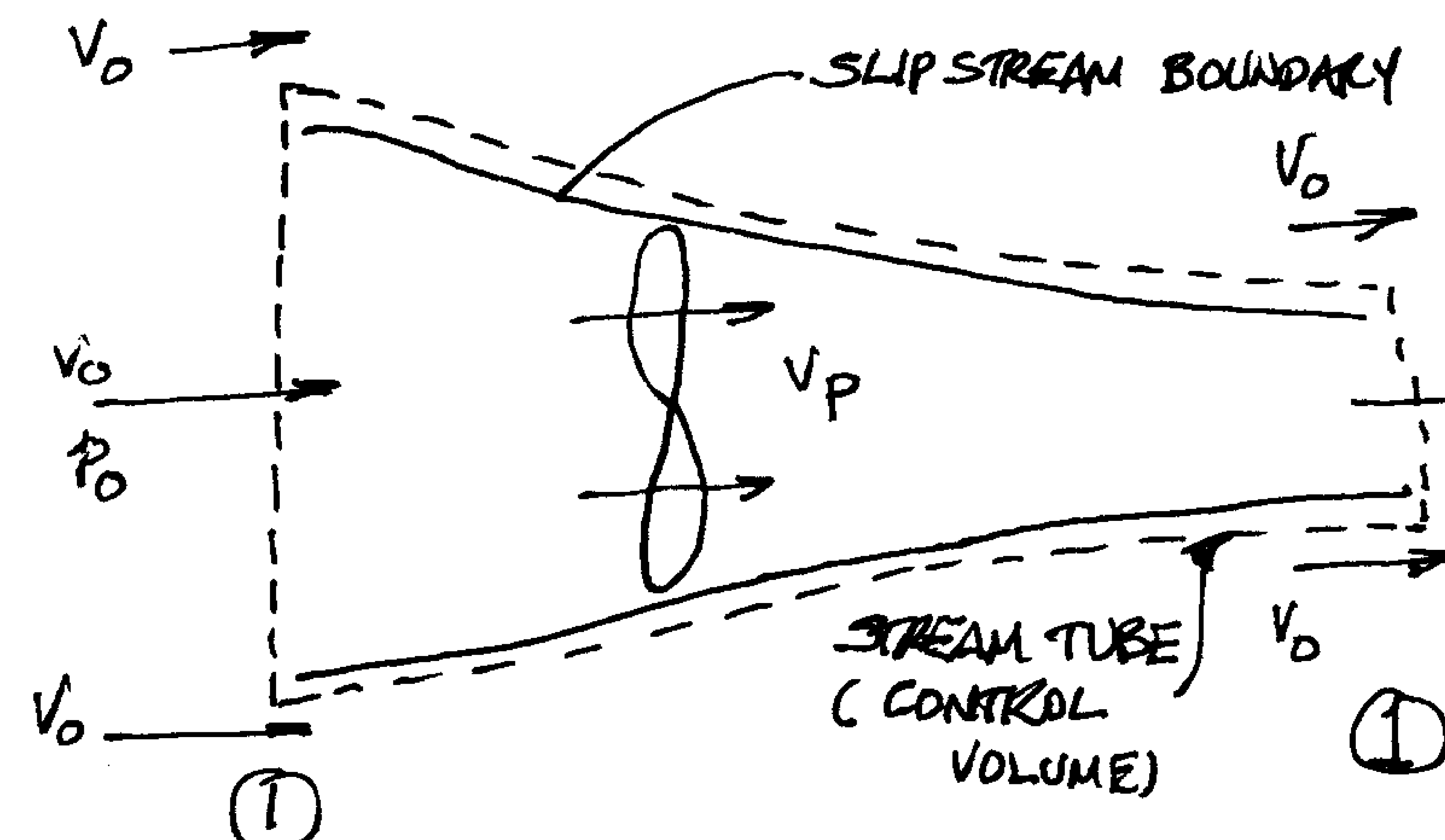
EXAMPLE PROBLEM

CALCULATE THE USEFUL WORK DONE PER UNIT TIME BY A PROPELLER AND THE THEORETICAL PROPELLER EFFICIENCY.

