Fluids – Lecture 2 Notes

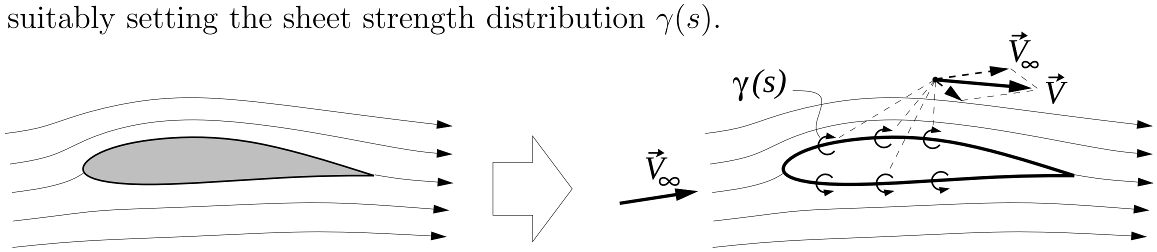
1. Airfoil Vortex Sheet Models
2. Thin-Airfoil Analysis Problem

Reading: Anderson 4.4, 4.7

Airfoil Vortex Sheet Models

Surface Vortex Sheet Model

An accurate means of representing the flow about an airfoil in a uniform flow is to place a vortex sheet on the airfoil surface. The total velocity $\vec{V}(x, z)$, which is the vector sum of the freestream velocity and the vortex-sheet velocity, can be forced parallel to the airfoil surface by suitably setting the sheet strength distribution $\gamma(s)$.



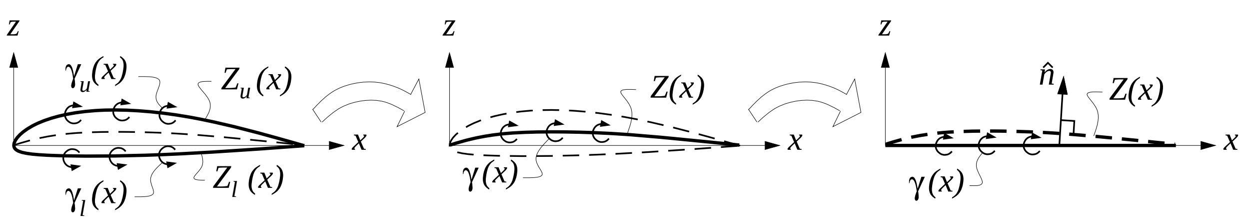
A panel method is normally used to numerically compute $\gamma(s)$. By using a sufficient number of panels, this result can be made as accurate as needed. The main drawback of such numerical calculations is that they give limited insight into how the flow is influenced by changes in the angle of attack or the airfoil geometry. Such insight, which is important for effective aerodynamic design and engineering, is much better provided by simple approximate analytic solutions. The panel method can still be used for accuracy when it's needed.

Single Vortex Sheet Model

In order to simplify the problem sufficiently to allow analytic solution, we make the following assumptions and approximations:

1) The airfoil is assumed to be *thin*, with small maximum camber and thickness relative to the chord, and is assumed to operate at a small angle of attack, $\alpha \ll 1$.

2) The upper and lower vortex sheets are superimposed together into a single vortex sheet $\gamma = \gamma_u + \gamma_\ell$, which is placed on the x axis rather than on the curved mean camber line $Z = (Z_u + Z_\ell)/2$.



3) The flow-tangency condition $\vec{V} \cdot \hat{n} = 0$ is applied on the x-axis at $z = 0$, rather than on the camber line at $z = Z$. But the normal vector $\hat{n}$ is normal to the <u>actual</u> camber line shape, as shown in the figure.

4) Small-angle approximations are assumed. The freestream velocity is then written as follows.

$$\vec{V}_\infty = V_\infty \left[(\cos\alpha)\hat{\imath} + (\sin\alpha)\hat{k} \right] \simeq V_\infty \left(\hat{\imath} + \alpha\hat{k} \right)$$

On the x-axis where the vortex sheet lies, the sheet's velocity $w(x)$, which is strictly in the z-direction, is given by integrating all the contributions along the sheet.

