LIFT AND PITCHING MOMENT IN STEADY, TWO-DIMENSIONAL POTENTIAL FLOW

FLOW PAST A SLIGHTLY CAMBERED AIRFOIL

THIN AIRFOIL THEORY

GENERAL COMMENTS

WE NOTE THAT THE PHYSICAL DIFFERENCE BETWEEN FLOW OVER A CYLINDER WITH FINITE CIRCULATION Γ AND FLOW OVER A CYLINDER WITH ZERO CIRCULATION ($\Gamma = 0$) IS THE ADDED CONSTANT VELOCITY COMPONENT THAT IS PROPORTIONAL TO THE CIRCULATION. RECALL THE FOLLOWING EXPRESSIONS:

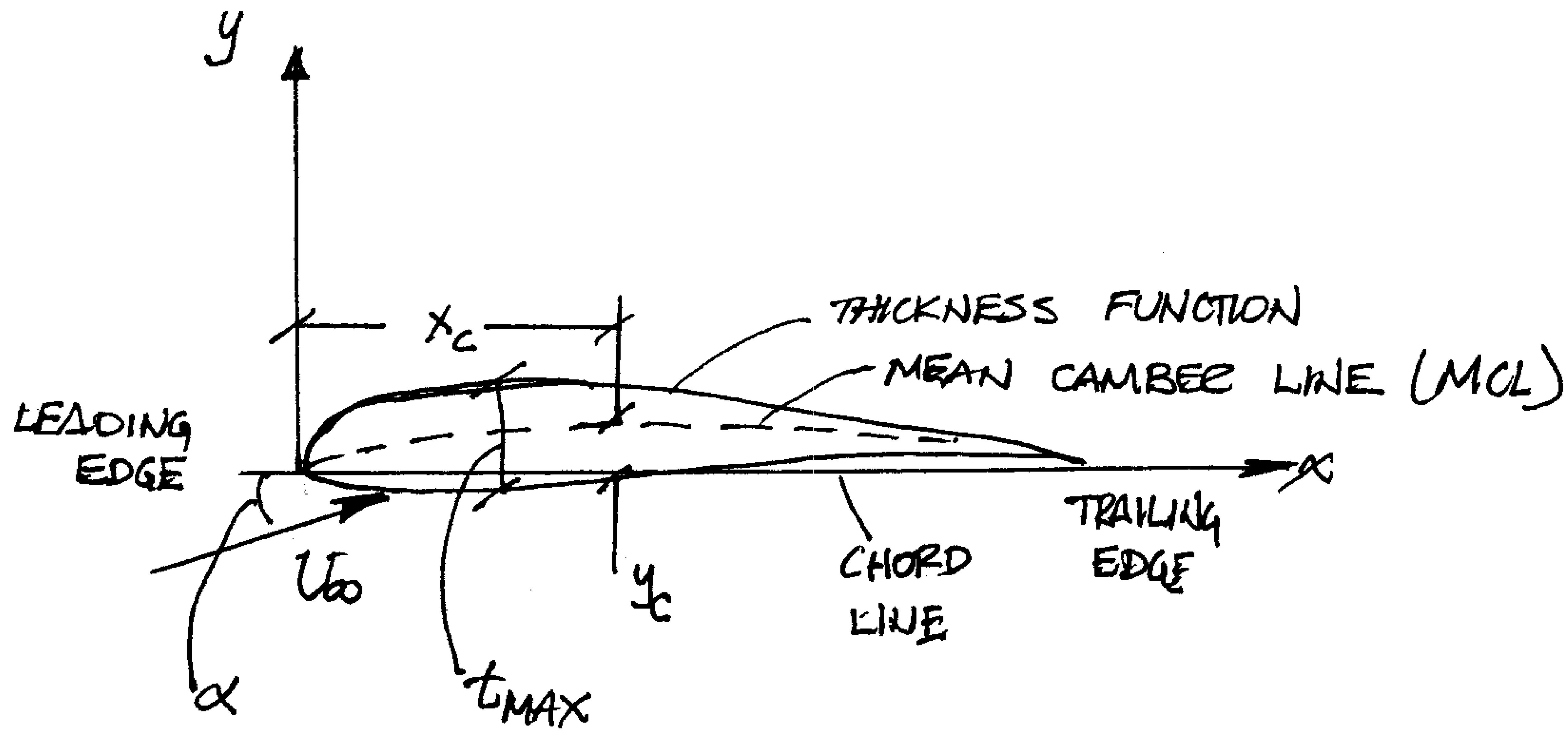
$$u_{\theta_{\substack{\Gamma=0\\ r=a}}} = 2U_\infty \sin\theta \qquad\qquad u_{\theta_{\substack{\Gamma=\Gamma_1\\ r=a}}} = 2U_\infty \sin\theta + \frac{\Gamma_1}{2\pi a}$$

