Fluids – Lecture 15 Notes

1. Mach Number Relations
2. Normal-Shock Properties

Reading: Anderson 8.4, 8.6

Mach Number Relations

Local Mach number

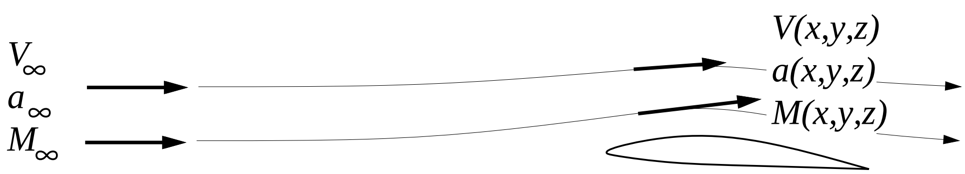
For a perfect gas, the speed of sound can be given in a number of ways.

$$a \;=\; \sqrt{\gamma R T} \;=\; \sqrt{\frac{\gamma p}{\rho}} \;=\; \sqrt{(\gamma - 1)\, h} \tag{1}$$

The dimensionless *local Mach number* can then be defined.

$$M \;\equiv\; \frac{V}{a} \;=\; \sqrt{\frac{\rho(u^2+v^2+w^2)}{\gamma p}} \;=\; \sqrt{\frac{u^2+v^2+w^2}{(\gamma-1)\, h}}$$

It's important to note that this is a field variable $M(x, y, z)$, and is distinct from the freestream Mach number M_∞. Likewise for V and a.



The local stagnation enthalpy can be given in terms of the static enthalpy and the Mach number, or in terms of the speed of sound and the Mach number.

$$h_o \;=\; h \,+\, \frac{1}{2}V^2 \;=\; h\left(1 + \frac{1}{2}\frac{V^2}{h}\right) \;=\; h\left(1 + \frac{\gamma-1}{2}M^2\right) \;=\; \frac{a^2}{\gamma-1}\left(1 + \frac{\gamma-1}{2}M^2\right) \tag{2}$$

This now allows the isentropic relations

$$\frac{p_o}{p} \;=\; \left(\frac{\rho_o}{\rho}\right)^{\gamma} \;=\; \left(\frac{h_o}{h}\right)^{\gamma/(\gamma-1)}$$

to be put in terms of the Mach number rather than the speed as before.

$$\frac{\rho_o}{\rho} \;=\; \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)}$$

$$\frac{p_o}{p} \;=\; \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}$$

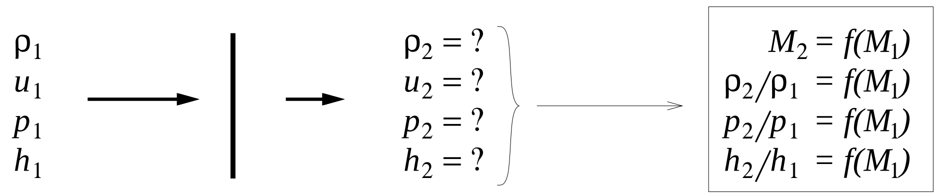
The following relation is also sometimes useful.

$$1 \,-\, \frac{V^2}{2h_o} \;=\; \left(1 + \frac{\gamma-1}{2}M^2\right)^{-1}$$

Normal-Shock Properties

Mach jump relations

We now seek to determine the properties ρ_2, u_2, p_2, h_2 downstream of the shock, as functions of the known upstream properties ρ_1, u_1, p_1, h_1. In practice, it is sufficient and much more convenient to merely determine the downstream Mach number M_2 and the variable ratios, since these are strictly functions of the upstream Mach number M_1.



The starting point is the normal shock equations obtained earlier, with $V = u$ for this 1-D case. They are also known as the *Rankine-Hugoniot* shock equations.

$$\rho_1 u_1 = \rho_2 u_2 \tag{3}$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \tag{4}$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2 \tag{5}$$

$$p_2 = \frac{\gamma - 1}{\gamma}\rho_2 h_2 \tag{6}$$

Dividing the momentum equation (4) by the continuity equation (3) gives

$$u_1 + \frac{p_1}{\rho_1 u_1} = u_2 + \frac{p_2}{\rho_2 u_2}$$

or

$$u_1 - u_2 = \frac{1}{\gamma}\left(\frac{a_2^2}{u_2} - \frac{a_1^2}{u_1}\right) \tag{7}$$

where we have substituted $p/\rho = a^2/\gamma$ and rearranged the terms.

Now we make use of the energy equation (5). For algebraic convenience we first define the constant total enthalpy in terms of the known upstream quantities

$$h_1 + \frac{1}{2}u_1^2 \equiv h_o = h_2 + \frac{1}{2}u_2^2$$

which then gives a_1^2 and a_2^2 in terms of u_1 and u_2, respectively.

$$a_1^2 = (\gamma - 1)h_1 = (\gamma - 1)\left(h_o - \frac{1}{2}u_1^2\right)$$

$$a_2^2 = (\gamma - 1)h_2 = (\gamma - 1)\left(h_o - \frac{1}{2}u_2^2\right)$$

Substituting these energy relations into the combined momentum/mass relation (7) gives, after some further manipulation

$$u_1 - u_2 = \frac{\gamma - 1}{\gamma}\left(\frac{h_o}{u_2} - \frac{h_o}{u_1} + \frac{1}{2}(u_1 - u_2)\right)$$