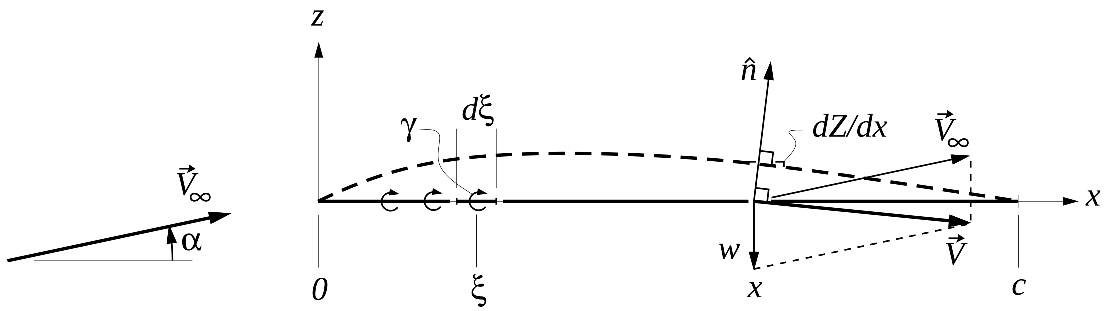
$$w(x) \;=\; -\int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)}$$

Adding this to the freestream velocity then gives the total velocity.

$$\vec{V}(x,0) \;=\; \vec{V}_\infty + w\hat{k} \;\simeq\; V_\infty \hat{\imath} \;+\; \left(V_\infty \alpha \;-\; \int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)} \right) \hat{k} \qquad (1)$$

The normal unit vector is obtained from the slope of the camberline shape $Z(x)$.

$$\vec{n}(x) \;=\; -\frac{dZ}{dx}\,\hat{\imath} \;+\; \hat{k} \qquad , \qquad \hat{n} \;=\; \frac{\vec{n}}{|\vec{n}|} \qquad (2)$$



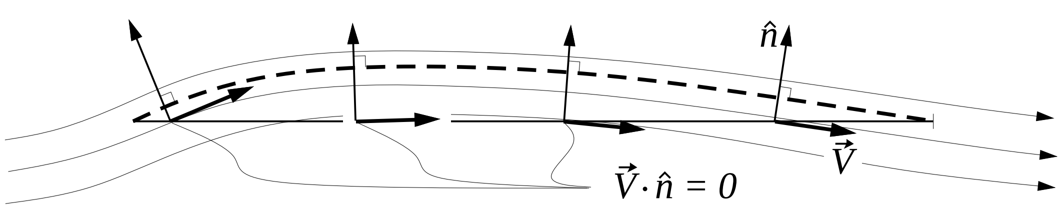
To force the total velocity to be parallel to the camberline, we now apply the flow tangency condition $\vec{V} \cdot \hat{n} = 0$. Performing this dot product between (1) and (2), and removing the unnecessary factor $1/|\vec{n}|$ gives the *fundamental equation of thin airfoil theory* .

$$V_\infty \left(\alpha - \frac{dZ}{dx} \right) \;-\; \int_0^c \frac{\gamma(\xi)\, d\xi}{2\pi(x-\xi)} \;=\; 0 \qquad\qquad (\text{for } 0 < x < c) \qquad (3)$$

Thin-Airfoil Analysis Problem

Flow tangency imposition

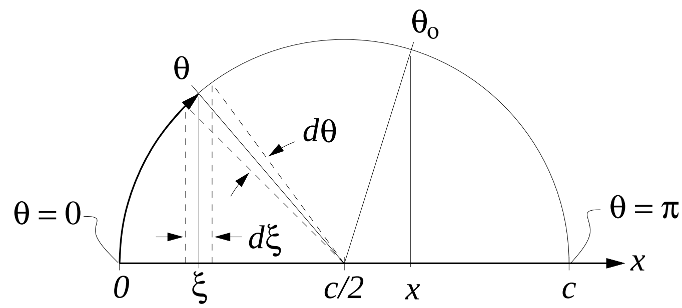
For a given camberline shape $Z(x)$ and angle of attack α, we now seek to determine the vortex strength distribution $\gamma(x)$ such that the fundamental equation (3) is satisfied at every x location. As shown in the figure, this will result in the total velocity $\vec{V}$ at every x-location to be approximately parallel to the local camberline, producing a physically-correct flow about this camberline. The thinner the airfoil, the closer the camberline is to the x-axis where the flow tangency is actually imposed, and the more accurate the approximation becomes. Compared to typical airfoils, the height of the camberline in the figure is exaggerated severalfold for the sake of illustration.