SOLUTION

THE PURPOSE OF THE PROPELLER IS TO PRODUCE A THRUST BY CHANGING THE MOMENTUM OF THE FLUID AROUND IT. TO DO THIS IT MOVES LARGE MASSES OF AIR THROUGH SMALL VELOCITY [COMPARED TO A JET ENGINE] CHANGES.

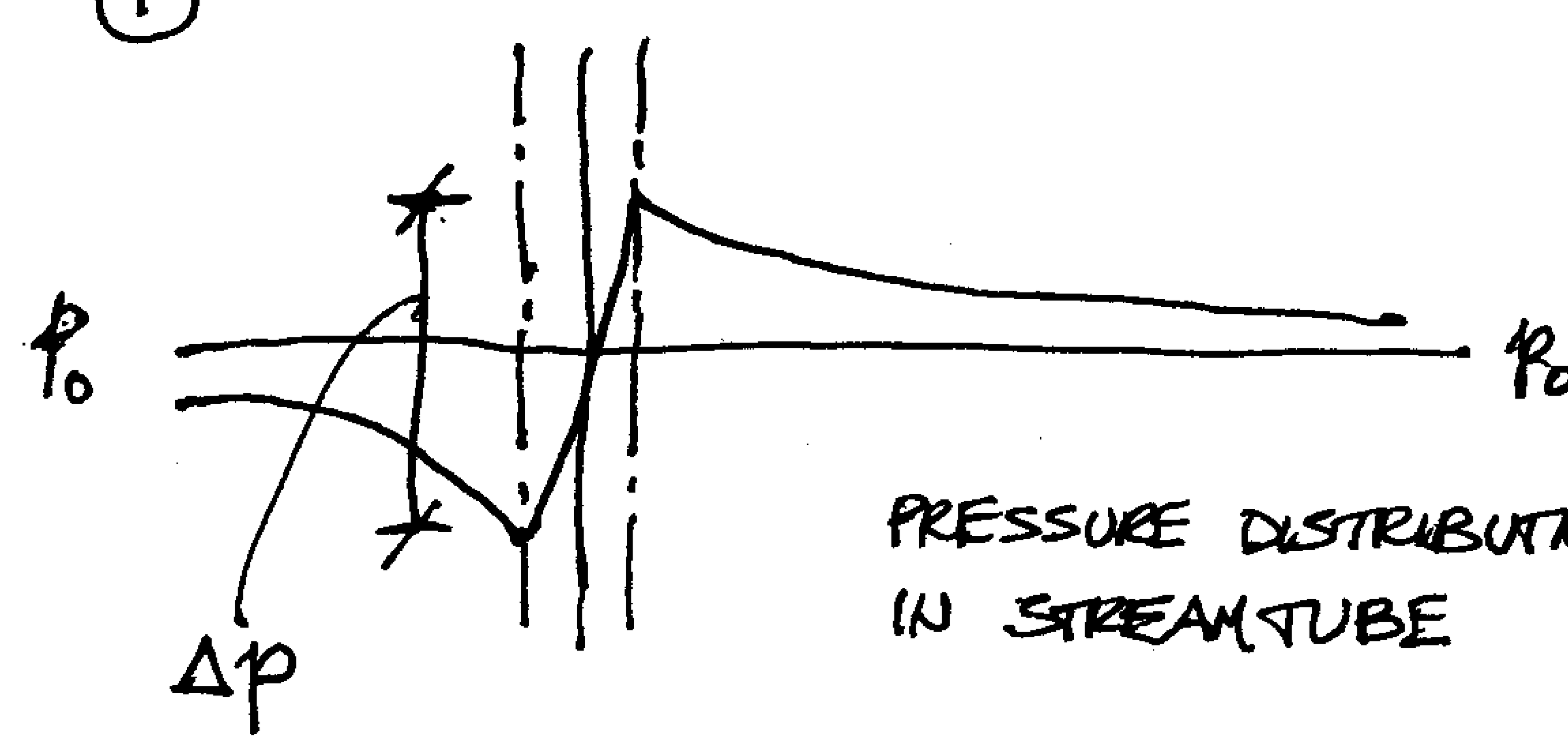
CONSIDER THE FOLLOWING FLOW THROUGH A PROPELLER. THIS WILL BE OUR MODEL OF WHAT TAKES PLACE IN A REAL PROPELLER



