

# Fluids – Lecture 3 Notes

## 1. Thin-Airfoil Analysis Problem (continued)

Reading: Anderson 4.8

### Cambered airfoil case

We now consider the case where the camberline  $Z(x)$  is nonzero. The general thin airfoil equation, which is a statement of flow tangency on the camberline, was derived previously.

$$\frac{1}{2\pi} \int_0^\pi \frac{\gamma(\theta) \sin \theta d\theta}{\cos \theta - \cos \theta_o} = V_\infty \left( \alpha - \frac{dZ}{dx} \right) \quad (\text{for } 0 < \theta_o < \pi) \quad (1)$$

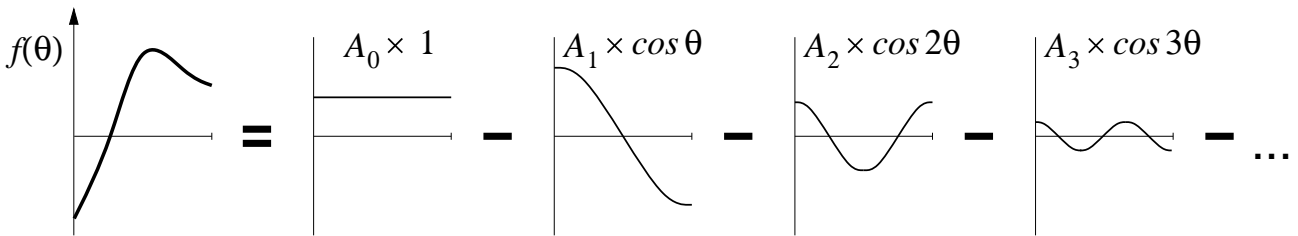
For an arbitrary camberline shape  $Z(x)$ , the slope  $dZ/dx$  varies along the chord, and in the equation it is negated and shifted by the constant  $\alpha$ . Let us consider this combination to be some general function of  $\theta_o$ .

$$\alpha - \frac{dZ}{dx} \equiv f(\theta_o)$$

For the purpose of computation, any such function can be conveniently represented or approximated by a *Fourier cosine series*,

$$f(\theta_o) = A_0 - \sum_{n=1}^N A_n \cos n\theta_o$$

which is illustrated in the figure. The negative sign in front of the sum could be absorbed into all the  $A_n$  coefficients, but is left outside for later algebraic simplicity.



The overall summation can be made arbitrarily close to a known  $f(\theta_o)$  by making  $N$  sufficiently large (i.e. using sufficiently many terms). The required coefficients  $A_0, A_1, \dots, A_N$  are computed one by one using *Fourier analysis*, which is the evaluation of the following integrals.

$$\begin{aligned} A_0 &= \frac{1}{\pi} \int_0^\pi f(\theta) d\theta \\ A_1 &= \frac{2}{\pi} \int_0^\pi f(\theta) \cos \theta d\theta \\ A_2 &= \frac{2}{\pi} \int_0^\pi f(\theta) \cos 2\theta d\theta \\ &\vdots \\ A_N &= \frac{2}{\pi} \int_0^\pi f(\theta) \cos N\theta d\theta \end{aligned}$$

For the particular  $f(\theta_o)$  used here, these integrals become

$$A_0 = \alpha - \frac{1}{\pi} \int_0^\pi \frac{dZ}{dx} d\theta$$

$$A_n = \frac{2}{\pi} \int_0^\pi \frac{dZ}{dx} \cos n\theta d\theta \quad (n = 1, 2, \dots)$$

In practice, the integrals can be evaluated either analytically or numerically. If  $dZ/dx$  is smooth, then the higher  $A_n$  coefficients will rapidly decrease, and at some point the remainder can be discarded (the series truncated) with little loss of accuracy.

