# Fluids – Lecture 15 Notes

1. Mach Number Relations
2. Normal-Shock Properties

Reading: Anderson 8.4, 8.6

## Mach Number Relations

### Local Mach number

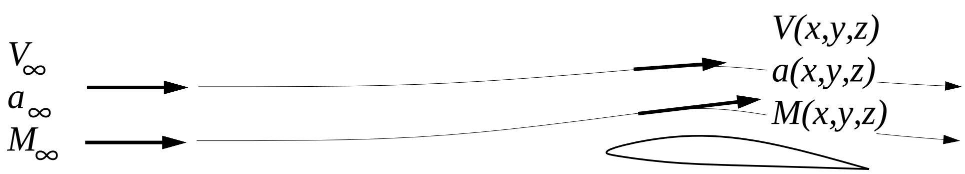
For a perfect gas, the speed of sound can be given in a number of ways.

$$a \;=\; \sqrt{\gamma RT} \;=\; \sqrt{\frac{\gamma p}{\rho}} \;=\; \sqrt{(\gamma-1)\,h} \tag{1}$$

The dimensionless *local Mach number* can then be defined.

$$M \;\equiv\; \frac{V}{a} \;=\; \sqrt{\frac{\rho(u^2+v^2+w^2)}{\gamma p}} \;=\; \sqrt{\frac{u^2+v^2+w^2}{(\gamma-1)\,h}}$$

It's important to note that this is a field variable $M(x,y,z)$, and is distinct from the freestream Mach number $M_\infty$. Likewise for $V$ and $a$.




The local stagnation enthalpy can be given in terms of the static enthalpy and the Mach number, or in terms of the speed of sound and the Mach number.

$$h_o \;=\; h + \frac{1}{2}V^2 \;=\; h\left(1+\frac{1}{2}\frac{V^2}{h}\right) \;=\; h\left(1+\frac{\gamma-1}{2}M^2\right) \;=\; \frac{a^2}{\gamma-1}\left(1+\frac{\gamma-1}{2}M^2\right) \tag{2}$$

This now allows the isentropic relations

$$\frac{p_o}{p} \;=\; \left(\frac{\rho_o}{\rho}\right)^{\gamma} \;=\; \left(\frac{h_o}{h}\right)^{\gamma/(\gamma-1)}$$

to be put in terms of the Mach number rather than the speed as before.

$$\frac{\rho_o}{\rho} \;=\; \left(1+\frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)}$$

$$\frac{p_o}{p} \;=\; \left(1+\frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}$$

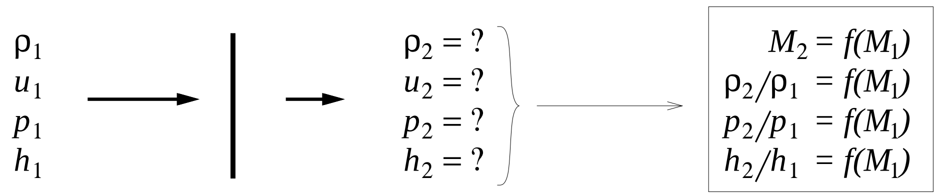
The following relation is also sometimes useful.

$$1 - \frac{V^2}{2h_o} \;=\; \left(1+\frac{\gamma-1}{2}M^2\right)^{-1}$$

## Normal-Shock Properties

### Mach jump relations

We now seek to determine the properties $\rho_2, u_2, p_2, h_2$ downstream of the shock, as functions of the known upstream properties $\rho_1, u_1, p_1, h_1$. In practice, it is sufficient and much more convenient to merely determine the downstream Mach number $M_2$ and the variable ratios, since these are strictly functions of the upstream Mach number $M_1$.




The starting point is the normal shock equations obtained earlier, with $V = u$ for this 1-D case. They are also known as the *Rankine-Hugoniot* shock equations.

$$\rho_1 u_1 = \rho_2 u_2 \tag{3}$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \tag{4}$$

$$h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2 \tag{5}$$

$$p_2 = \frac{\gamma - 1}{\gamma}\rho_2 h_2 \tag{6}$$

Dividing the momentum equation (4) by the continuity equation (3) gives

$$u_1 + \frac{p_1}{\rho_1 u_1} = u_2 + \frac{p_2}{\rho_2 u_2}$$

or

$$u_1 - u_2 = \frac{1}{\gamma}\left(\frac{a_2^2}{u_2} - \frac{a_1^2}{u_1}\right) \tag{7}$$

where we have substituted $p/\rho = a^2/\gamma$ and rearranged the terms.

Now we make use of the energy equation (5). For algebraic convenience we first define the constant total enthalpy in terms of the known upstream quantities

$$h_1 + \frac{1}{2}u_1^2 \equiv h_o = h_2 + \frac{1}{2}u_2^2$$

which then gives $a_1^2$ and $a_2^2$ in terms of $u_1$ and $u_2$, respectively.

$$a_1^2 = (\gamma-1)h_1 = (\gamma-1)\left(h_o - \frac{1}{2}u_1^2\right)$$

$$a_2^2 = (\gamma-1)h_2 = (\gamma-1)\left(h_o - \frac{1}{2}u_2^2\right)$$

Substituting these energy relations into the combined momentum/mass relation (7) gives, after some further manipulation

$$u_1 - u_2 = \frac{\gamma-1}{\gamma}\left(\frac{h_o}{u_2} - \frac{h_o}{u_1} + \frac{1}{2}(u_1 - u_2)\right)$$

