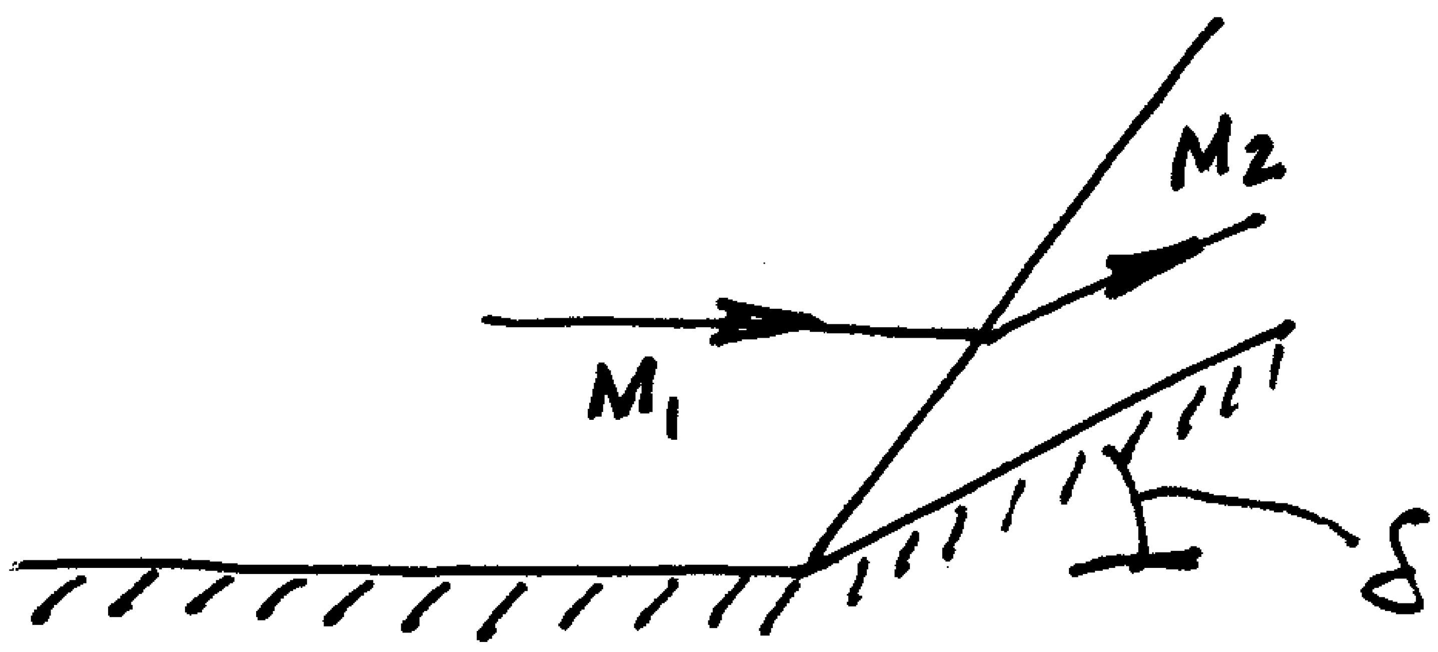
FOR THE CHANGE IN FLOW SPEED ACROSS AN OBLIQUE SHOCK FOR SMALL DEFLECTION ANGLE.