NOTE:

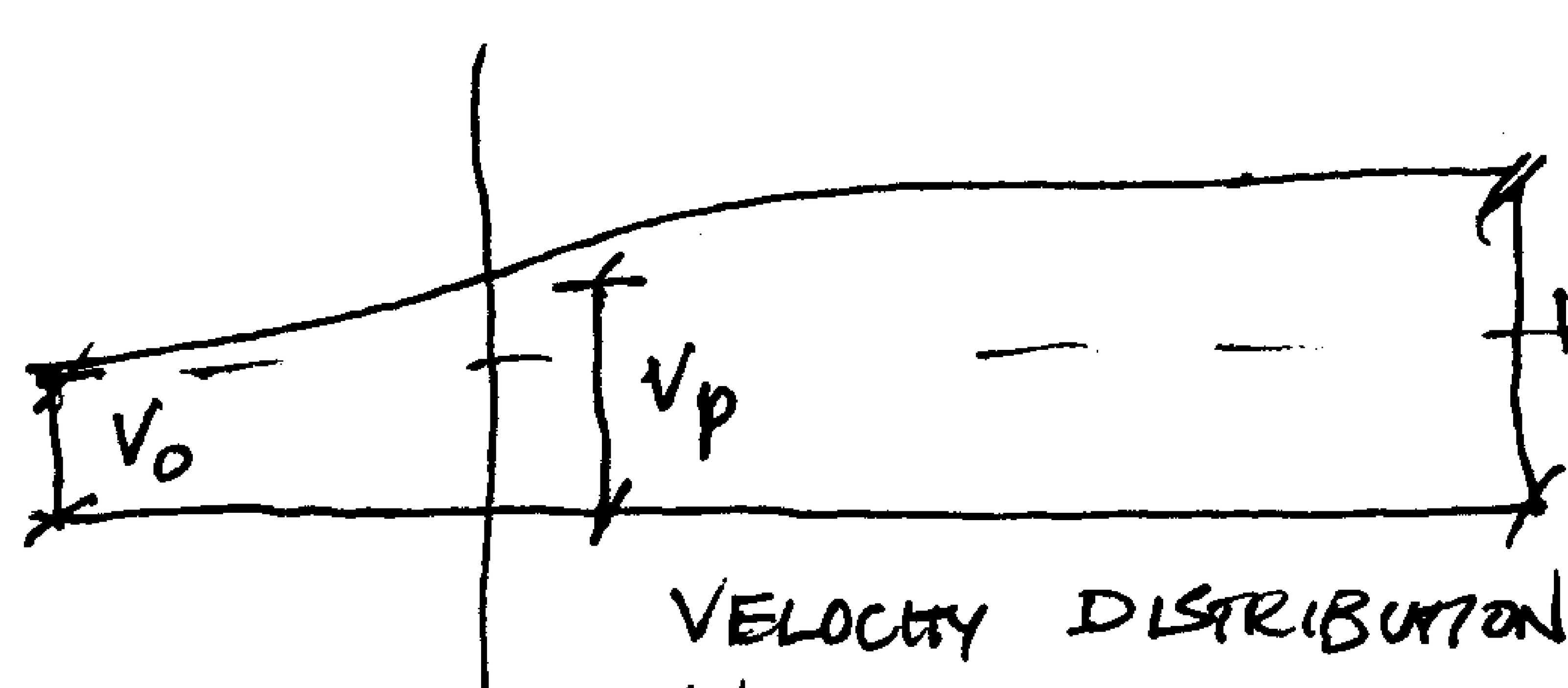
(1) THE PROPELLER IS MOVING IN AN UNDISTURBED STREAM AT SPEED V_o.