Dividing by $u_1 - u_2$ produces

$$1 \;=\; \frac{\gamma-1}{\gamma}\left(\frac{h_o}{u_1 u_2} + \frac{1}{2}\right) \tag{8}$$

$$\frac{(\gamma-1)h_o}{u_1 u_2} \;=\; \frac{\gamma+1}{2}$$

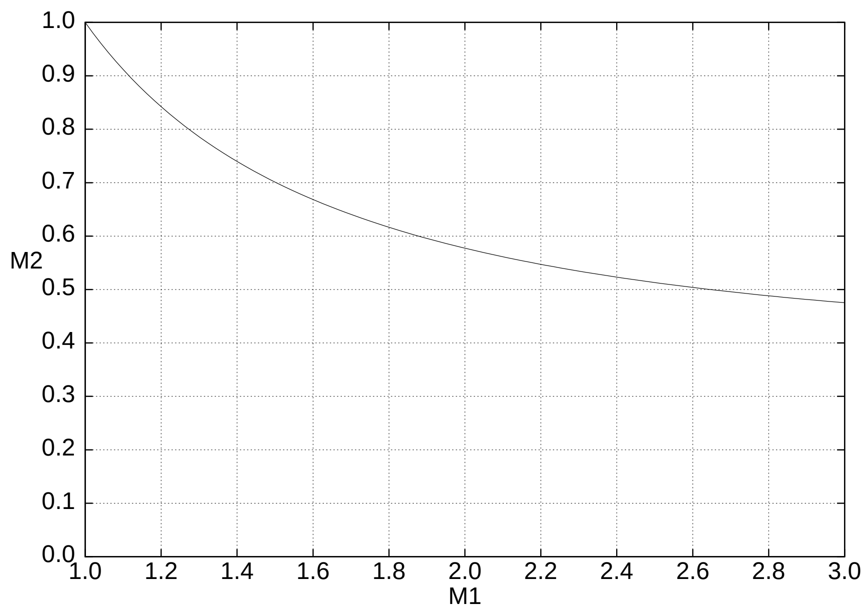
$$\frac{(\gamma-1)^2 h_o^2}{u_1^2\, u_2^2} \;=\; \left(\frac{\gamma+1}{2}\right)^2 \tag{9}$$

Since $h_o = h_{o_1} = h_{o_2}$, we can write

$$(\gamma-1)^2 h_o^2 \;=\; (\gamma-1)h_{o_1}\,(\gamma-1)h_{o_2} \;=\; a_1^2\left(1+\frac{\gamma-1}{2}M_1^2\right)\, a_2^2\left(1+\frac{\gamma-1}{2}M_2^2\right)$$

and using this to eliminate h_o^2 from equation (9), and solving for M_2, yields the desired $M_2(M_1)$ function. This is shown plotted for $\gamma = 1.4$.

$$M_2^2 \;=\; \frac{1 \,+\, \frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 \,-\, \frac{\gamma-1}{2}} \tag{10}$$



The $M_1 \rightarrow 1^+$, $M_2 \rightarrow 1^-$ limit corresponds to infinitesimal shock, or a sound wave. The $M_2(M_1)$ function is not shown for $M_1 < 1$, since this would correspond to an "expansion shock" which is physically impossible based on irreversibility considerations.

Static jump relations

The jumps in the static flow variables are now readily determined as ratios using the known M_2. From the mass equation (3) we have

$$\frac{\rho_2}{\rho_1} \;=\; \frac{u_1}{u_2} \;=\; \frac{u_1^2}{u_1 u_2}$$

From the Mach definition we have

$$u_1^2 \;=\; M_1^2 a_1^2 \;=\; M_1^2\,\frac{(\gamma-1)h_o}{1+\frac{\gamma-1}{2}M_1^2}$$

and from equation (8) we have

$$\frac{1}{u_1 u_2} \;=\; \frac{1}{(\gamma{-}1)h_o}\frac{\gamma{+}1}{2}$$

Combining these gives the shock density ratio in terms of M_1 alone.

$$\frac{\rho_2}{\rho_1} \;=\; \frac{(\gamma{+}1)M_1^2}{2 \,+\, (\gamma{-}1)M_1^2} \tag{11}$$

The combination of the momentum equation (4) and mass equation (3) gives

$$p_2 - p_1 \;=\; \rho_1 u_1^2 \;-\; \rho_2 u_2^2 \;=\; \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right) \;=\; \rho_1 u_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right)$$

which can be further simplified by using the general relation $\rho u^2 = \gamma p M^2$, dividing by p_1, and then using (11) to eliminate ρ_2/ρ_1 in terms of M_1. The final result for the shock static pressure ratio is

$$\frac{p_2}{p_1} \;=\; 1 \;+\; \frac{2\gamma}{\gamma{+}1}\left(M_1^2 - 1\right) \tag{12}$$

The static temperature or enthalpy ratio is now readily obtained from the pressure and density ratios via the state equation.

$$\frac{T_2}{T_1} \;=\; \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

The result is

$$\frac{T_2}{T_1} \;=\; \frac{h_2}{h_1} \;=\; \left[1 \;+\; \frac{2\gamma}{\gamma{+}1}\left(M_1^2 - 1\right)\right] \frac{2 \,+\, (\gamma{-}1)M_1^2}{(\gamma{+}1)M_1^2} \tag{13}$$

The three static quantity ratios (11), (12), (13), are shown plotted versus M_1.