Dividing by $u_1 - u_2$ produces

$$1 \;=\; \frac{\gamma-1}{\gamma}\left(\frac{h_o}{u_1 u_2} + \frac{1}{2}\right) \tag{8}$$

$$\frac{(\gamma-1)h_o}{u_1 u_2} \;=\; \frac{\gamma+1}{2}$$

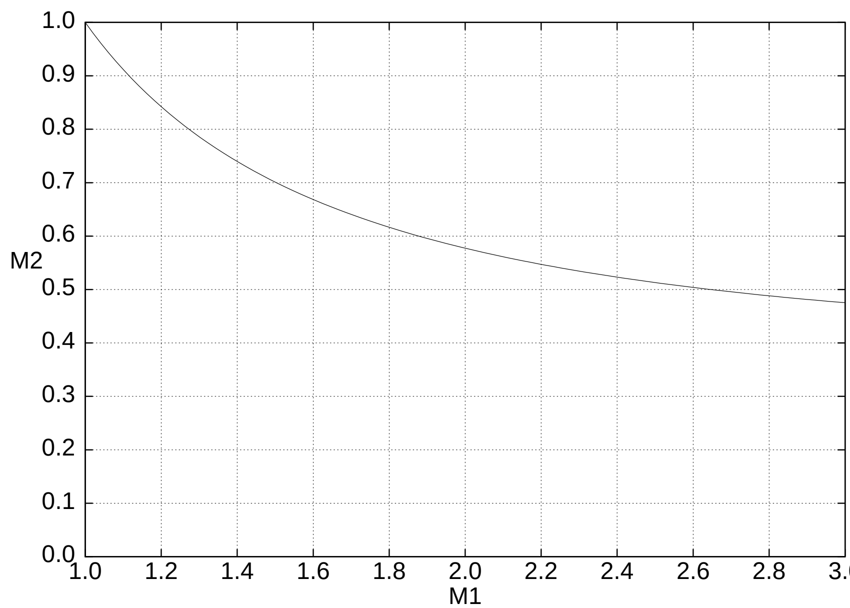
$$\frac{(\gamma-1)^2 h_o^2}{u_1^2\, u_2^2} \;=\; \left(\frac{\gamma+1}{2}\right)^2 \tag{9}$$

Since $h_o = h_{o_1} = h_{o_2}$, we can write

$$(\gamma-1)^2 h_o^2 \;=\; (\gamma-1)h_{o_1}\,(\gamma-1)h_{o_2} \;=\; a_1^2\left(1+\frac{\gamma-1}{2}M_1^2\right)\, a_2^2\left(1+\frac{\gamma-1}{2}M_2^2\right)$$

and using this to eliminate $h_o^2$ from equation (9), and solving for $M_2$, yields the desired $M_2(M_1)$ function. This is shown plotted for $\gamma = 1.4$ .

$$M_2^2 \;=\; \frac{1 \,+\, \frac{\gamma-1}{2}M_1^2}{\gamma M_1^2 \,-\, \frac{\gamma-1}{2}} \tag{10}$$




The $M_1 \rightarrow 1^+$, $M_2 \rightarrow 1^-$ limit corresponds to infinitesimal shock, or a sound wave. The $M_2(M_1)$ function is not shown for $M_1 < 1$, since this would correspond to an "expansion shock" which is physically impossible based on irreversibility considerations.

### Static jump relations

The jumps in the static flow variables are now readily determined as ratios using the known $M_2$. From the mass equation (3) we have

$$\frac{\rho_2}{\rho_1} \;=\; \frac{u_1}{u_2} \;=\; \frac{u_1^2}{u_1 u_2}$$

From the Mach definition we have

$$u_1^2 \;=\; M_1^2 a_1^2 \;=\; M_1^2\,\frac{(\gamma-1)h_o}{1+\frac{\gamma-1}{2}M_1^2}$$

and from equation (8) we have

$$\frac{1}{u_1 u_2} \;=\; \frac{1}{(\gamma{-}1)h_o}\frac{\gamma{+}1}{2}$$

Combining these gives the shock density ratio in terms of $M_1$ alone.

$$\frac{\rho_2}{\rho_1} \;=\; \frac{(\gamma{+}1)M_1^2}{2 \,+\, (\gamma{-}1)M_1^2} \tag{11}$$

The combination of the momentum equation (4) and mass equation (3) gives

$$p_2 - p_1 \;=\; \rho_1 u_1^2 \,-\, \rho_2 u_2^2 \;=\; \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right) \;=\; \rho_1 u_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right)$$

which can be further simplified by using the general relation $\rho u^2 = \gamma p M^2$, dividing by $p_1$, and then using (11) to eliminate $\rho_2/\rho_1$ in terms of $M_1$. The final result for the shock static pressure ratio is

$$\frac{p_2}{p_1} \;=\; 1 \;+\; \frac{2\gamma}{\gamma{+}1}\left(M_1^2 - 1\right) \tag{12}$$

The static temperature or enthalpy ratio is now readily obtained from the pressure and density ratios via the state equation.

$$\frac{T_2}{T_1} \;=\; \frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$$

The result is

$$\frac{T_2}{T_1} \;=\; \frac{h_2}{h_1} \;=\; \left[1 \;+\; \frac{2\gamma}{\gamma{+}1}\left(M_1^2 - 1\right)\right] \frac{2 \,+\, (\gamma{-}1)M_1^2}{(\gamma{+}1)M_1^2} \tag{13}$$

The three static quantity ratios (11), (12), (13), are shown plotted versus $M_1$.