Coordinate transformation

To enable solution of equation (3), it is necessary to first perform a trigonometric substitution for the coordinate x, and the dummy variable of integration ξ.

$$x = \frac{c}{2}(1 - \cos\theta_o)$$
$$\xi = \frac{c}{2}(1 - \cos\theta)$$
$$d\xi = \frac{c}{2}\sin\theta \, d\theta$$



As shown in the figure, θ runs from 0 at the leading edge, to π at the trailing edge. Since x and θ are interchangable, functions of x can now be treated as functions of θ. Equation (3) then becomes

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\ \sin\theta \, d\theta}{\cos\theta - \cos\theta_o} = V_\infty\left(\alpha - \frac{dZ}{dx}\right) \qquad (\text{for } 0 < \theta_o < \pi) \qquad (4)$$

where the known camberline slope dZ/dx is now considered a function of θ_o. This is an *integral equation* which must be solved for the unknown $\gamma(\theta)$ distribution, with the additional requirement that it satisfy the Kutta condition at the trailing edge point,

$$\gamma(\pi) = 0$$

Symmetric airfoil case

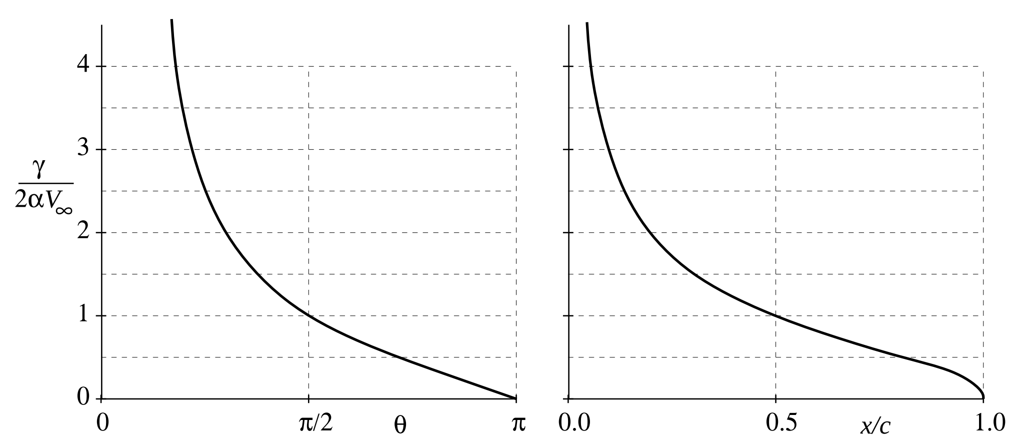
In practice, the camberline slope dZ/dx can have any arbitrary distribution along the chord. For simplicity, we will first consider a *symmetric airfoil* . This has a flat camberline, with $Z = 0$ and $dZ/dx = 0$. Equation (4) then simplifies to

$$\frac{1}{2\pi}\int_0^\pi \frac{\gamma(\theta)\ \sin\theta \, d\theta}{\cos\theta - \cos\theta_o} = V_\infty\alpha \qquad (5)$$

Solution of this equation is still formidable, and is beyond scope here. Let us simply state that the solution is

$$\gamma(\theta) = 2\alpha V_\infty \frac{1 + \cos\theta}{\sin\theta} \qquad \text{or} \qquad \gamma(x) = 2\alpha V_\infty \sqrt{\frac{c - x}{x}}$$

The shape of these distributions is shown in the figure below.



Note that at the trailing edge, $\gamma = 0$ as required by the Kutta condition, and that at the leading edge $\gamma \to \infty$. The latter is of course not physical, although the singularity is weak (integrable), and the integrated results for c_ℓ and c_m are in fact valid.

