

## CONSERVATION OF LINEAR MOMENTUM

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OUR OBJECTIVE IS TO GENERALIZE EULER'S EQUATIONS TO INCLUDE BOTH BODY AND SURFACE FORCES. WE SHALL DEFINE A BODY OR VOLUME FORCE TO BE A FORCE THAT ACTS ON A SYSTEM BECAUSE OF THE MASS EXTENT OF THE SYSTEM. A BODY OR VOLUME FORCE IS PROPORTIONAL TO THE MASS OF THE SYSTEM. THE SURFACE FORCE ACTS ON THE EXTENT OF THE SURFACE AREA OF A SYSTEM. THE NORMAL SURFACE FORCE PER UNIT AREA IS CALLED PRESSURE.

NOW LET :

$-\bar{F}_B \equiv$  THE FORCE APPLIED TO THE FLUID IN REACTION TO THE AERODYNAMIC FORCES ON THE BODY

$\bar{F}_B \equiv$  THE TOTAL AERODYNAMIC FORCE PRODUCED ON THE BODY (AIRCRAFT) IN CONTACT WITH THE FLUID

$$\bar{F}_B = \bar{F}_{\text{VISCOUS}} + \bar{F}_{\text{GRAVITY}} + \bar{F}_{\text{BOUYANCY}} + \bar{F}_{\text{PRESSURE}}$$

IT IS CONVENIENT TO WORK WITH BODY FORCE PER UNIT VOLUME AND SURFACE FORCE PER UNIT AREA. HENCE, WE HAVE

$\bar{f}_V \equiv$  EXTERNALLY APPLIED BODY FORCE PER UNIT VOLUME

$\bar{f}_S \equiv$  EXTERNALLY APPLIED SURFACE FORCE PER UNIT AREA

$$\bar{f}_S = -\bar{n} \cdot \bar{P} = \bar{n} \cdot \left( p \bar{I} - \bar{T} \right)$$

$\bar{n}$  = SURFACE UNIT NORMAL VECTOR

$\frac{2}{I}$  = UNIT DIAGONAL TENSOR

$\frac{2}{\tau}$  = VISCOUS STRESS TENSOR

$p$  = THERMODYNAMIC PRESSURE

WITHOUT PROOF, WE STATE THE CONSERVATION OF LINEAR MOMENTUM IN INTEGRAL FORM FOR A FIXED FLUID ELEMENT;

$$\begin{aligned} \iiint_{\text{Vol.}} \frac{\partial}{\partial t} (\rho \bar{v}) d\text{Vol.} + \iint_{\text{SURF.}} \bar{n} \cdot (\rho \bar{v} \bar{v}) d\text{SURF.} \\ = \iiint_{\text{Vol.}} \bar{f}_v d\text{Vol.} - \iint_{\text{SURF.}} \bar{n} \cdot \bar{P} d\text{SURF.} \end{aligned}$$

USING GAUSS' THEOREM, WE RE-WRITE THE ABOVE EQUATION

$$\iiint_{\text{Vol.}} \left[ \frac{\partial}{\partial t} (\rho \bar{v}) + \nabla \cdot (\rho \bar{v} \bar{v}) - \bar{f}_v + \nabla \cdot \bar{P} \right] d\text{Vol.} = 0$$

SINCE  $d\text{Vol.}$  IS ARBITRARY, WE MAY WRITE:

$$\frac{\partial}{\partial t} (\rho \bar{v}) + \nabla \cdot (\rho \bar{v} \bar{v}) = \bar{f}_v - \nabla \cdot \bar{P}$$

$$\rho \left[ \frac{\partial \bar{v}}{\partial t} + \bar{v} \cdot \nabla \bar{v} \right] = \bar{f}_v - \nabla \cdot \bar{P}$$

$$\rho \frac{D\bar{v}}{Dt} = \bar{f}_v - \nabla \cdot \bar{P}$$

$$\rho \frac{D\bar{v}}{Dt} = \bar{f}_v - \nabla p + \nabla \cdot \frac{2}{\tau}$$