SUPERSONIC COMPRESSION BY TURNING

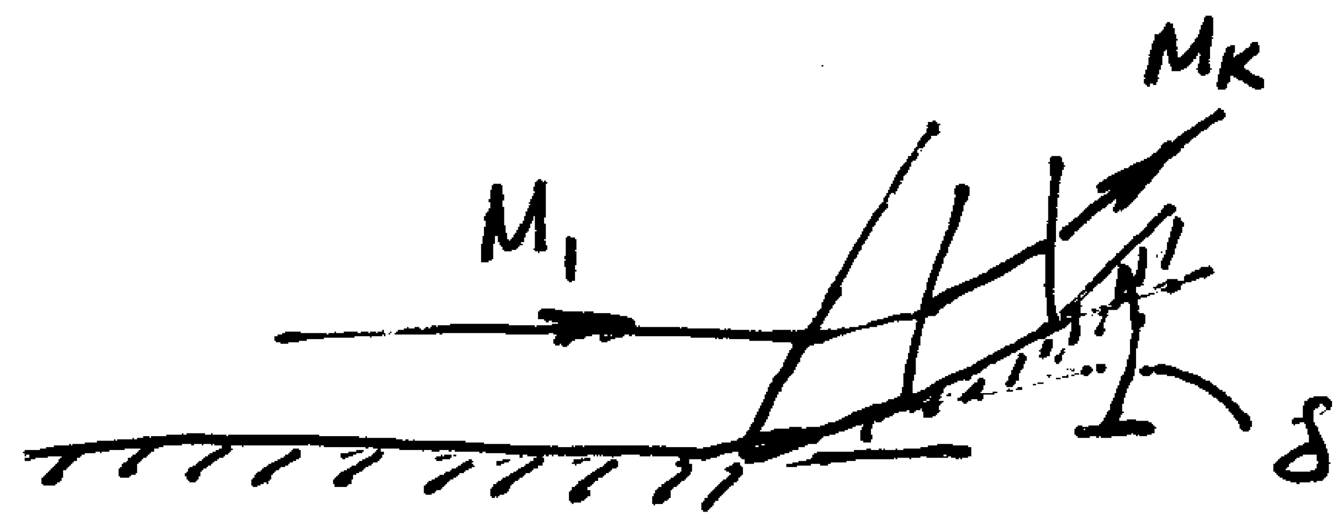
SHOCK WAVES INCREASE THE PRESSURE, DENSITY, AND TEMPERATURE OF THE FLUID PASSING THROUGH IT. SHOCK WAVES COMPRESS THE FLOW.

CONSIDER THE FOLLOWING THREE MEANS OF COMPRESSING A FLOW IN A SUPERSONIC STREAM.

