# CONFORMAL TRANSFORMATION

A CONFORMAL TRANSFORMATION CONSISTS OF MAPPING A REGION OF ONE PLANE ON ANOTHER PLANE IN SUCH A MANNER THAT THE DETAILED SHAPE OF INFINITESIMAL ELEMENTS OF AREA ARE NOT CHANGED.

RECALL THAT EQUIPOTENTIAL LINES AND STREAMLINES INTERSECT AT RIGHT ANGLES, THUS DIVIDING THE FLOW FIELD INTO A NUMBER OF RECTANGLES.

SINCE

$$w_1 = f(z), \quad z = x + iy$$

$$w_2 = g(\zeta), \quad \zeta = \xi + i\eta$$

REPRESENT TWO DIFFERENT FLOW PATTERNS. NOW IF WE CONSIDER

$$w_1 = f(z) = w_2 = g(\zeta)$$

$$f(z) = g(\zeta)$$

THEN THE EQUIPOTENTIAL LINES AND STREAMLINES IN EITHER PLANE DIVIDE THE PLANE INTO RECTANGLES. THESE RECTANGLES ARE SIMILAR AT CORRESPONDING POINTS IN BOTH PLANES.

THE EQUATION

$$f(z) = g(\zeta)$$

REPRESENTS THE CONFORMAL TRANSFORMATION FROM THE $z$-PLANE TO THE $\zeta$-PLANE OR THE CONVERSE. WE SHALL ASSUME THE FLOW IN THE $z$-PLANE, $f(z)$, IS KNOWN AND THE CORRESPONDING FLOW $g(\zeta)$ IN THE $\zeta$-PLANE IS DESIRED. TO PLOT THE FLOW KNOWN ON THE $z$-PLANE ONTO THE $\zeta$-PLANE, IT IS REQUIRED TO SOLVE THE CONFORMAL TRANSFORMATION FOR $\zeta$, VIZ.

$$\zeta = h(z)$$

THE VELOCITIES ARE RELATED THROUGH THE TRANSFORMATION:

$$\frac{dw}{dz} = u - iv$$

$$\frac{dw}{d\zeta} = \frac{dw}{dz}\,\frac{dz}{d\zeta}$$

EXAMPLE PROBLEM

CONSIDER THE FOLLOWING TRANSFORMATION

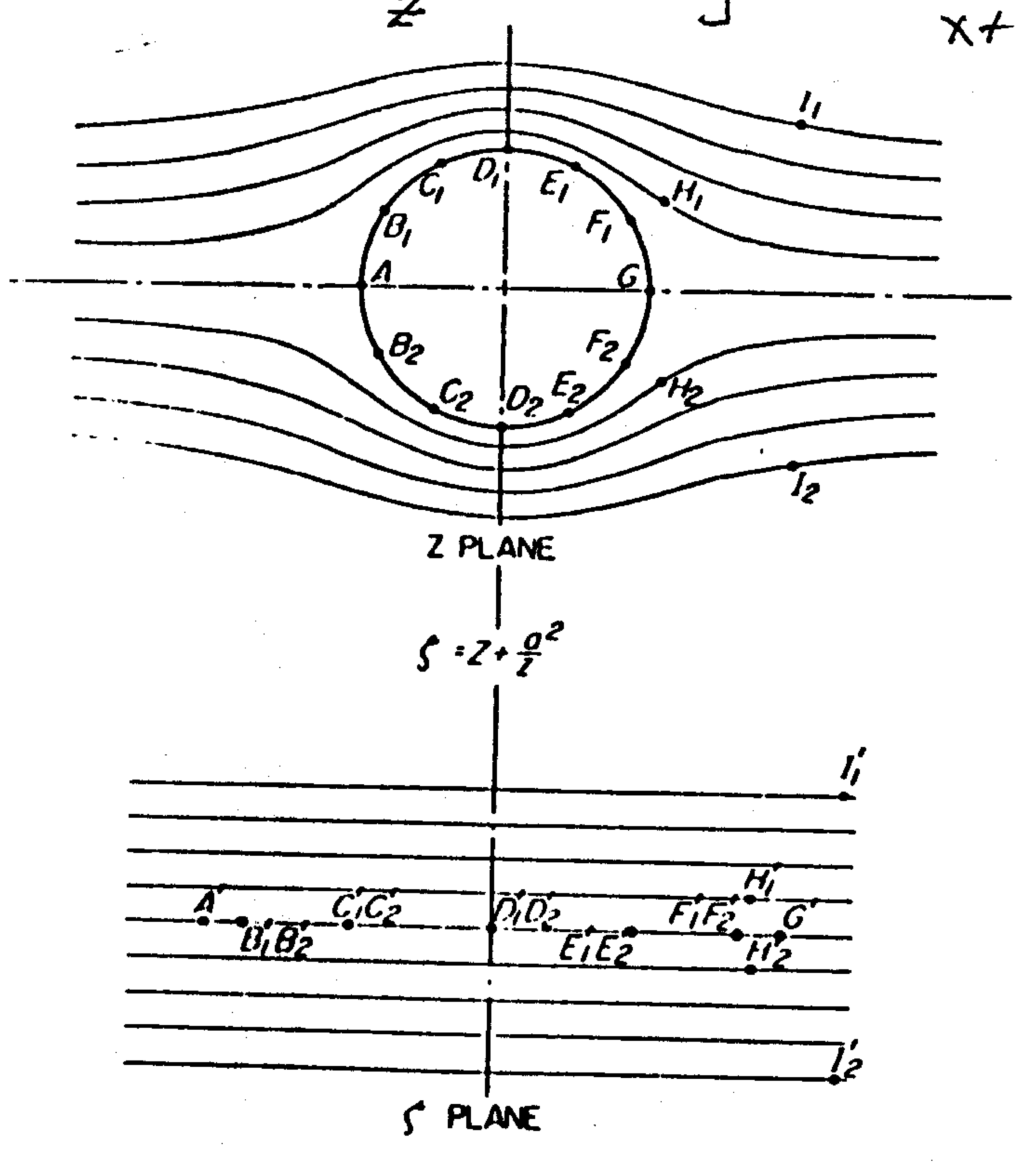
$$w = U_\infty \left(z + \frac{a^2}{z}\right) = U_\infty \zeta$$