Replacing  $\alpha - dZ/dx$  in equation (1) with its Fourier series gives the integral equation

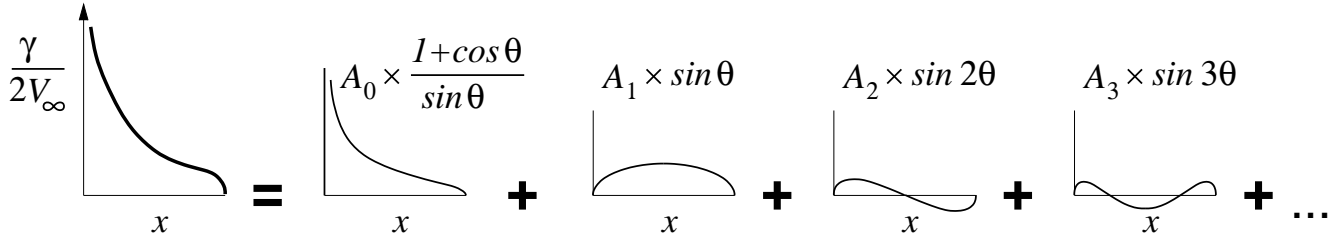
$$\frac{1}{2\pi} \int_0^\pi \frac{\gamma(\theta) \sin \theta d\theta}{\cos \theta - \cos \theta_o} = V_\infty \left( A_0 - \sum_{n=1}^N A_n \cos n\theta_o \right) \quad (2)$$

which is to be solved for the unknown  $\gamma(\theta)$  distribution. As before, the solution of this integral equation is beyond scope here. Again, let us simply state the solution.

$$\gamma(\theta) = 2V_\infty \left( A_0 \frac{1 + \cos \theta}{\sin \theta} + \sum_{n=1}^N A_n \sin n\theta \right)$$

The leading term is the same as for the zero-camber case, but with  $A_0$  replacing  $\alpha$ . The remaining coefficients  $A_1, \dots, A_N$  in the summation depend only on the shape of the camberline, and in particular are independent of  $\alpha$ .

The figure shows the contributions of the various terms towards  $\gamma$ , all plotted versus the physical  $x$  coordinate rather than versus  $\theta$ . Note that here the coefficients  $A_0, A_1 \dots A_N$  have



already been determined, and are now merely used to construct  $\gamma(\theta)$  by simple summation of the series. This  $\gamma(\theta)$  will now be integrated to obtain the lift force and moment.

### Force calculation

The circulation and lift/span are computed in the same manner as with the symmetric airfoil case.

$$\Gamma = \int_0^c \gamma(\xi) d\xi \quad , \quad L' = \rho V_\infty \Gamma$$

The integral is again most easily performed in the trigonometric coordinate  $\theta$ .

$$\Gamma = \frac{c}{2} \int_0^\pi \gamma(\theta) \sin \theta d\theta = cV_\infty \left[ A_0 \int_0^\pi (1 + \cos \theta) d\theta + \sum_{n=1}^N A_n \int_0^\pi \sin n\theta \sin \theta d\theta \right]$$

The first integral in the brackets is easily evaluated.

$$\int_0^\pi (1 + \cos \theta) d\theta = \pi$$

The integrals inside the summation can be evaluated by using the *orthogonality property* of the sine functions.

$$\int_0^\pi \sin n\theta \sin m\theta d\theta = \begin{cases} \pi/2 & (\text{if } n = m) \\ 0 & (\text{if } n \neq m) \end{cases}$$

We see that only the  $n = 1$  integral inside the summation evaluates to  $\pi/2$ , and all the others are zero. The final result is

$$\begin{aligned} \Gamma &= cV_\infty \left( \pi A_0 + \frac{\pi}{2} A_1 \right) \\ L' &= \rho V_\infty^2 \Gamma = \rho V_\infty^2 c \pi \left( A_0 + \frac{1}{2} A_1 \right) \\ c_\ell &= \frac{L'}{\frac{1}{2} \rho V_\infty^2 c} = \pi (2A_0 + A_1) \end{aligned}$$

It's informative to substitute the previously-obtained expressions for  $A_0$  and  $A_1$ , giving

$$c_\ell = 2\pi \left[ \alpha - \frac{1}{\pi} \int_0^\pi \frac{dZ}{dx} (1 - \cos \theta_o) d\theta_o \right]$$

The integral term inside the brackets depends only on the camberline shape, and is independent of the angle of attack. Hence the lift slope is

$$\frac{dc_\ell}{d\alpha} = 2\pi$$

which is the same as for the symmetrical airfoil case. We therefore reach the important conclusion that camber has no influence on the lift slope. A terse and convenient way to represent the  $c_l(\alpha)$  function is therefore

$$c_\ell = \frac{dc_\ell}{d\alpha} (\alpha - \alpha_{L=0})$$

where  $\alpha_{L=0}$  is called the *zero-lift angle*, which depends only on the camberline shape.