A. SINGLE SHOCK OF STRENGTH δ



B. SEVERAL WEAKER SHOCKS, EACH OF STRENGTH $\Delta\delta$

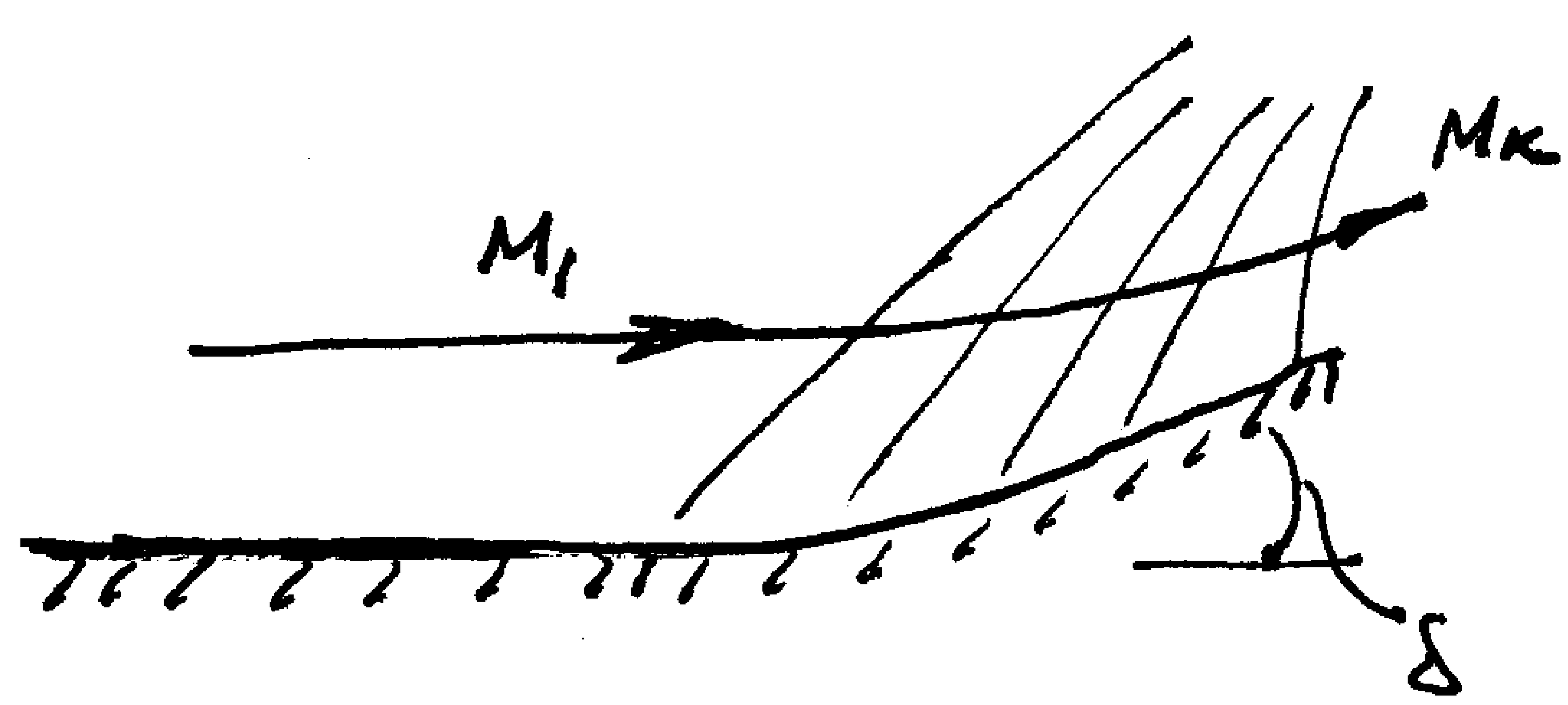


$\Delta p \cong \delta$

$\Delta s \cong S\left(\frac{\delta}{k}\right)^2$

COMPRESSION OCCURS THROUGH A SUCCESSION OF OBLIQUE SHOCKS BY DIVIDING THE TOTAL TURNING ANGLE INTO SMALLER SEGMENTS, $\Delta\delta$. EACH REGION IS INDEPENDENT OF THE FOLLOWING ONE — LIMITED UPSTREAM INFLUENCE, EXCEPT IN SUBSONIC REGION

C. SMOOTH CONTINUOUS COMPRESSION



$\Delta\delta \to 0$

(ISENTROPIC)

