

Fluids – Lecture 14 Notes

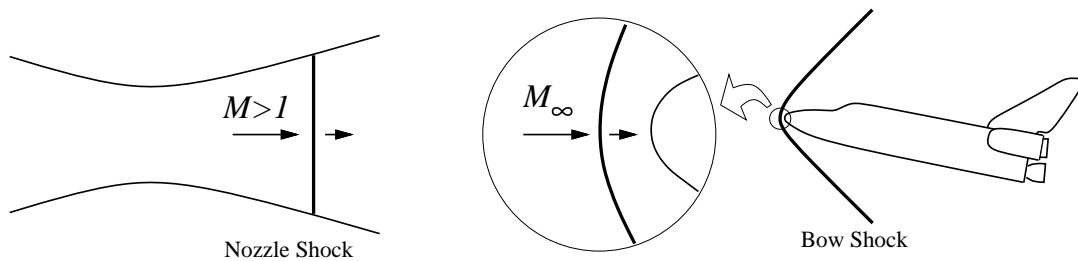
1. Normal Shock Waves
2. Speed of Sound

Reading: Anderson 8.1 – 8.3

Normal Shock Waves

Occurance of normal shock waves

A *normal shock wave* appears in many types of supersonic flows. Two examples are shown in the figure. Any blunt-nosed body in a supersonic flow will develop a curved *bow shock*, which is normal to the flow locally just ahead of the stagnation point. Another common example is a supersonic nozzle flow, which is typically found in a jet or rocket engine. A normal shock can appear in the diverging part of the nozzle under certain conditions.

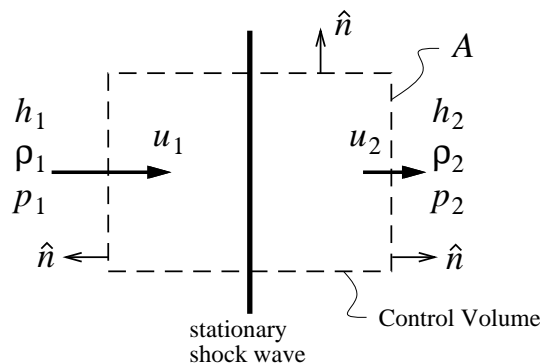


Shock jump relations

We examine the flow in the frame in which the shock is stationary. The upstream and downstream flow properties are denoted by the subscripts $()_1$ and $()_2$ as shown in the figure. A control volume is defined straddling the shock. The flow in the shock has the following properties:

1. Flow is steady, so $\partial()/\partial t = 0$ in all equations.
2. Flow is adiabatic, so $\dot{q} = 0$.
3. Body forces such as gravity are negligible, so \vec{g} is neglected.

The flow is also assumed irreversible due to viscous forces acting in the extremely large velocity gradients in the thin shock, although this doesn't explicitly influence the analysis. We now apply the integral conservation equations to the control volume. The flow is 1-dimensional in the x -direction normal to the shock, so that $\vec{V} = u\hat{i}$. There is no flow through the top and bottom boundaries, since $\vec{V} \cdot \hat{n}$ there.



Mass continuity

$$\begin{aligned}\oint \rho \vec{V} \cdot \hat{n} dA &= 0 \\ -\rho_1 u_1 A + \rho_2 u_2 A &= 0 \\ \rho_1 u_1 &= \rho_2 u_2\end{aligned}\tag{1}$$

x-Momentum

$$\begin{aligned}\oint \rho \vec{V} \cdot \hat{n} u dA + \oint p \hat{n} \cdot \hat{i} dA &= 0 \\ -\rho_1 u_1^2 A + \rho_2 u_2^2 A - p_1 A + p_2 A &= 0 \\ \rho_1 u_1^2 + p_1 &= \rho_2 u_2^2 + p_2\end{aligned}\tag{2}$$

Energy

$$\begin{aligned}\oint \rho \vec{V} \cdot \hat{n} h_o dA &= 0 \\ -\rho_1 u_1 h_{o1} A + \rho_2 u_2 h_{o2} A &= 0 \\ h_{o1} &= h_{o2} \\ h_1 + \frac{1}{2} u_1^2 &= h_2 + \frac{1}{2} u_2^2\end{aligned}\tag{3}$$