CIRCULAR CYLINDER
$z$-PLANE

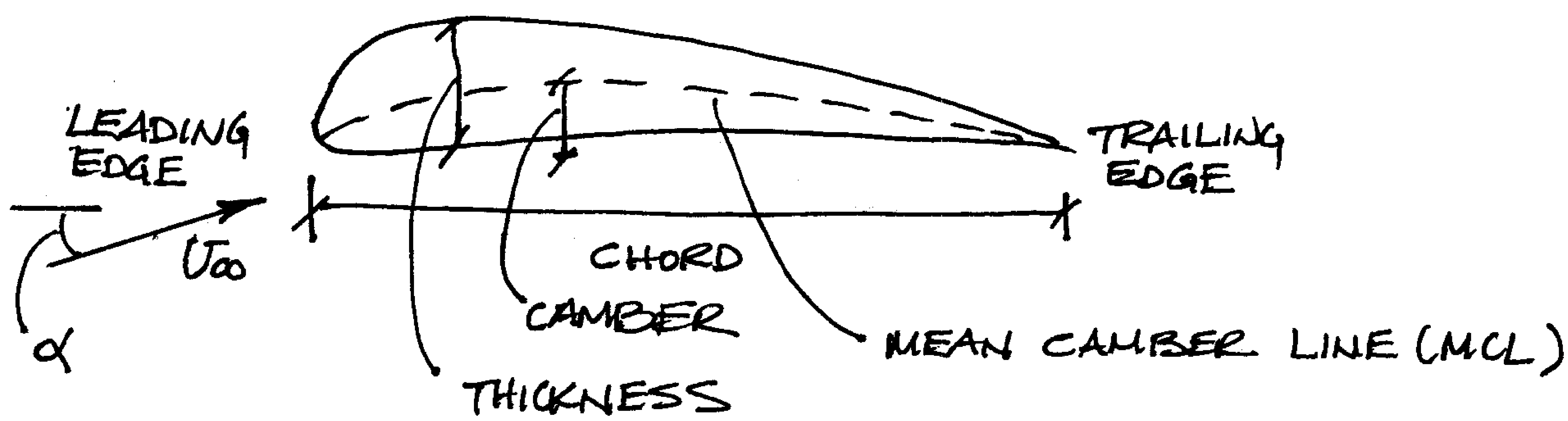
UNIFORM FLOW PARALLEL
TO $\xi$-AXIS
$\zeta$-PLANE