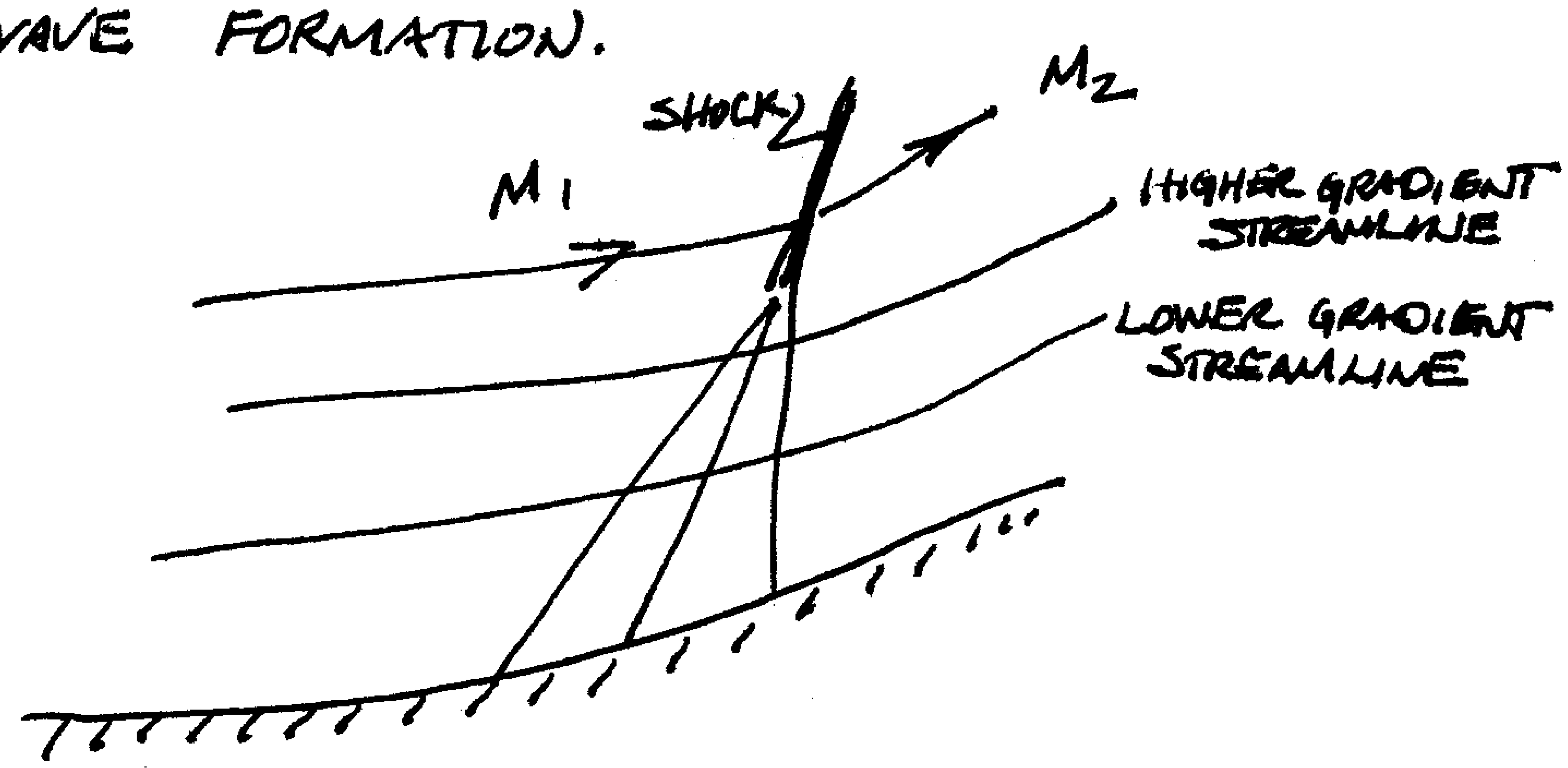
SUMMARY

- LIMITING OF POSITIONS OF VANISHING WEAK SHOCKS ARE STRAIGHT MACH LINES

- EACH SEGMENT OF UNIFORM FLOW BECOMES VANISHING NARROW AND FINALLY COINCIDES WITH A MACH LINE
  - FLOW INCLINATION AND MACH NUMBER ARE CONSTANT ON EACH MACH LINE

