

# F4 – Lecture Notes

## 1. Thin Airfoil Theory Application: Analysis Example

Reading: Anderson 4.8, 4.9

### Analysis Example

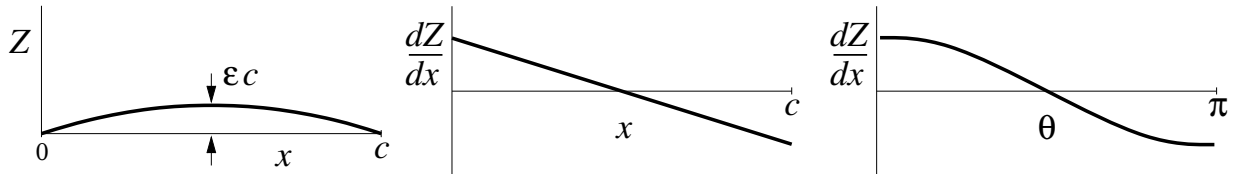
#### Airfoil camberline definition

Consider a thin airfoil with a simple parabolic-arc camberline, with a maximum camber height  $\varepsilon c$ .

$$Z(x) = 4\varepsilon x \left(1 - \frac{x}{c}\right)$$

The camberline slope is then a linear function in  $x$ , or a cosine function in  $\theta$ .

$$\frac{dZ}{dx} = 4\varepsilon \left(1 - 2\frac{x}{c}\right) = 4\varepsilon \cos \theta_o$$



#### Fourier coefficient calculation

Substituting the above  $dZ/dx$  into the general expressions for the Fourier coefficients gives

$$A_0 = \alpha - \frac{1}{\pi} \int_0^\pi \frac{dZ}{dx} d\theta = \alpha - \frac{1}{\pi} \int_0^\pi 4\varepsilon \cos \theta d\theta$$

$$A_n = \frac{2}{\pi} \int_0^\pi \frac{dZ}{dx} \cos n\theta d\theta = \frac{2}{\pi} \int_0^\pi 4\varepsilon \cos \theta \cos n\theta d\theta$$

The integral in the  $A_0$  expression easily evaluates to zero. The integral in the  $A_n$  expression can be evaluated by using the *orthogonality property* of the cosine functions.

$$\int_0^\pi \cos n\theta \cos m\theta d\theta = \begin{cases} \pi & (\text{if } n = m = 0) \\ \pi/2 & (\text{if } n = m \neq 0) \\ 0 & (\text{if } n \neq m) \end{cases}$$

For our case we have  $m = 1$ , and then set  $n = 1, 2, 3, \dots$  to evaluate  $A_1, A_2, A_3, \dots$ . The final results are

$$\begin{aligned} A_0 &= \alpha \\ A_1 &= 4\varepsilon \\ A_2 &= 0 \\ A_3 &= 0 \\ &\vdots \end{aligned}$$

so only  $A_0$  and  $A_1$  are nonzero for this case.

## Lift and moment coefficients

The coefficients can now be computed directly using their general expressions derived previously.

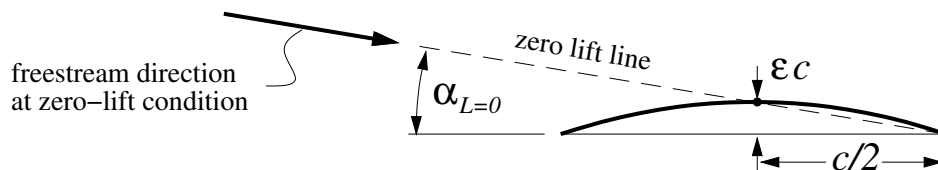
$$c_\ell = \pi(2A_0 + A_1) = 2\pi(\alpha + 2\varepsilon)$$

$$c_{m,c/4} = \frac{\pi}{4}(A_2 - A_1) = -\pi\varepsilon$$

From the  $c_\ell(\alpha)$  expression above, the zero-lift angle is seen to be

$$\alpha_{L=0} = -2\varepsilon$$

which is also the angle of the *zero lift line*. In the present case of a parabolic camber line, the zero lift line passes through the maximum-camber point and the trailing edge point.



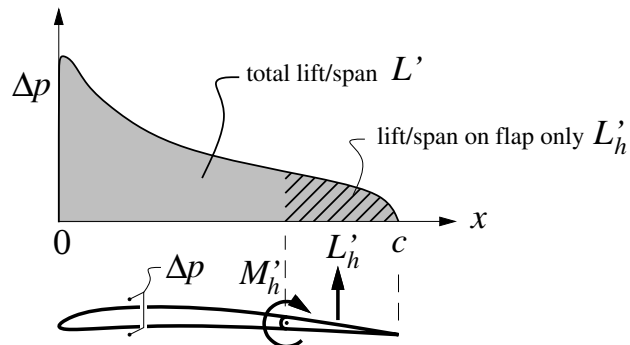
As a possible shortcut, the zero-lift angle could also have been computed directly from its explicit equation derived earlier.

$$\alpha_{L=0} = \frac{1}{\pi} \int_0^\pi \frac{dZ}{dx} (1 - \cos \theta_o) d\theta_o = \frac{1}{\pi} \int_0^\pi 4\varepsilon \cos \theta_o (1 - \cos \theta_o) d\theta_o = -2\varepsilon$$

But this integral is just the combination of the integrals for  $A_0$  and  $A_1$ , so there is no real simplification here.

