

# F11 – Lecture Notes

1. Introduction to Compressible Flows
2. Thermodynamics Concepts

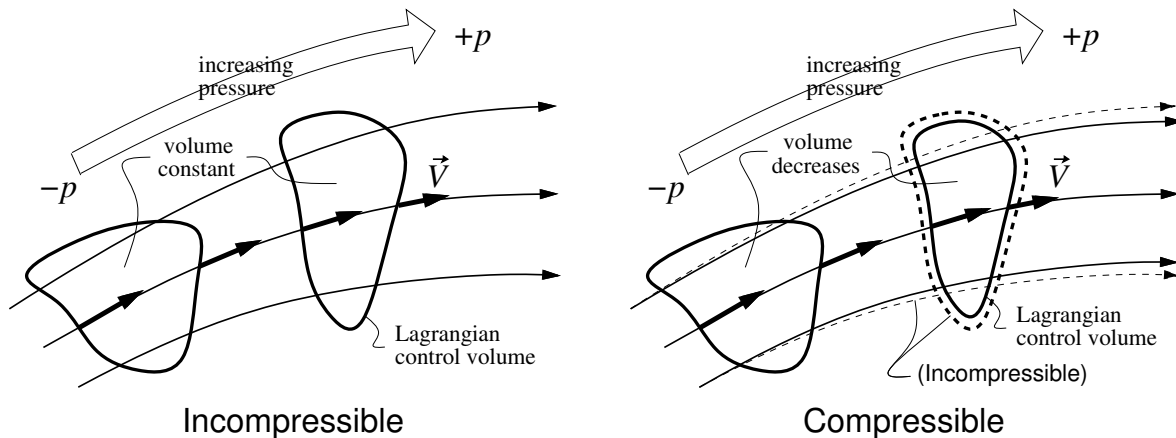
Reading: Anderson 7.1 – 7.2

## Introduction to Compressible Flows

### Definition and implications

A *compressible flow* is a flow in which the fluid density  $\rho$  varies significantly within the flowfield. Therefore,  $\rho(x, y, z)$  must now be treated as a field variable rather than simply a constant. Typically, significant density variations start to appear when the flow Mach number exceeds 0.3 or so. The effects become especially large when the Mach number approaches and exceeds unity.

The figure shows the behavior of a moving Lagrangian Control Volume (CV) which by definition surrounds a fixed mass of fluid  $m$ . In incompressible flow the density  $\rho$  does not change, so the CV's volume  $\mathcal{V} = m/\rho$  must remain constant. In the compressible flow case, the CV is squeezed or expanded significantly in response to pressure changes, with  $\rho$  changing in inverse proportion to  $\mathcal{V}$ . Since the CV follows the streamlines, changes in the CV's volume must be accompanied by changes in the streamlines as well. Above Mach 1, these volumetric changes dominate the streamline pattern.



Many of the relations developed for incompressible (i.e. low speed) flows must be revisited and modified. For example, the Bernoulli equation is no longer valid,

$$p + \frac{1}{2}\rho V^2 \neq \text{constant}$$

since  $\rho = \text{constant}$  was assumed in its derivation. However, concepts such as stagnation pressure  $p_o$  are still usable, but their definitions and relevant equations will be different from the low speed versions.

Some flow solution techniques used in incompressible flow problems will no longer be applicable to compressible flows. In particular, the technique of superposition will no longer be generally applicable, although it will still apply in some restricted situations.

### Perfect gas

A *perfect gas* is one whose individual molecules interact only via direct collisions, with no

other intermolecular forces present. For such a perfect gas,  $p$ ,  $\rho$ , and the temperature  $T$  are related by the following *equation of state*

$$p = \rho RT$$

where  $R$  is the specific gas constant. For air,  $R = 287\text{J/kg}\cdot\text{K}^\circ$ . It is convenient at this point to define the *specific volume* as the limiting volume per unit mass,

$$v \equiv \lim_{\Delta\mathcal{V} \rightarrow 0} \frac{\Delta\mathcal{V}}{\Delta m} = \frac{1}{\rho}$$

which is merely the reciprocal of the density. In general, the nomenclature “specific X” is synonymous with “X per unit mass”. The equation of state can now be written as

$$pv = RT$$

which is the more familiar thermodynamic form.

The appearance of the temperature  $T$  in the equation of state means that it must vary within the flowfield. Therefore,  $T(x, y, z)$  must be treated as a new field variable in addition to  $\rho(x, y, z)$ . In the moving CV scenario above, the change in the CV’s volume is not only accompanied by a change in density, but by a change in temperature as well.

The appearance of the temperature also means that thermodynamics will need to be addressed. So in addition to the conservation of mass and momentum which were employed in low speed flows, we will now also need to consider the conservation of energy. The following table compares the variables and equations which come into play in the two cases.

	Incompressible flow	Compressible flow
variables:	$\vec{V}, p$	$\vec{V}, p, \rho, T$
equations:	mass, $\vec{\text{momentum}}$	mass, $\vec{\text{momentum}}$ , energy, state

## Thermodynamics Concepts

### Internal Energy and Enthalpy

The law of conservation of energy involves the concept of *internal energy*, which is the sum of the energies of all the molecules of a system. In fluid mechanics we employ the *specific internal energy*, denoted by  $e$ , which is defined for each point in the flowfield. A related quantity is the *specific enthalpy*, denoted by  $h$ , and related to the other variables by

$$h = e + pv$$

The units of  $e$  and  $h$  are (velocity)<sup>2</sup>, or  $\text{m}^2/\text{s}^2$  in SI units.