- LIMITED UPSTREAM INFLUENCE IS PRESERVED
  - DOWNSTREAM CHANGES IN THE WALL GEOMETRY HAVE NO UPSTREAM EFFECT

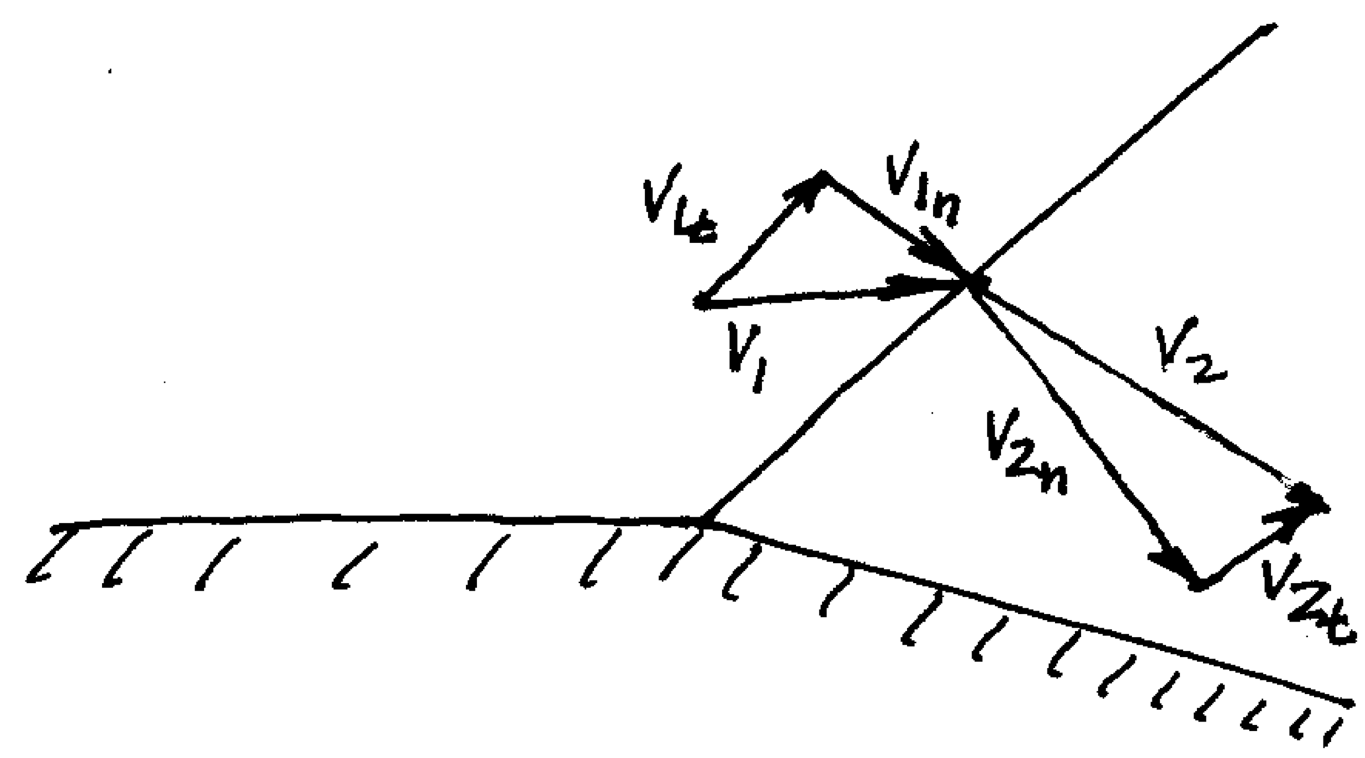
- CONVERGENCE OF MACH LINES IN FARFIELD GIVES RISE TO SHOCK WAVE FORMATION.



SUPERSONIC EXPANSION BY TURNING

WE WILL NOW CONSIDER SUPERSONIC FLOW THROUGH A CONVEX TURN ("AWAY FROM THE ON COMING STREAM).

NOTE THAT A COMPRESSION IS NOT POSSIBLE SINCE SUCH WOULD LEAD TO A DECREASE IN ENTROPY :



$V_{1t} = V_{2t}$
$V_{1n} < V_{2n}$

COMPRESSION AT A CONVEX TURN : $\Delta S < 0$!!