(2) RELATIVE TO THE PROPELLER, THE ENTIRE STREAM TO THE LEFT OF THE PROPELLER MOVES TOWARD IT AT SPEED V_o AS SHOWN



PRESSURE DISTRIBUTION IN STREAMTUBE

(3) THE PROPELLER SUCKS THE FLUID TOWARD IT AND IN THE PROCESS ACCELERATES THE FLOW TO A SPEED V_p AT THE PROPELLER.



VELOCITY DISTRIBUTION IN STREAM TUBE

(4) V_p IS CONSTANT ACROSS THE PROPELLER.

(5) THE PRESSURE CHANGES ACROSS THE PROPELLER BY AN AMOUNT Δp.

(6) DOWNSTREAM, THE VELOCITY IN THE STREAMTUBE INCREASES TO A VALUE V_1 WHILE THE PRESSURE DECREASES BACK TO ATMOSPHERIC PRESSURE p_0.

(7) CONSERVATION OF MASS REQUIRES THAT THE STREAMTUBE AREA DECREASES AS IT PROCEEDS DOWNSTREAM [HIGHER VELOCITY, CONSTANT DENSITY).

(8) THE PRESSURE AT ALL POINTS ON THE CONTROL VOLUME SURFACE IS CONSTANT, p_0. HENCE, THERE IS NO RESULTANT PRESSURE FORCE ACTING ON THE CONTROL VOLUME.

(9) SLIP STREAM SURFACE — A SURFACE THAT SUSTAINS A JUMP IN PARALLEL VELOCITY ACROSS ITS BOUNDARY

(10) THE THRUST IMPARTED BY THE PROPELLER ON THE FLUID MUST EQUAL THE CHANGE OF LINEAR MOMENTUM OF THE FLUID FROM STATION 0 TO STATION 1.

(11) ASSUMPTIONS

- STEADY FLOW: $\dfrac{\partial(\ )}{\partial t} = 0$

- INCOMPRESSIBLE FLUID: $\rho = \text{CONSTANT}$

- NEGLECT GRAVITY

WITH THESE ASSUMPTIONS APPLIED TO THIS PROBLEM, THE CONSERVATION OF LINEAR MOMENTUM APPLIED TO OUR CONTROL VOLUME TAKES THE FORM

$$\iint_{SURF} \bar{n}\cdot(\rho\bar{V}\bar{V})\,dSURF = \text{THRUST} = T$$

$$\iint_{A_0} -\bar{\imath}\cdot(\rho V_0 V_0 \bar{\imath}\bar{\imath})\,dSURF + \iint_{A_1} \bar{\imath}\cdot(\ V_1 V_1 \bar{\imath}\bar{\imath})\,dSURF = T\vec{\imath}$$

$$(-\rho V_0^2 A_0 + \rho V_1^2 A_1)\vec{\imath} = T\vec{\imath}$$

NOW APPLY THE CONSERVATION OF MASS PRINCIPLE :

$$\rho V_0 A_0 = \rho V_1 A_1 = \rho V_p A_p = \dot{m} = \text{CONSTANT}$$

COMBINING THESE TWO EQUATIONS, WE OBTAIN:

$$-(\rho V_0 A_0)V_0 + (\rho V_1 A_1)V_1 = T$$

$$(V_1 - V_0)\dot{m} = T$$

$$\boxed{T = \dot{m}(V_1 - V_0)}$$

ACROSS THE PROPELLER DISC, THE THR PER UNIT AREA IS EQUAL TO THE PRESSURE DIFFERENCE ACROSS THE PROPELLER, VIZ.