THERE IS OBVIOUSLY AN INCREASE IN VELOCITY ON THE UPPER SURFACE AND A DECREASE IN VELOCITY ON THE LOWER SURFACE OF THE CYLINDER CORRESPONDING TO $\Gamma = \Gamma_1$. THE PRESSURE ON THESE SURFACES OF THE CYLINDER WILL BE LOWER AND HIGHER RESPECTIVELY, THAN THEIR VALUES FOR THE CASE OF NO ROTATION, $\Gamma = 0$.

THE VORTEX PRODUCES AN ADDITIONAL INCREMENT OF VELOCITY IN THE SURROUNDING FLOW FIELD.

WE SHALL EXTEND THIS POINT VORTEX MODEL TO A VORTEX SHEET IN ORDER TO ANALYZE FLOW PAST AN AIRFOIL.

NOW FOR A FEW DEFINITIONS, WE SHALL DEFINE OUR AIRFOIL AS FOLLOWS:



THE CHORD LINE IS DEFINED AS THE STRAIGHT LINE CONNECTING THE LEADING AND TRAILING EDGES.

THE GEOMETRIC ANGLE OF ATTACK IS THE ANGLE BETWEEN THE CHORD LINE AND THE DIRECTION OF THE UNDISTURBED "FREE STREAM" FLOW

THE LOCUS OF THE POINTS MIDWAY BETWEEN THE UPPER SURFACE AND THE LOWER SURFACE, AS MEASURED PERPENDICULAR TO THE CHORD LINE, DEFINES THE MEAN CAMBER LINE (MCL).

t_{MAX} IS THE MAXIMUM THICKNESS OF THE AIRFOIL AS MEASURED PERPENDICULAR TO THE CHORD LINE.

y_c IS THE MAXIMUM MEAN CAMBER REFERENCED TO THE CHORD LINE

x_c IS THE LOCATION OF y_c AFT THE LEADING EDGE AS MEASURED ALONG THE CHORD LINE.

IN THIN AIRFOIL THEORY, THE AIRFOIL IS REPLACED BY ITS MEAN CAMBER LINE.

THE THIN AIRFOIL MODEL REPLACES THE AIRFOIL BY ITS MEAN CAMBER LINE WHICH IN TURN IS REPRESENTED BY A VORTEX SHEET CONSISTING OF A CONTINUOUS DISTRIBUTION OF POINT VORTICES, $\gamma(x)$.

THIS MODEL IS CONSTRAINED BY THE USUAL <u>BOUNDARY CONDITIONS</u>:

1. UNIFORM STREAM, U, AND INDUCED FLOW FIELD DUE TO THE VORTEX SHEET MUST SATISFY CONTINUITY AND IRROTATIONALITY.

2. AT INFINITY, $U = U_\infty$

3. KUTTA CONDITION: $\gamma(x = x_{TE}) = 0$

4. VORTEX SHEET CORRESPONDS TO A STREAMLINE; HENCE, THE NORMAL COMPONENTS OF VELOCITY MUST TOTAL TO ZERO.

WE NOW WILL PUT OUR MODEL TOGETHER, ONE COMPONENT AT A TIME UNTIL WE HAVE THE COMPLETE PUZZLE/PICTURE.

CONSTRUCTION OF THE THIN AIRFOIL THEORY MODEL

CONSIDER A SLIGHTLY CAMBERED AIRFOIL AT A SMALL ANGLE OF ATTACK OR INCIDENCE.