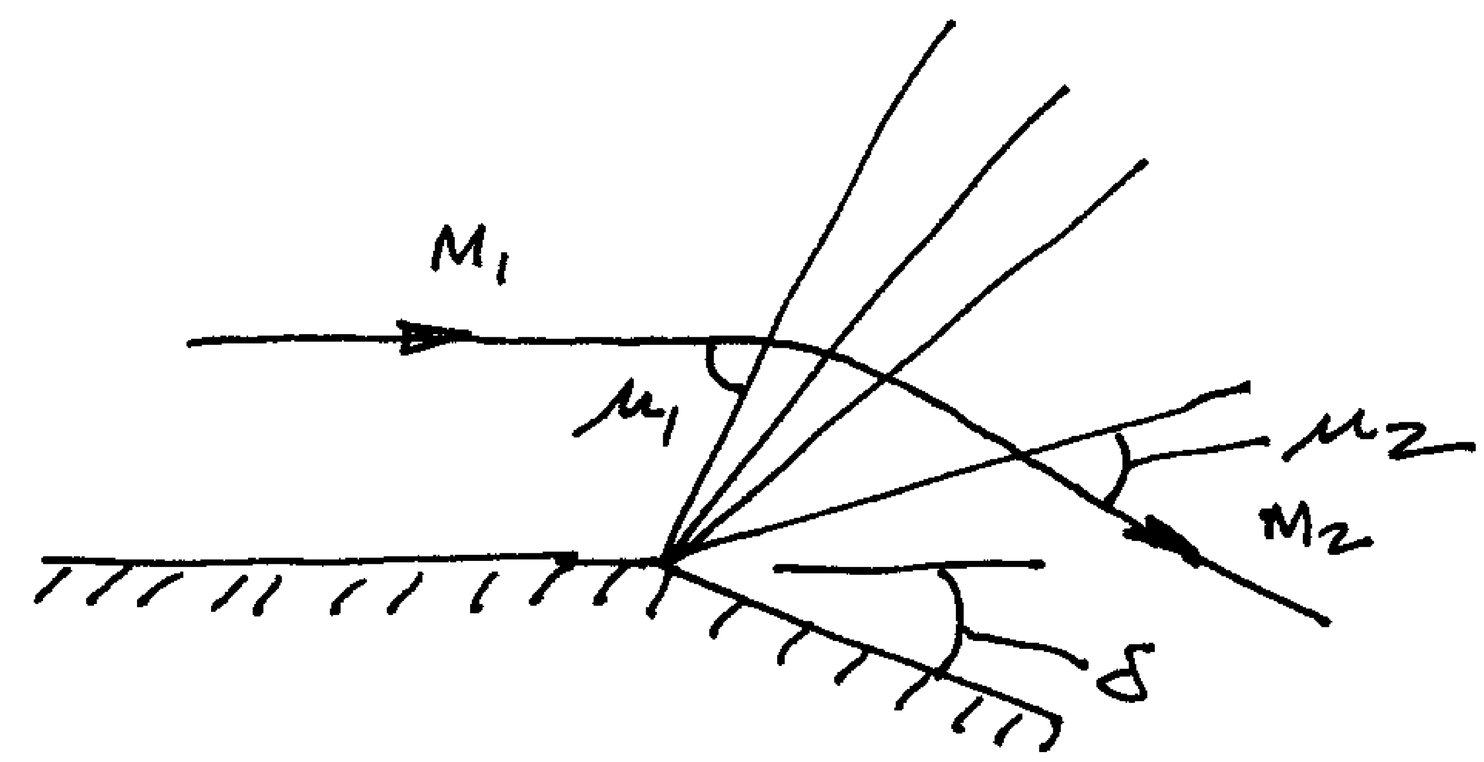
AT THE CONVEX TURN, THE FLOW PROCESS IS ENTIRELY ISENTROPIC. THE MACH LINES MUST <u>DIVERGE</u> AS THE FLOW MOVES THROUG THE CONVEX TURN. FOR SHARP TURNS, WE HAVE A CENTERED WAVE, A FAN, OR A <u>PRANDTL-MEYER FAN</u>. HENCE

TYPE OF TURN ($M_1 > 1$)	WAVE STRUCTURE	ΔS	ΔT_0
CONCAVE	OBLIQUE SHOCK	>0	$=0$
CONVEX	EXPANSION FAN	$=0$	$=0$

THE APPROXIMATE EXPRESSION FOR THE CHANGE OF SPEED ACROSS A WEAK SHOCK (ISENTROPIC TURN) IS:

$$\frac{dV}{V} = -\frac{d\delta}{\sqrt{M^2-1}}$$

INTEGRATING THIS EQUATION,



$$-\delta + \text{CONSTANT}$$

$$= \int (M^2 - 1)^{1/2} \frac{dV}{V}$$

$$= \nu(M)$$

WHERE

$$V = aM$$

$$\frac{a_0^2}{a^2} = 1 + \frac{\gamma - 1}{2} M^2$$

THEREFORE

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a}$$

$$\frac{dV}{V} = \frac{dM}{M} \left(\frac{1}{1 + \frac{\gamma-1}{2} M^2} \right)$$