CORRESPONDING POINTS OF BOTH PLANES ARE OBTAINED FROM THE RELATION

$$\zeta = z + \frac{a^2}{z} = x + iy + \frac{a^2}{x+iy}$$



# AIRFOIL TERMS, DEFINITIONS




$\alpha$ = ANGLE OF ATTACK

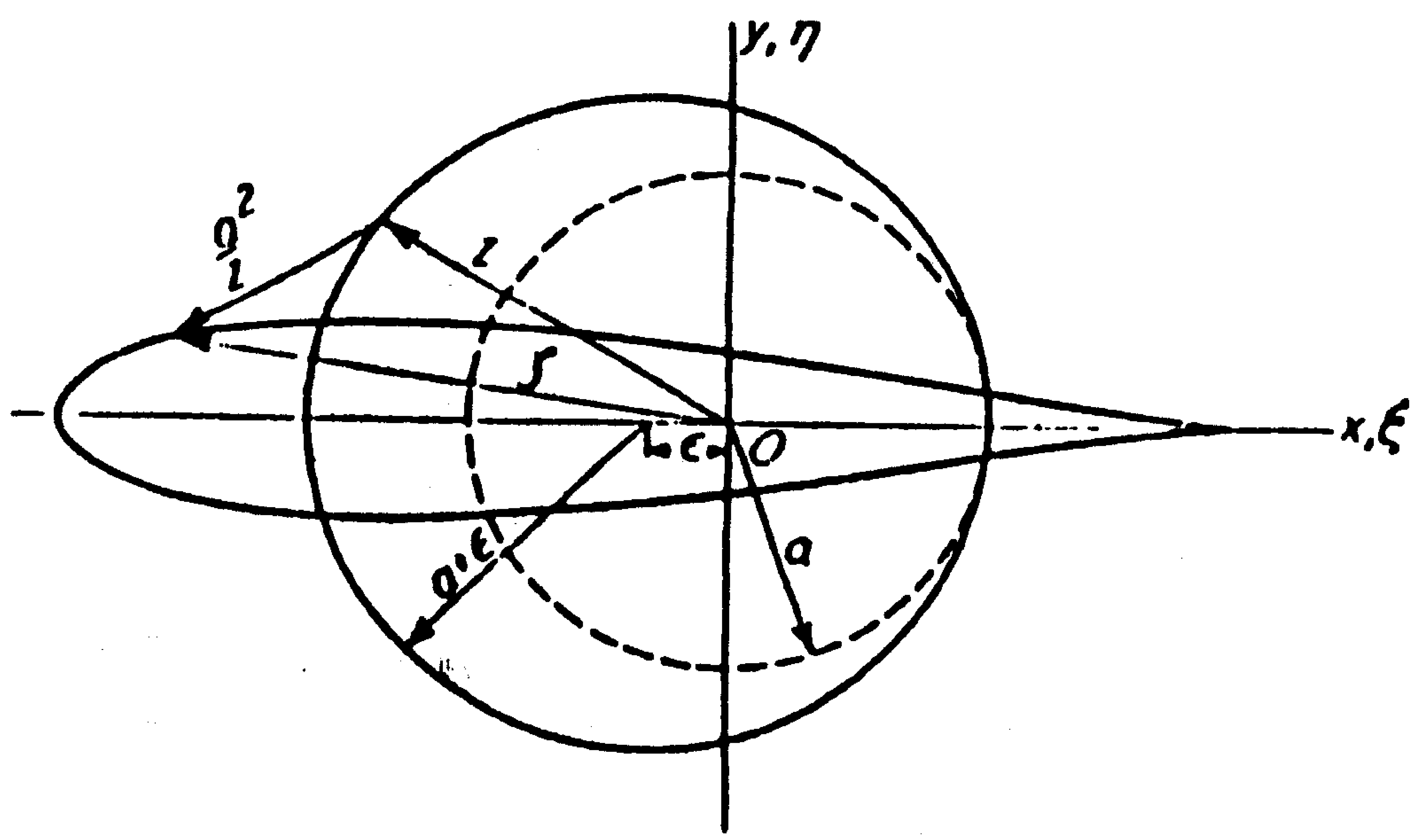
CAMBER IS THE MAXIMUM DISTANCE THE MEAN CAMBER LINE IS DISPLACED ABOVE/FROM THE CHORD LINE

## TRANSFORMATION OF A CIRCLE INTO A WING SECTION (AIRFOIL)

A CIRCLE CAN BE TRANSFORMED INTO A SHAPE RESEMBLING THAT OF A WING SECTION BY THE SUBSTITUTION OF THE VARIABLE

$$\zeta = z + \frac{a^2}{z}$$

INTO THE EXPRESSION FOR THE FLOW ABOUT A CIRCULAR CYLINDER HAVING A RADIUS SLIGHTLY LARGER THAN $a$, AND SO PLACED THAT THE CIRCUMFERENCE PASSES THROUGH THE POINT $x = a$. IF, IN ADDITION, THE CENTER OF THE LARGER CYLINDER IS PLACED ON THE X-AXIS, THE TRANSFORMED CURVE WILL BE THAT OF A SYMMETRICAL AIRFOIL.



EXAMPLE

LET THE CENTER OF THE LARGER CYLINDER BE PLACED AT THE POINT $x = -\epsilon$ WHERE $\epsilon$ IS A REAL, POSITIVE, SMALL QUANTITY. THE RADIUS OF THE LARGER CYLINDER IS $a+\epsilon$. THE EQUATION OF FLOW ABOUT A CYLINDER WITH CIRCULATION $\Gamma$:

$$w = U_\infty \left[ (z^* + \epsilon) + \frac{(a+\epsilon)^2}{(z^*+\epsilon)} \right] + i\frac{\Gamma}{2\pi} \ln \frac{(z^*+\epsilon)}{(a+\epsilon)}$$

THE MORE GENERAL EXPRESSION FOR THE FLOW ABOUT THE CIRCULAR CYLINDER WITH THE FLOW AT INFINITY INCLINED AT AN ANGLE $\alpha_0$ TO THE X-AXIS IS FOUND BY SUBSTITUTING THE EXPRESSION

$$z+\epsilon = (z^* + \epsilon) e^{i\alpha_0}$$

$$w = U_\infty \left[ (z+\epsilon) e^{-i\alpha_0} + \frac{(a+\epsilon)^2 e^{i\alpha_0}}{(z+\epsilon)} \right] + i\frac{\Gamma}{2\pi} \ln \frac{(z+\epsilon) e^{-i\alpha_0}}{(a+\epsilon)}$$

SUBSTITUTING THE TRANSFORMATION EQUATION

$$\zeta = z + \frac{a^2}{z}$$

INTO THE ABOVE EQUATION YIELDS THE DESIRED FLOW ABOUT AN AIRFOIL; BUT THIS WOULD BE A COMPLICATED EXPRESSION.

A SIMPLE METHOD OF OBTAINING THE AIRFOIL SHAPE IS TO SELECT VALUES OF $z$ CORRESPONDING TO POINTS ON THE LARGER CYLINDER AND THEN USE THE TRANSFORMATION EQUATION TO FIND CORRESPONDING POINTS IN THE $\zeta$-PLANE.