For a *calorically perfect gas*, which is an excellent model for air at moderate temperatures, both  $e$  and  $h$  are directly proportional to the temperature. Therefore we have

$$\begin{aligned} e &= c_v T \\ h &= c_p T \end{aligned}$$

where  $c_v$  and  $c_p$  are specific heats at constant volume and constant pressure, respectively.

$$h - e = pv = (c_p - c_v)T$$

and comparing to the equation of state, we see that

$$c_p - c_v = R$$

Defining the *ratio of specific heats*,  $\gamma \equiv c_p/c_v$ , we can with a bit of algebra write

$$c_v = \frac{1}{\gamma - 1} R$$

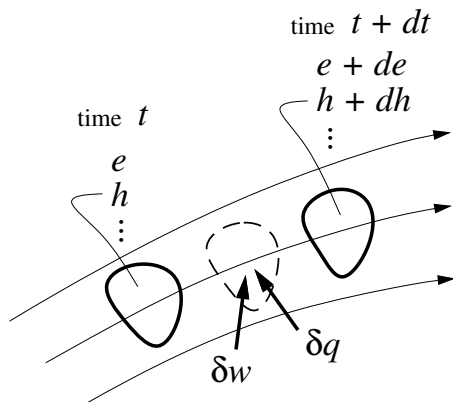
$$c_p = \frac{\gamma}{\gamma - 1} R$$

so that  $c_v$  and  $c_p$  can be replaced with the equivalent variables  $\gamma$  and  $R$ . For air, it is handy to remember that

$$\gamma = 1.4 \qquad \frac{1}{\gamma - 1} = 2.5 \qquad \frac{\gamma}{\gamma - 1} = 3.5 \qquad (\text{air})$$

### First Law of Thermodynamics

Consider a *thermodynamic system* consisting of a small Lagrangian control volume (CV) moving with the flow.



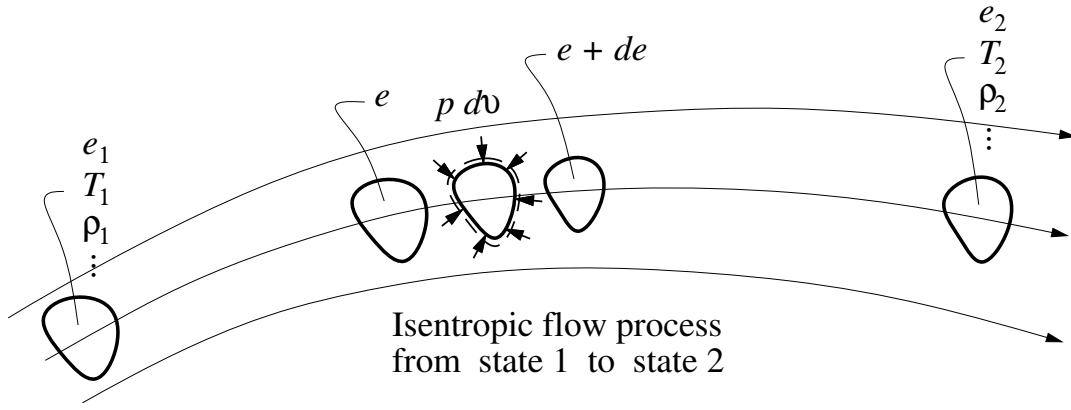
Over the short time interval  $dt$ , the CV undergoes a *process* where it receives work  $\delta w$  and heat  $\delta q$  from its surroundings, both per unit mass. This process results in changes in the state of the CV, described by the increments  $de$ ,  $dh$ ,  $dp \dots$ . The first law of thermodynamics for the process is

$$\delta q + \delta w = de \tag{1}$$

This states that whatever energy is added to the system, whether by heat or by work, it must appear as an increase in the internal energy of the system.

The first law only makes a statement about  $de$ . We can deduce the changes in the other state variables, such as  $dh$ ,  $dp$ ,  $\dots$  if we assume special types of processes.

1. *Adiabatic process*, where no heat is transferred, or  $\delta q = 0$ . This rules out heating of the CV via conduction through its boundary, or by combustion inside the CV.
2. *Reversible process*, no dissipation occurs, implying that work must be only via volumetric compression, or  $dw = -p dv$ . This rules out work done by friction forces.
3. *Isentropic process*, which is both adiabatic and reversible, implying  $-p dv = de$ .



### Isentropic relations

Aerodynamic flows are effectively inviscid outside of boundary layers. This implies they have negligible heat conduction and friction forces, and hence are isentropic. Therefore, along the pathline followed by the CV in the figure above, the isentropic version of the first law applies.

$$-p dv = de \quad (2)$$

This relation can be integrated after a few substitutions. First we note that

$$dv = d\left(\frac{1}{\rho}\right) = -\frac{d\rho}{\rho^2}$$

and with the perfect gas relation

$$de = c_v dT = \frac{1}{\gamma - 1} R dT$$

the isentropic first law (2) becomes

$$\begin{aligned} p \frac{d\rho}{\rho^2} &= \frac{1}{\gamma - 1} R dT \\ \frac{d\rho}{\rho} &= \frac{1}{\gamma - 1} \frac{\rho R}{p} dT \\ \frac{d\rho}{\rho} &= \frac{1}{\gamma - 1} \frac{dT}{T} \end{aligned}$$

The final form can now be integrated from any state 1 to any state 2 along the pathline.

$$\begin{aligned} \ln \rho &= \frac{1}{\gamma - 1} \ln T + \text{const.} \\ \rho &= \text{const.} \times T^{1/(\gamma-1)} \\ \frac{\rho_2}{\rho_1} &= \left(\frac{T_2}{T_1}\right)^{1/(\gamma-1)} \end{aligned} \quad (3)$$

From the equation of state we also have

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1} \frac{T_1}{T_2}$$

which when combined with (3) gives the alternative isentropic relation

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)} \quad (4)$$