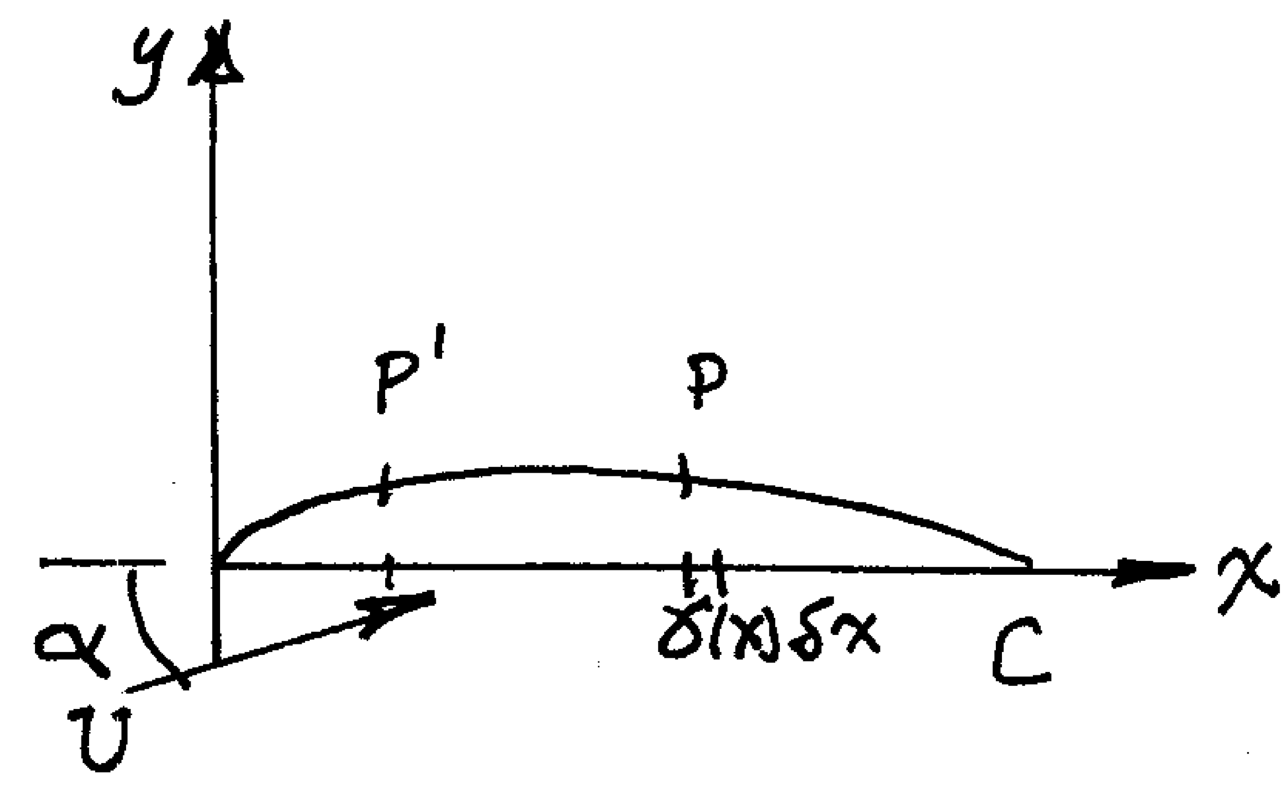
THE CHALLENGE IS TO DETERMINE $\gamma(x)$ IN TERMS OF THE GEOMETRIC PROPERTIES OF THE AIRFOIL AND THEN TO COMPUTE THE PRESSURE DISTRIBUTION. ONCE THE PRESSURE DISTRIBUTION IS KNOWN, THE LIFT AND MOMENT DISTRIBUTIONS ARE CALCULATED DIRECTLY.

ASSUME:

$\alpha \ll 1$

$y_c/c \ll 1$

$t_{max}/c \ll 1$



NOW $\gamma(x)$ IS DISTRIBUTED ALONG THE CHORD LINE AND NOT THE MEAN CAMBER LINE.

THE INDUCED VELOCITY COMPONENTS AT THE POINT P' ARE u AND v WHERE

$$\frac{u}{U} \ll 1 \quad , \quad \frac{v}{U} \ll 1$$

THEREFORE THE SLOPE OF THE MEAN CAMBER LINE (MCL) OR THAT OF THE RESULTING FLOW AT THE POINT P' RELATIVE TO THE X-AXIS IS:

$$\text{SLOPE (MCL)} = \frac{U\sin\alpha + v}{U\cos\alpha + u}$$

$$\approx \frac{U\alpha + v}{U}$$

$$\approx \alpha + \frac{v}{U} + \text{H.O.T.}$$

OR

$$\left(\frac{dy}{dx}\right)_{P'}^{MCL} = \alpha + \frac{v}{U} \qquad (A)$$

WE NOW RELATE v TO $\gamma(x)$. ACCORDING TO THE BIOT-SAVART LAW A VORTEX ELEMENT AT THE POINT P INDUCES A FIELD AT THE POINT P' GIVEN BY

$$dv = \frac{\gamma(x)\,\delta x}{2\pi\,(x - x')}$$

HENCE, THE NET INDUCED VELOCITY AT THE POINT P' DUE TO THE DISTRIBUTION OF $\gamma(x)$ IS:

$$v = \int_0^c \frac{\gamma(x)\,dx}{2\pi(x - x')} \qquad (B)$$

COMBINING EQNS (A) AND (B), WE HAVE:

$$\int_0^c \frac{\gamma(x)\,dx}{2\pi(x - x')} = U\left[\left(\frac{dy}{dx}\right)_{p'}^{MCL} - \alpha\right] \qquad (C)$$

NOTE $\left(dy/dx\right)_{p'}^{MCL} = 0$ DEFINES A <u>SYMMETRIC AIRFOIL</u>.