## Surface loading (further details)

In many applications, obtaining just the  $c_\ell$  and  $c_m$  of the entire airfoil is sufficient. But in some cases, we may also want to know the force and moment on only a portion of the airfoil. For example, the force and moment on a flap are of considerable interest, since the flap hinge and flap control linkage must be designed to withstand these loads. We therefore need to know how the loading  $\Delta p(x)$  is distributed over the chord, and over the flap in particular.



The loading  $\Delta p$  is directly related to the vortex sheet strength  $\gamma(x)$ , and can also be given in terms of the dimensionless pressure coefficient.

$$\Delta p(x) = \rho V_\infty \gamma(x) = \frac{1}{2} \rho V_\infty^2 \Delta C_p(x) \quad (1)$$

The general expression for the sheet strength, obtained previously, is

$$\gamma(\theta) = 2V_\infty \left( A_0 \frac{1 + \cos \theta}{\sin \theta} + \sum_{n=1}^N A_n \sin n\theta \right)$$

Substituting the Fourier coefficients obtained for the present case gives

$$\gamma(\theta) = 2V_\infty \left( \alpha \frac{1 + \cos \theta}{\sin \theta} + 4\varepsilon \sin \theta \right)$$

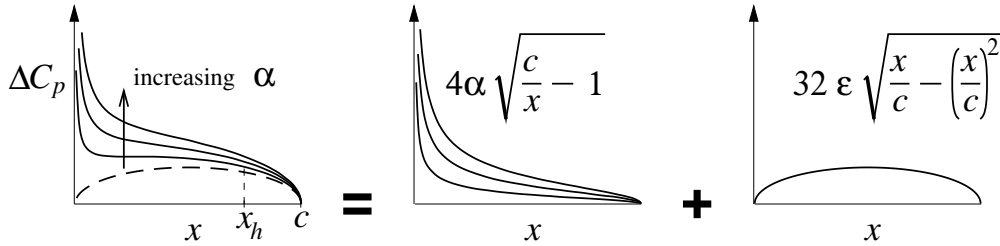
$$\text{or } \Delta C_p(\theta) = 2 \frac{\gamma(\theta)}{V_\infty} = 4\alpha \frac{1 + \cos \theta}{\sin \theta} + 16\varepsilon \sin \theta$$

The integration of  $\Delta C_p$  over the flap can be conveniently performed in the  $\theta$  coordinate as usual, using the above expression. But it is also of some interest to examine this distribution in the physical  $x$  coordinate. The relevant relations between  $\theta$  and  $x$  are

$$\begin{aligned} \cos \theta &= 1 - 2x/c \\ \sin \theta &= \sqrt{1 - \cos^2 \theta} = \sqrt{1 - (1 - 2x/c)^2} = 2\sqrt{x/c - (x/c)^2} \end{aligned}$$

which can be substituted into the above  $\Delta C_p(\theta)$  expression to put it in terms of  $x$ .

$$\Delta C_p(x) = 4\alpha \sqrt{\frac{c}{x} - 1} + 32\varepsilon \sqrt{\frac{x}{c} - \left(\frac{x}{c}\right)^2}$$



Define  $x_h$  as the location of the flap hinge, so the flap extends from  $x = x_h$ , to the trailing edge at  $x = c$ . The corresponding  $\theta$  locations are  $\theta = \arccos(1 - 2x_h/c) \equiv \theta_h$ , and  $\theta = \pi$ , respectively. The load/span and moment/span coefficients on the flap hinge can now be computed by integrating the pressure loading.

$$c_{\ell_h} \equiv \frac{L'_h}{\frac{1}{2}\rho V_\infty^2 c} = \frac{1}{c} \int_{x_h}^c \Delta C_p(x) dx = \frac{1}{2} \int_{\theta_h}^{\pi} \Delta C_p(\theta) \sin \theta d\theta$$

$$c_{m_h} \equiv \frac{M'_h}{\frac{1}{2}\rho V_\infty^2 c^2} = \frac{1}{c^2} \int_{x_h}^c \Delta C_p(x) (x_h - x) dx = \frac{1}{4} \int_{\theta_h}^{\pi} \Delta C_p(\theta) (\cos \theta - \cos \theta_h) \sin \theta d\theta$$

Here, integrations in  $\theta$  are simpler, but still somewhat tedious, and are best left for symbolic integration methods such as Maple.