The load distribution $p_\ell - p_u$ is obtained using the Bernoulli equation, together with the tangential velocity jump properties across the vortex sheet.

$$p_\ell - p_u \;=\; \left(p_o - \frac{1}{2}\rho V_\ell^2\right) \;-\; \left(p_o - \frac{1}{2}\rho V_u^2\right) \;=\; \rho\frac{1}{2}\left(V_u + V_\ell\right)\left(V_u - V_\ell\right) \;=\; \rho V_\infty \gamma$$

The lift/span on an element $d\xi$ of the sheet is

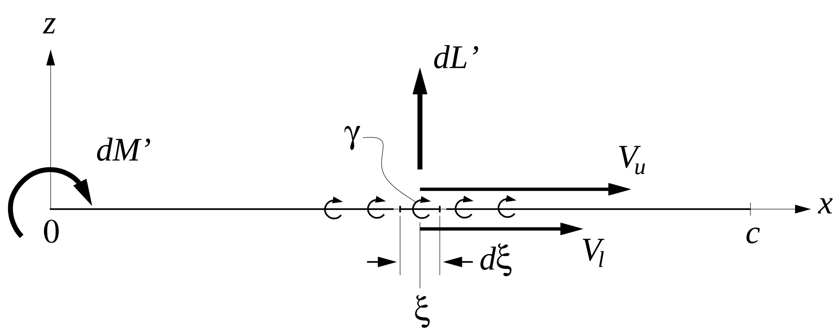
$$dL' \;=\; (p_\ell - p_u)\; d\xi \;=\; \rho V_\infty \gamma\, d\xi$$

and the total lift/span is then obtained by integrating this load distribution.

$$L' \;=\; \rho V_\infty \int_0^c \gamma(\xi)\, d\xi$$

The integral is also seen to be the overall circulation, making this lift result consistent with the Kutta-Joukowsky Theorem.

$$\Gamma \;=\; \int_0^c \gamma(\xi)\, d\xi \quad , \qquad L' \;=\; \rho V_\infty \Gamma$$



The actual integration of the loading or sheet strength is most easily performed in the trigonometric coordinate θ. By direct substitution, we have

$$\Gamma \;=\; \frac{c}{2}\int_0^\pi \gamma(\theta)\; \sin\theta\, d\theta \;=\; \alpha c V_\infty \int_0^\pi (1+\cos\theta)\, d\theta \;=\; \pi\alpha c V_\infty$$

The lift/span is then

$$L' \;=\; \rho V_\infty \Gamma \;=\; \pi\alpha\, c\rho V_\infty^2$$

and the corresponding lift coefficient is

$$c_\ell \;=\; \frac{L'}{\frac{1}{2}\rho V_\infty^2\, c} \;=\; 2\pi\,\alpha$$

This is a very important result, showing that the lift is proportional to the angle of attack, with a *lift slope* of

$$\frac{dc_\ell}{d\alpha} \;=\; 2\pi$$

These results very closely match the results of more complex panel method calculations, as well as experimental data.

The pitching moment/span on the sheet element, taken about the leading edge, is obtained by weighting the lift by the moment arm ξ.

$$dM'_{\text{LE}} \;=\; -\xi \, dL' \;=\; -\rho V_\infty \gamma \, \xi \, d\xi$$

The overall moment/span is then

$$M'_{\text{LE}} \;=\; -\rho V_\infty \int_0^c \gamma \, \xi \, d\xi$$

We can again most easily integrate this in the trigonometric coordinate.

$$M'_{\text{LE}} \;=\; -\alpha \frac{c^2}{2} \rho V_\infty^2 \int_0^\pi (1+\cos\theta)(1-\cos\theta)\, d\theta \;=\; -\alpha \frac{c^2}{2} \rho V_\infty^2 \int_0^\pi (1-\cos^2\theta)\, d\theta \;=\; -\pi\alpha \frac{c^2}{4} \rho V_\infty^2$$

The moment coefficient about the leading edge point is

$$c_{m,\text{le}} \;=\; \frac{M'_{\text{LE}}}{\frac{1}{2}\rho V_\infty^2 \, c^2} \;=\; -\frac{\pi\alpha}{2} \;=\; -\frac{c_\ell}{4}$$

and the equivalent moment coefficient about the standard quarter-chord point at $x/c = 1/4$ is

$$c_{m,c/4} \;=\; c_{m,\text{le}} \,+\, \frac{1}{4} c_\ell \;=\; 0$$

This very important result shows that a symmetric airfoil has zero moment about the quarter-chord point, for any angle of attack.