Equation of State

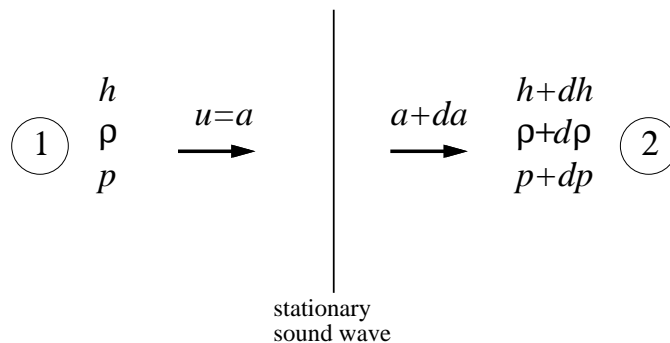
$$p_2 = \frac{\gamma - 1}{\gamma} \rho_2 h_2\tag{4}$$

Simplification of the energy equation (3) makes use of the mass equation (1). In most shock flow analysis situations, the upstream supersonic flow quantities at station 1 are known, either from the freestream conditions or from the flow about some upstream body. The four equations (1)–(4) then are sufficient to determine the four downstream flow quantities ρ_2 , u_2 , p_2 , and h_2 . The temperatures T_1 and T_2 can be considered additional variables, but for a perfect gas these are trivially related to h_1 and h_2 through $h = c_p T$.

Speed of Sound

Sound wave

Before solution of (1)–(4) is carried out for a general shock wave, we first consider an infinitesimally weak shock wave, also known as a *sound wave*. Because the velocity gradients and hence the viscous action is small, the flow process through the wave is *isentropic*.



Rather than treating the 1 and 2 variables, we instead examine their infinitesimal differences dh , $d\rho$, dp , \dots . We also define u for this case to be the speed of sound (yet unknown), and denote it with a separate symbol a . The objective here is to determine this a in terms of the other variables by applying equations (1) – (4).

Speed of sound derivation

The mass equation (1) for the sound wave case becomes

$$\begin{aligned}\rho a &= (\rho + d\rho)(a + da) = \rho a + a d\rho + \rho da \\ da &= -\frac{a}{\rho} d\rho\end{aligned}\tag{5}$$

where the higher-order term $d\rho da$ has been dropped. Similarly for x -momentum we have

$$\begin{aligned}\rho a^2 + p &= (\rho + d\rho)(a + da)^2 + (p + dp) = \rho a^2 + a^2 d\rho + 2a\rho da + p + dp \\ 0 &= 2a\rho da + a^2 d\rho + dp\end{aligned}\tag{6}$$

Using equation (5) to eliminate da from (6) gives

$$\begin{aligned}0 &= 2a\rho \left(-\frac{a}{\rho} d\rho\right) + a^2 d\rho + dp \\ 0 &= -a^2 d\rho + dp \\ a^2 &= \frac{dp}{d\rho}\end{aligned}\tag{7}$$

We could now relate p and ρ and thus get $dp/d\rho$ using the energy and state equations (3) and (4). But an algebraically simpler approach is to use one of the isentropic relations instead, which are valid for this weak wave. The simplest relation for this purpose is

$$\begin{aligned}\frac{p_2}{p_1} &= \left(\frac{\rho_2}{\rho_1}\right)^\gamma \\ \text{or} \quad \frac{p + dp}{p} &= \left(\frac{\rho + d\rho}{\rho}\right)^\gamma \\ 1 + \frac{dp}{p} &= \left(1 + \frac{d\rho}{\rho}\right)^\gamma = 1 + \gamma \frac{d\rho}{\rho} + \text{h.o.t.} \\ \frac{dp}{p} &= \gamma \frac{d\rho}{\rho} \\ \frac{dp}{d\rho} &= \gamma \frac{p}{\rho} = \gamma RT\end{aligned}$$

Combining this with equation (7) gives the speed of sound as

$$a = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT}$$

which can be seen to depend on the temperature alone.