THE VELOCITY AT ANY POINT ON THE WING SECTION OR AIRFOIL IS COMPUTED AS FOLLOWS:

$$\frac{dW}{d\zeta} = \frac{dw}{dz}\frac{dz}{d\zeta} = \left[U_\infty\left(e^{-i\alpha_0} - \frac{(a+\epsilon)^2 e^{i\alpha_0}}{(z+\epsilon)^2}\right) + i\frac{\Gamma}{2\pi(z+\epsilon)}\right]\left(\frac{z^2}{z^2-a^2}\right)$$

KUTTA CONDITION

FROM THE EXPRESSION FOR $\frac{dw}{d\zeta}$ ABOVE, THE VELOCITY AT THE POINT $z = a$ IS INFINITE UNLESS THE FIRST FACTOR IS ZERO. THE POINT $z = a$ IS THE TRAILING EDGE OF THE AIRFOIL. THE KUTTA CONDITION STATES THAT THE VALUE OF THE CIRCULATION, $\Gamma$, IS SUCH THAT THE FIRST FACTOR EQUALS TO ZERO. THIS CONDITION ENSURES SMOOTH FLOW AT THE TRAILING EDGE, $z=a$. HENCE:

$$U_\infty\left[e^{-i\alpha_0} - \frac{(a+\epsilon)^2}{(a+\epsilon)^2}e^{i\alpha_0}\right] + i\frac{\Gamma}{2\pi(a+\epsilon)} = 0$$

OR

$$\Gamma = 4\pi(a+\epsilon)\,U_\infty \sin\alpha_0$$

THE LEADING EDGE OF THE AIRFOIL CORRESPONDS TO THE POINT:

$$\zeta = -a - 2\epsilon - \frac{a^2}{a+2\epsilon}$$

OR

$$\zeta \cong -2a$$

NOTE THAT THE TRAILING EDGE OF THE AIRFOIL CORRESPONDS TO THE POINT

$$\zeta = 2a$$

HENCE, THE AIRFOIL CHORD IS $4a$.

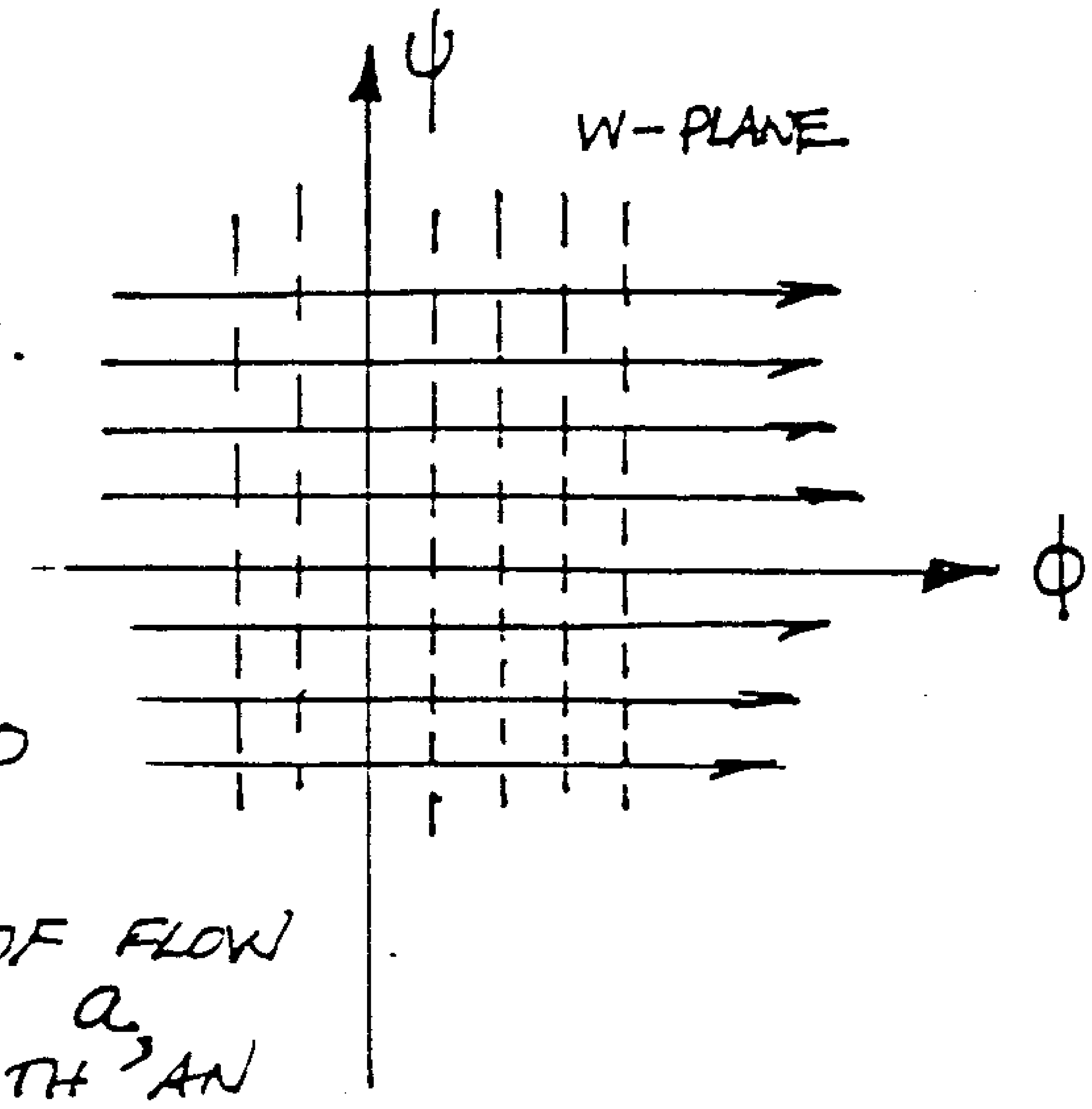
ALSO, RECALL THAT THE LIFT IS GIVEN BY THE EXPRESSION