SUBSTITUTING

$$\nu(M) = \int (M^2 - 1)^{1/2} \left(1 + \frac{\gamma-1}{2} M^2\right)^{-1} \frac{dM}{M}$$

EVALUATING THIS INTEGRAL WE OBTAIN:

$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1} (M^2 - 1)} - \tan \sqrt{M^2 - 1}$$

$\nu(M) \equiv$ PRANDTL-MEYER FUNCTION

EXPANSION CONVENTION:

$$\nu_2 = \nu_1 + |\delta_2 - \delta_1|$$

EXAMPLE PROBLEMS

A. COMPUTE THE RATIO L/D FOR A FLAT PLATE, CHORD c, AT $\alpha = 10°$ IN A SUPERSONIC FLOW $M_1 = 3.0$.

ANS: $L = 1.72\, p_1 c \cos\alpha$; $D = 1.72\, p_1 c \sin\alpha$;

$$L/D = 5.76$$

B. COMPUTE THE RATIO L/D FOR A SYMMETRICAL DIAMOND AIRFOIL OF MAXIMUM THICKNESS $2t$ AT $\alpha = 0°$ IN A SUPERSONIC FLOW $M_1 = 3.0$. AIRFOIL NOSE ANGLE $2\delta = 20°$.

ANS: $L = -0.57 \frac{c}{2} p_1 \cos\delta$; $D = 1.7 \frac{c}{2} p_1 \sin\delta$;

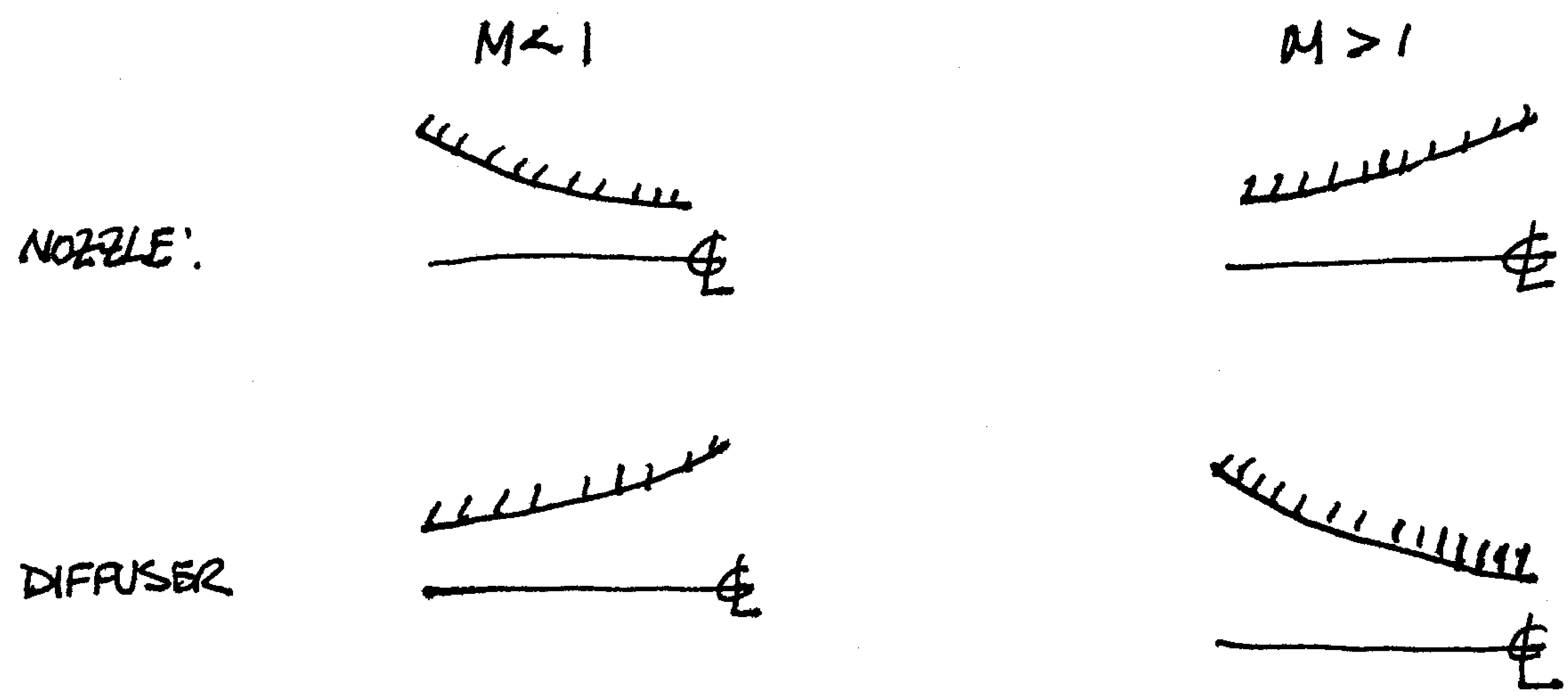
$$L/D = -0.34 \frac{\cos\delta}{\sin\delta}$$