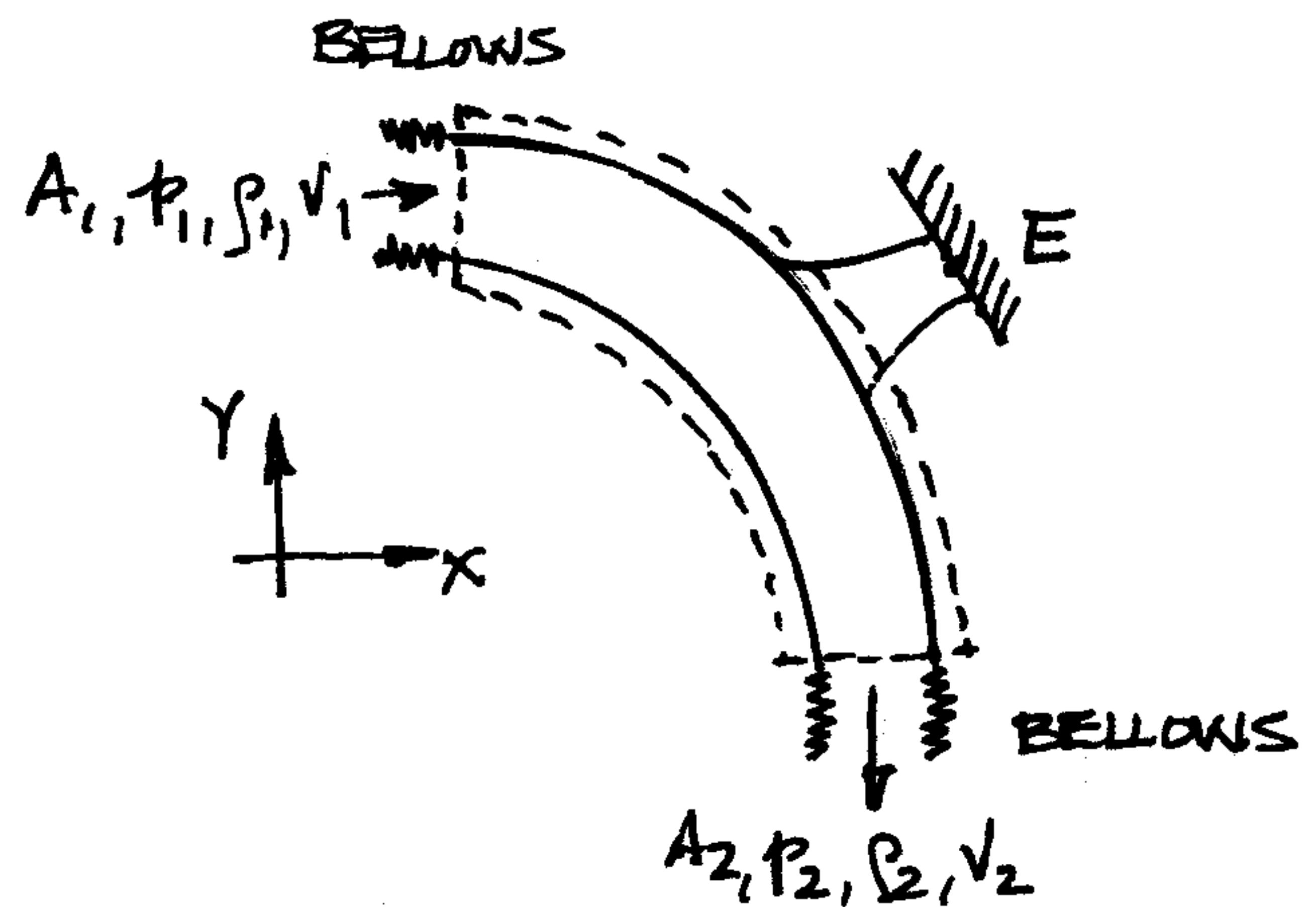
IN THIS COURSE WE WILL CONSIDER NEWTONIAN FLUIDS.  
FOR ALL SUCH FLUIDS:

$$\frac{\tau}{L} \sim \frac{\tau}{E}$$

$$\frac{\tau}{E} \equiv \text{STRAIN RATE TENSOR}$$

## EXAMPLE PROBLEM

CALCULATE THE FORCE REQUIRED TO PRODUCE A CHANGE IN FLOW DIRECTION BY APPLYING LINEAR MOMENTUM CONSERVATION PRINCIPLES TO THE FLOW IN A PIPE BEND AS SHOWN. THE BEND IS CONNECTED WITH THE REST OF THE PIPING THROUGH PERFECTLY FLEXIBLE BELLOWS SO THAT THE EXTERNAL FORCE  $\bar{F}_E$  IS EXERTED ONLY AT THE SUPPORT E.



## SOLUTION

WE WILL USE THE INTEGRAL EQUATION FORM OF THE CONSERVATION OF LINEAR MOMENTUM EQUATION.

$$-\bar{F}_B = \frac{\partial}{\partial t} \iiint_{VOL} \bar{v} \rho dVol + \iint_{SURF} \bar{v} \rho \bar{v} \cdot \bar{n} dSURF.$$

WHERE

$-\bar{F}_B$  = THE FORCE APPLIED TO THE FLUID BY EXTERNAL SOURCES.

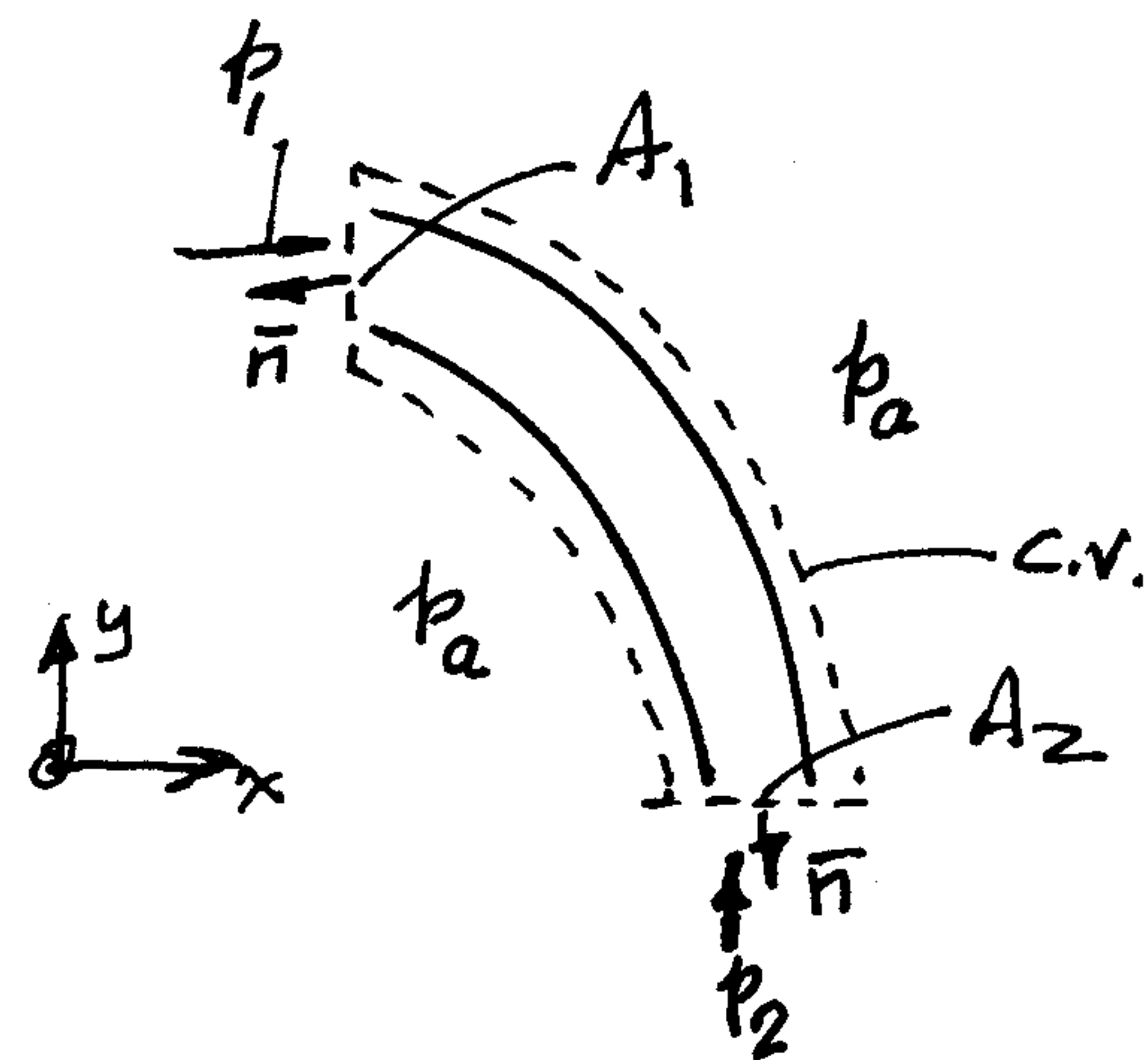
IT IS NOTED THAT THERE IS NO VELOCITY GRADIENT NORMAL TO THE CONTROL VOLUME SURFACE. THIS MEANS (À LA NEWTON) THAT THERE IS NO SHEAR STRESS AT BOUNDARY OF THE CONTROL BOUNDARY. ONLY PRESSURE FORCES ACT ON THE FLUID AT THE CONTROL VOLUME SURFACE.