$$L = \rho U_\infty \Gamma$$

HENCE:

$$C_\ell = 2\pi \left(1 + \frac{\epsilon}{a}\right) \sin\alpha_0$$

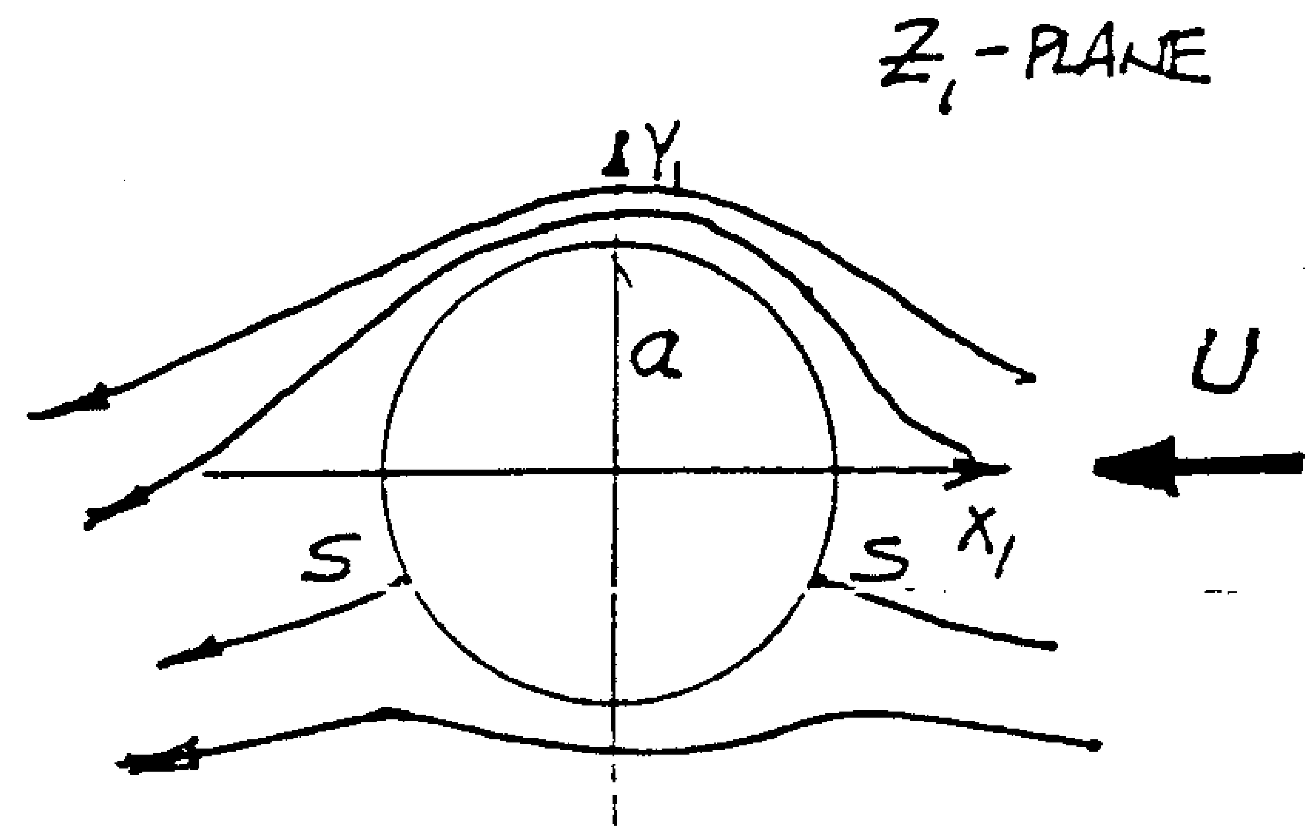
FLOW PAST A JOUKOWSKI AIRFOIL WITH ANGLE OF ATTACK $\alpha$ AND CIRCULATION $\Gamma = K$.




i) W-PLANE TO $Z_1$-PLANE

$$W(z_1) = -U\left(z_1 + \frac{a^2}{z_1}\right) - i\frac{K}{2\pi}\ln z_1\,, \quad K > 0$$