$$\alpha_{L=0} = \frac{1}{\pi} \int_0^\pi \frac{dZ}{dx} (1 - \cos \theta_o) d\theta_o$$

The moment/span about the leading edge is again computed using the trigonometric coordinate.

$$M'_{LE} = -\rho V_\infty \int_0^c \gamma \xi d\xi = -\rho V_\infty \frac{c^2}{4} \int_0^\pi \gamma(\theta) (1 - \cos \theta) \sin \theta d\theta = -\rho V_\infty^2 \frac{c^2}{4} \pi \left( A_0 + A_1 - \frac{1}{2} A_2 \right)$$

The moment/span and corresponding moment coefficient about the  $x = c/4$  quarter-chord point are

$$\begin{aligned} M'_{c/4} &= M'_{LE} + \frac{c}{4} L' = \rho V_\infty^2 \frac{c^2}{4} \frac{\pi}{2} (A_2 - A_1) \\ c_{m,c/4} &= \frac{M'_{c/4}}{\frac{1}{2} \rho V_\infty^2 c^2} = \frac{\pi}{4} (A_2 - A_1) \end{aligned}$$

An important result is that this  $c_{m,c/4}$  depends only on the camberline shape, but not on the angle of attack. Therefore, the quarter-chord location is the *aerodynamic center* for any airfoil, defined as the location about which the moment is independent of  $\alpha$ , or

$$\frac{dc_{m,c/4}}{d\alpha} = 0$$

## Summary

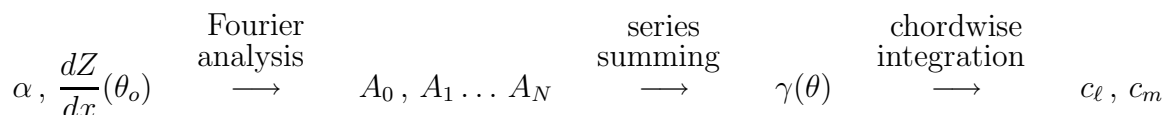
For airfoil analysis, Thin Airfoil Theory takes in the following inputs:

- $\alpha$  angle of attack
- $dZ/dx$  camberline slope distribution along chord

The outputs are:

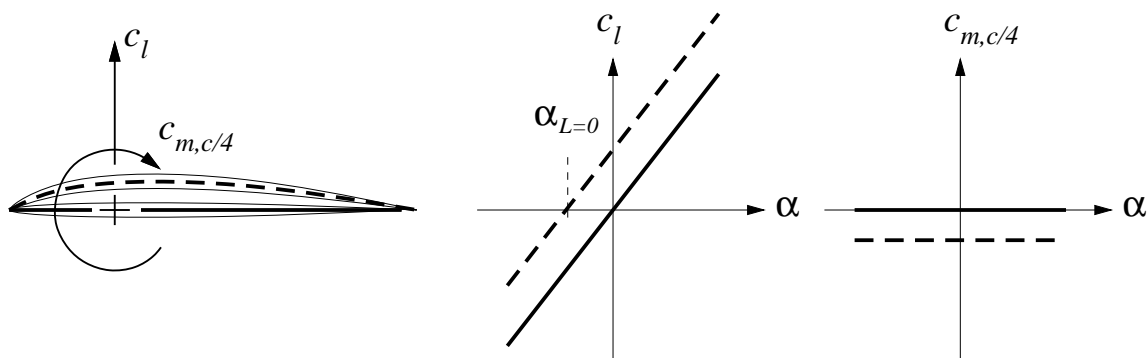
- $c_\ell$  lift coefficient
- $c_m$  moment coefficient, about  $c/4$  or any other location

The information propagates as follows.



The Fourier coefficients  $A_n$  and the vortex sheet strength distribution  $\gamma(\theta)$  are intermediate results.

The influence of camber on the airfoil  $c_\ell(\alpha)$  and  $c_{m,c/4}(\alpha)$  curves is illustrated in the figure.



These results are subject to the assumptions inherent in thin airfoil theory. In practice, they are surprisingly accurate even for relatively thick or highly-cambered airfoils. It appears to be better at predicting trends (with camber,  $\alpha$ , etc) than absolute numbers. When used merely as a conceptual framework for understanding airfoil behavior rather than for quantitative predictions, thin airfoil theory is highly applicable to almost any airfoil.