A GOOD APPROXIMATION FOR $\gamma(x)$ IS ONE THAT INVOLVES THE SHAPE OF THE MCL, ANGLE OF ATTACK, AND IS FINITE EVERYWHERE INCLUDING THE POINT AT THE LEADING EDGE.

EQUATION (C) IS THE MASTER EQUATION OF THIN AIRFOIL THEORY.

TO SOLVE EQN (C), WE INTRODUCE THE FOLLOWING COORDINATE TRANSFORMATION:

$$x = \frac{c}{2}\left(1 - \cos\theta\right) = c\sin^2(\theta/2) \qquad (D)$$

WE ASSUME:

$$\frac{\gamma(\theta)}{2U} = A_0 \cot\frac{\theta}{2} + \sum_{n=1}^{\infty} A_n \sin(n\theta) \qquad (E)$$

WE ARE ASSURED THAT THE ABOVE EXPRESSION SATISFIES THE KUTTA CONDITION AT THE TRAILING EDGE,

$$\gamma(x = x_{TE}) = 0$$

COMBINING EQNS. (C), (D), AND (E) AND USING FOURIER ANALYSES, WE OBTAIN

$$\left(\frac{dy}{dx}\right)_\theta = (\alpha - A_0) + \sum_1^\infty A_n \cos(n\theta)$$

$$A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \left(\frac{dy}{dx}\right)_\theta d\theta$$

$$A_1 = \frac{2}{\pi}\int_0^\pi \left(\frac{dy}{dx}\right)_\theta \cos\theta \, d\theta$$

$$A_n = \frac{2}{\pi}\int_0^\pi \left(\frac{dy}{dx}\right)_\theta \cos(n\theta) d\theta$$

WE NOW DIRECTLY CALCULATE THE FOLLOWING:

$$l' = \rho U \Gamma = \rho U \int_0^c \gamma(x) dx = \rho U \int_0^\pi \gamma(\theta) \frac{c}{2} \sin\theta \, d\theta$$

$$= \rho U^2 c \int_0^\pi \left[A_0 (1 + \cos\theta) + \sum_1^\infty A_n \sin(n\theta) \sin\theta \right] d\theta$$

$$l' = \rho U^2 c \left(\pi A_0 + \frac{\pi}{2} A_1 \right)$$

$$c_\ell = \frac{\ell'}{\frac{1}{2}\rho U^2 c} = \pi(2A_0 + A_1)$$

$$c_\ell = 2\pi\alpha + \int_0^\pi \left(\frac{dy}{dx}\right)_\theta (\cos\theta - 1)\,d\theta$$

$$\frac{dc_\ell}{d\alpha} = 2\pi$$

$$c_\ell = 2\pi(\alpha - \alpha_0) \quad ; \quad \alpha_0 \equiv \text{ANGLE OF ZERO LIFT}$$

$$\alpha_0 = -\frac{1}{\pi}\int_0^\pi \left(\frac{dy}{dx}\right)_\theta (\cos\theta - 1)\,d\theta$$

$$\overset{\curvearrowleft +}{m}_{LE} = \int_0^c \rho U \gamma(x)\cdot x \cdot dx$$

$$= \frac{1}{2}\rho U^2 c^2 \int_0^\pi (\cos\theta - 1)\left[A_0(1+\cos\theta) + \sum_1^\infty A_n \sin(n\theta)\sin\theta\right] d\theta$$

$$\overset{\curvearrowleft}{m}_{LE} = -\frac{\pi}{4}\rho U^2 c^2 \left(A_0 + A_1 - \frac{A_2}{2}\right)$$

$$c_{m_{LE}} = \frac{\overset{\curvearrowleft}{m}_{LE}}{\frac{1}{2}\rho U^2 c^2} = -\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2}\right)$$

$$c_{m_{LE}} = -\frac{c_\ell}{4} + c_{m_0}$$

$$c_{m_0} = \frac{\pi}{4}(A_2 - A_1)$$

$c_{m_0} \equiv$ PITCHING MOMENT COEFFICIENT FOR ZERO LIFT

$$c_{m_{ac}} = c_{m_0}$$

$$\frac{dC_m}{dC_\ell} = -\frac{1}{4}$$ $\longrightarrow$ THE AERODYNAMIC CENTER IS 0.25 C AFT THE LEADING EDGE, AT THE QUARTER CHORD POINT.