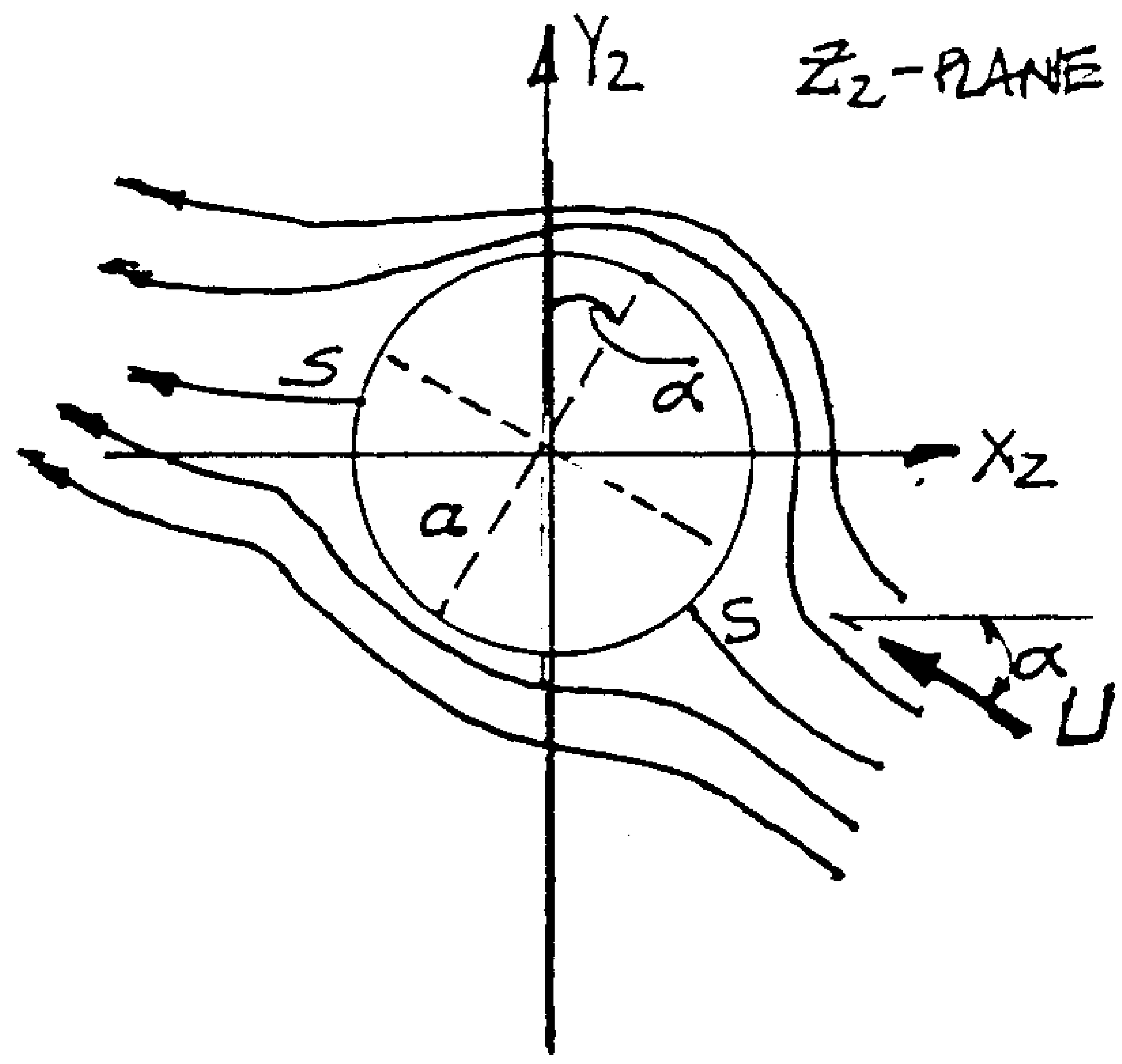
THIS TRANSFORMATION YIELDS THE PATTERN OF FLOW PAST A CIRCULAR CYLINDER OF RADIUS $a$, AT THE ORIGIN OF THE $Z_1$-PLANE, WITH AN ANTICLOCKWISE CIRCULATION, $K$.