WIND TUNNELS

DEFINITIONS

NOZZLE: $\dfrac{dv}{dx} > 0$, FLOW IS ACCELERATED IN PASSAGE

DIFFUSER: $\dfrac{dv}{dx} < 0$, FLOW IS DECELERATED IN PASSAGE



RECALL THE VELOCITY, AREA, MACH NUMBER RELATIONSHIP:

$$\frac{dA}{A} = (M^2 - 1)\frac{dv}{v}$$

THIS RELATIONSHIP IS VALID FOR:

- ONE-DIMENSIONAL FLOW
- STEADY FLOW
- ISENTROPIC FLOW
- CALORICALLY PERFECT GAS
- INVISCID FLOW

CONSIDER THE CONSERVATION OF ENERGY EQUATION FOR WIND TUNNEL (NOZZLE AND/OR DIFFUSER) FLOWS

$$c_p T_0 = c_p T \left(1 + \frac{\gamma - 1}{2} M^2 \right) = h_0$$

$$= h + \frac{1}{2} V^2$$

$$\therefore \; T_0 = T \left(1 + \frac{\gamma - 1}{2} M^2 \right) = \text{CONSTANT}$$

FOR EXAMPLE, AT $M = 1$, $A = A^*$, $dA = 0$, $T = T^*$

$$T_0 = T \left(1 + \frac{\gamma - 1}{2} M^2 \right)$$

$$T_0 = T^* \left(1 + \frac{\gamma - 1}{2} (1)^2 \right) = T^* \left(\frac{\gamma + 1}{2} \right)$$

THEREFORE WE MAY WRITE

$$\frac{T^*}{T} = \frac{2}{\gamma + 1} \left(1 + \frac{\gamma - 1}{2} M^2 \right)$$

$$\frac{T^*}{T_0} = \frac{2}{\gamma + 1} = 0.833 \quad (\text{AIR}, \gamma = 1.4)$$

$$\frac{p^*}{p} = \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma - 1}} \left(1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma}{\gamma - 1}}$$

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma - 1}} = 0.528 \quad (\text{AIR}, \gamma = 1.4)$$

CONSIDER THE CONSERVATION OF MASS EQUATION FOR WIND TUNNEL FLOWS

$$\dot{m} = \rho V A = \text{CONSTANT}$$

THIS EQUATION MAY BE EXPRESSED AS $\dot{m}/A$, i.e., MASS FLOW PER UNIT AREA,

$$\frac{\dot{m}}{A} = \frac{p_0}{\sqrt{T_0}} \sqrt{\frac{\gamma}{R}} \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{A^*}{A}$$

IN A CONVERGING-DIVERGING NOZZLE, A^* IS THE SMALLEST AREA. HENCE, FOR FIXED VALUES OF p_0 AND T_0, MAXIMUM MASS FLOW PER UNIT AREA OCCURS AT THE THROAT, $A = A^*$.

ONE MAY PROVE THAT THE MAXIMUM FLOW RATE PER UNIT AREA OCCURS AT THE THROAT. THIS IS CALLED CHOKED FLOW.

RECALL FROM YOUR NOTES THE FOLLOWING:

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}}$$