THE EXTERNAL PRESSURE,  $p_a$  IS UNIFORM. THIS  $p_a$  PRODUCES NO RESULTANT FORCE ACTING ON THE FLUID IN THE CONTROL VOLUME, C.V..

FROM PROBLEM STATEMENT IT IS REASONABLE TO ASSUME:

STEADY FLOW :  $\frac{\partial}{\partial t} ( ) = 0$

NEGLECT GRAVITY :  $\vec{F}_{GRAVITY} = 0$



THE CONSERVATION OF LINEAR MOMENTUM TAKES THE FORM

$$-\vec{F}_B = -\vec{F}_E - \vec{F}_{PRESSURE} = \iint_{SURF} \vec{V} \rho \vec{V} \cdot \vec{n} dSURF$$

$$-\vec{F}_E - \iint_{SURF} p \vec{n} dSURF = -\rho_2 V_2 \vec{j} [ -\vec{j} V_2 \cdot (-\vec{j}) ] A_2 + \rho_1 V_1 \vec{i} [ \vec{i} V_1 \cdot (-\vec{i}) ] A_1$$

$$-\vec{F}_E + \vec{i} (p_1 - p_a) A_1 + \vec{j} (p_2 - p_a) A_2 = -\rho_2 V_2^2 A_2 \vec{j} - \rho_1 V_1^2 A_1 \vec{i}$$

$$\vec{F}_E = \vec{i} ( \rho_1 V_1^2 A_1 + (p_1 - p_a) A_1 ) + \vec{j} ( \rho_2 V_2^2 A_2 + (p_2 - p_a) A_2 )$$

CONSERVATION OF MASS REQUIRES :

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \dot{m} = \text{CONSTANT}$$

HENCE

$$\vec{F}_E = \vec{i} ( \dot{m} V_1 + (p_1 - p_a) A_1 ) + \vec{j} ( \dot{m} V_2 + (p_2 - p_a) A_2 )$$

FINALLY, WE NOTE THAT VISCOSITY HAS NOT BEEN  
NEGLECTED AND WILL HAVE SUBSTANTIAL IMPACT  
ON THE PRESSURE, DENSITY, AND VELOCITY. VISCOSITY  
WILL IMPACT  $\bar{F}_E$ .