ii) $Z_1$-PLANE TO $Z_2$-PLANE

$$z_2 = z_1 e^{-i\alpha}$$

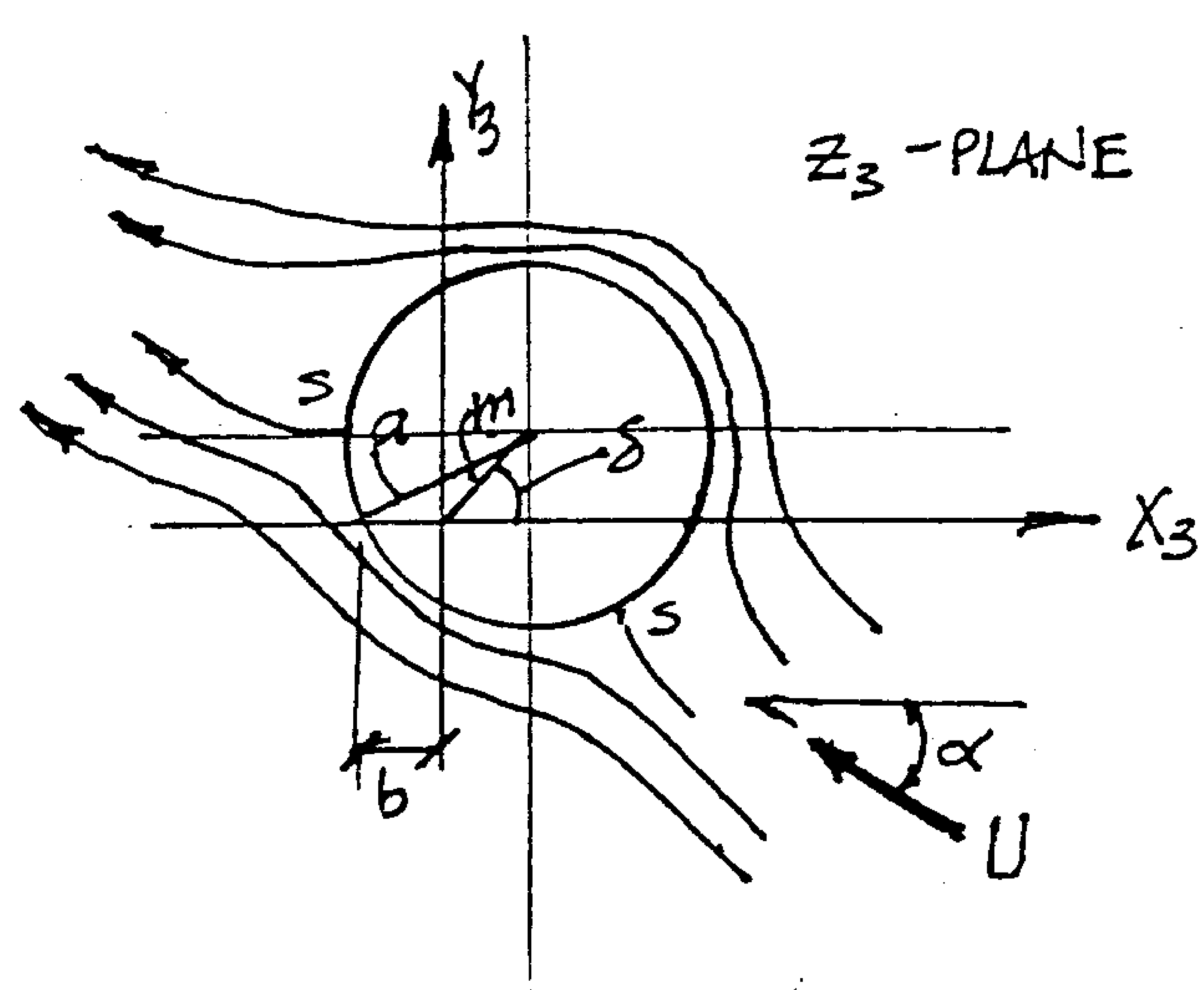
THIS TRANSFORMATION ROTATES THE PATTERN THROUGH THE ANGLE, $-\alpha$.