$$\frac{T}{A_p} = \Delta p$$

$$\frac{\dot{m}(V_1 - V_0)}{A_p} = \Delta p$$

$$\frac{\rho V_p A_p (V_1 - V_0)}{A_p} = \Delta p$$

$$\rho V_p (V_1 - V_0) = \Delta p$$

TO SOLVE FOR Δp, WE APPLY BERNOULLI'S EQUATION TWICE. FIRST, CONSIDER THE UPSTREAM FLOW REGION:

$$p_0 + \tfrac{1}{2}\rho V_0^2 = \left(p_0 - \tfrac{1}{2}\Delta p\right) + \tfrac{1}{2}\rho V_p^2$$

$$\therefore \Delta p = \rho(V_p^2 - V_0^2)$$

SECOND, CONSIDER THE DOWNSTREAM FLOW REGION

$$\left(p_0 + \tfrac{1}{2}\Delta p\right) + \tfrac{1}{2}\rho V_p^2 = p_0 + \tfrac{1}{2}\rho V_1^2$$

$$\therefore \Delta p = \rho(V_1^2 - V_p^2)$$

SOLVING FOR Δp BY ADDING THE TWO EQUATIONS FOR Δp:

$$\Delta p = \tfrac{1}{2}\rho(V_1^2 - V_0^2)$$

EQUATIONS Δp (BERNOULLI BASED AND THRUST DEFINITION BASE) ARE SET EQUAL:

$$\tfrac{1}{2}\rho(V_1^2 - V_0^2) = \rho V_p (V_1 - V_0)$$

$$\frac{1}{2}\rho(V_1+V_0)(V_1-V_0) = \rho V_p(V_1-V_0)$$

SOLVE FOR V_p:

$$V_p = \frac{1}{2}(V_1+V_0)$$

$$2V_p = V_1+V_0$$

$$V_p - V_0 = V_1 - V_p$$

$$V_p - V_0 = \frac{1}{2}(V_1-V_0)$$

THIS RESULT INDICATES THAT THE INCREASE OF VELOCITY (V_p-V_0) DUE THE PROPELLER IS ONE-HALF OF THE TOTAL VELOCITY INCREASE, (V_1-V_0).

BY DEFINITION, THE USEFUL WORK OR POWER OUTPUT IS $\mathcal{P}$:

$$\mathcal{P} = TV_0$$

$$\mathcal{P} = \dot{m}(V_1-V_0)V_0$$

THE TOTAL ENERGY PER UNIT TIME GIVEN TO THE FLUID IS THE DIFFERENCE IN KINETIC ENERGIES FROM STATE 1 TO STATE 0. THIS IS THE AVAILABLE POWER A_{POWER}:

$$A_{POWER} = \frac{1}{2}\dot{m}\left(V_1^2 - V_0^2\right)$$

WHERE

AVAILABLE POWER = USEFUL PROPELLER POWER

+ KINETIC ENERGY LEFT IN STREAM TUBE

SUBSTITUTING

$$\frac{1}{2}\dot{m}\left(V_1^2 - V_0^2\right) = \dot{m}(V_1 - V_0)V_0 + \frac{1}{2}\dot{m}\left(V_1 - V_0\right)^2$$

THE THEORETICAL PROPELLER EFFICIENCY IS THE RATIO OF THE USEFUL PROPELLER POWER TO THE AVAILABLE POWER

$$\eta = \frac{\dot{m}(V_1 - V_0)V_0}{\frac{1}{2}\dot{m}\left(V_1^2 - V_0^2\right)}$$

$$\eta = \frac{2V_0}{V_0 + V_1}$$

$$\eta = \frac{V_0}{V_p}$$