iii) $Z_2$-PLANE TO $Z_3$-PLANE

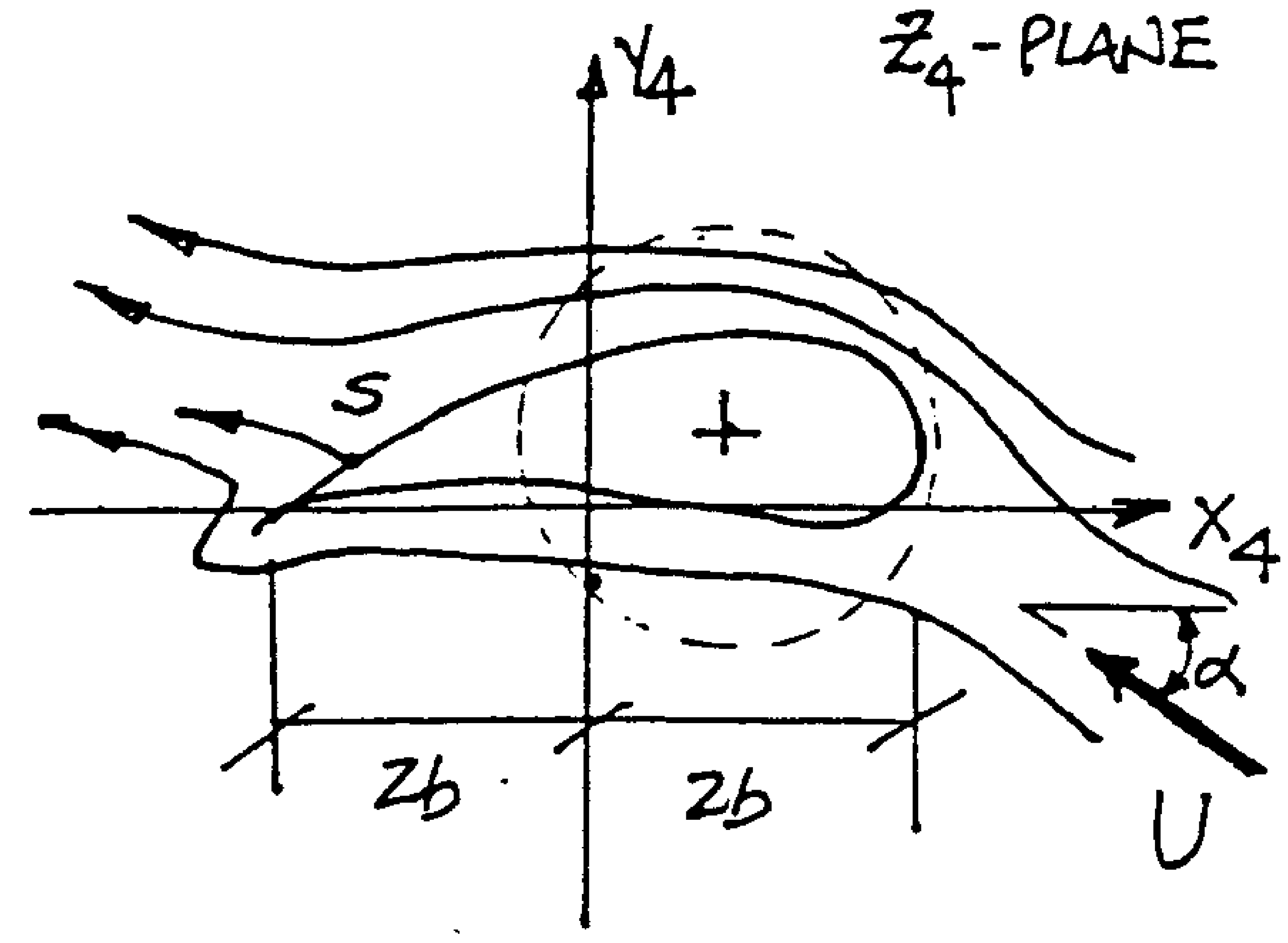
$$z_3 = z_2 + m e^{i\delta}$$

THIS TRANSFORMATION REMOVES THE $a$-CIRCLE A DISTANCE $m$ FROM THE ORIGIN IN THE DIRECTION $\delta$.



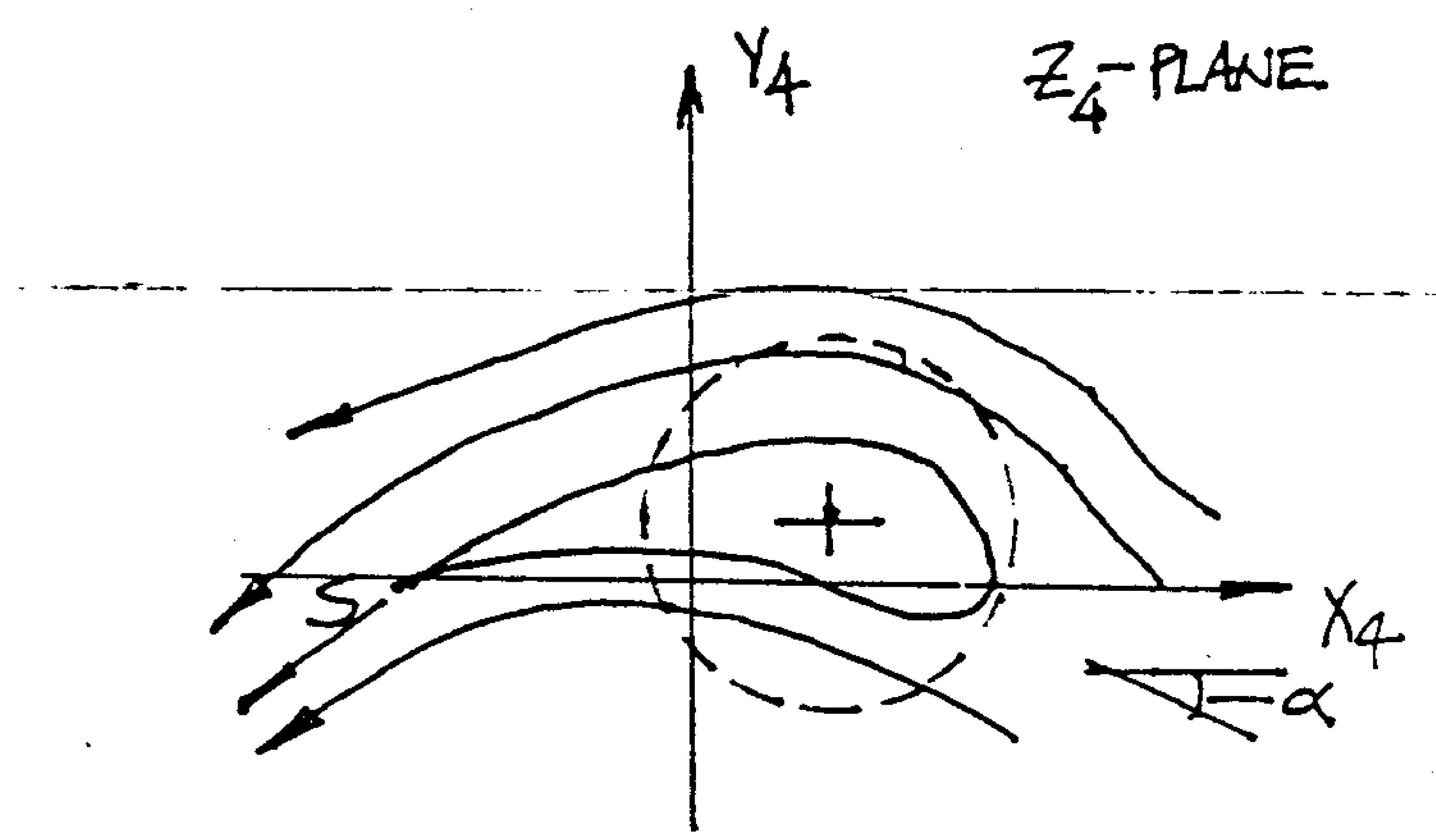
iv) $z_3$-PLANE TO $z_4$-PLANE

$$z_4 = z_3 + \frac{b^2}{z_3} \quad , \quad b < a$$




THIS TRANSFORMATION TAKES THE a-CIRCLE TO AN AIRFOIL. IF APPROPRIATE VALUE OF K IS USED, THE REAR STAGNATION POINT OCCURS AT THE TRAILING EDGE.

FIND THIS PARTICULAR VALUE OF K.



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