Mach Number Relations

Local Mach number

For a perfect gas, the speed of sound can be given in a number of ways.

\[ a = \sqrt{\gamma RT} = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{(\gamma - 1) h} \]  

(1)

The dimensionless local Mach number can then be defined.

\[ M \equiv \frac{V}{a} = \sqrt{\frac{\rho(u^2 + v^2 + w^2)}{\gamma p}} = \sqrt{\frac{u^2 + v^2 + w^2}{(\gamma - 1) h}} \]

It’s important to note that this is a field variable \( M(x, y, z) \), and is distinct from the freestream Mach number \( M_\infty \). Likewise for \( V \) and \( a \).

The local stagnation enthalpy can be given in terms of the static enthalpy and the Mach number, or in terms of the speed of sound and the Mach number.

\[ h_o = h + \frac{1}{2} V^2 = h \left( 1 + \frac{1}{2} \frac{V^2}{h} \right) = h \left( 1 + \frac{\gamma - 1}{2} M^2 \right) = \frac{a^2}{\gamma - 1} \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \]  

(2)

This now allows the isentropic relations

\[ \frac{p_o}{p} = \left( \frac{\rho_o}{\rho} \right)^\gamma = \left( \frac{h_o}{h} \right)^{\gamma/(\gamma - 1)} \]

to be put in terms of the Mach number rather than the speed as before.

\[ \frac{\rho_o}{\rho} = \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{1/(\gamma - 1)} \]

\[ \frac{p_o}{p} = \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\gamma/(\gamma - 1)} \]

The following relation is also sometimes useful.

\[ 1 - \frac{V^2}{2 h_o} = \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{-1} \]
Normal-Shock Properties

Mach jump relations

We now seek to determine the properties $\rho_2, u_2, p_2, h_2$ downstream of the shock, as functions of the known upstream properties $\rho_1, u_1, p_1, h_1$. In practice, it is sufficient and much more convenient to merely determine the downstream Mach number $M_2$ and the variable ratios, since these are strictly functions of the upstream Mach number $M_1$.

\[
\begin{align*}
\rho_1 & \quad \rightarrow \quad \rho_2 = ? \\
u_1 & \quad \rightarrow \quad u_2 = ? \\
p_1 & \quad \rightarrow \quad p_2 = ? \\
h_1 & \quad \rightarrow \quad h_2 = ? \\
\end{align*}
\]

The starting point is the normal shock equations obtained earlier, with $V = u$ for this 1-D case. They are also known as the Rankine-Hugoniot shock equations.

\[
\begin{align*}
\rho_1 u_1 &= \rho_2 u_2 \quad (3) \\
\rho_1 u_1^2 + p_1 &= \rho_2 u_2^2 + p_2 \quad (4) \\
h_1 + \frac{1}{2} u_1^2 &= h_2 + \frac{1}{2} u_2^2 \quad (5) \\
p_2 &= \frac{\gamma - 1}{\gamma} \rho_2 h_2 \quad (6)
\end{align*}
\]

Dividing the momentum equation (4) by the continuity equation (3) gives

\[
u_1 + \frac{p_1}{\rho_1 u_1} = u_2 + \frac{p_2}{\rho_2 u_2}
\]

or

\[
u_1 - u_2 = \frac{1}{\gamma} \left( \frac{a_2^2}{u_2} - \frac{a_1^2}{u_1} \right) \quad (7)
\]

where we have substituted $p/\rho = a^2/\gamma$ and rearranged the terms.

Now we make use of the energy equation (5). For algebraic convenience we first define the constant total enthalpy in terms of the known upstream quantities

\[
h_1 + \frac{1}{2} u_1^2 \equiv h_o = h_2 + \frac{1}{2} u_2^2
\]

which then gives $a_1^2$ and $a_2^2$ in terms of $u_1$ and $u_2$, respectively.

\[
a_1^2 = (\gamma - 1)h_1 = (\gamma - 1) \left( h_o - \frac{1}{2} u_1^2 \right) \\
a_2^2 = (\gamma - 1)h_2 = (\gamma - 1) \left( h_o - \frac{1}{2} u_2^2 \right)
\]

Substituting these energy relations into the combined momentum/mass relation (7) gives, after some further manipulation

\[
u_1 - u_2 = \frac{\gamma - 1}{\gamma} \left( \frac{h_o}{u_2} \frac{h_o}{u_1} + \frac{1}{2} (u_1 - u_2) \right)
\]
Dividing by \( u_1 - u_2 \) produces

\[
1 = \frac{\gamma - 1}{\gamma} \left( \frac{h_o}{u_1 u_2} + \frac{1}{2} \right) \tag{8}
\]

\[
\frac{(\gamma - 1)h_o}{u_1 u_2} = \frac{\gamma + 1}{2}
\]

\[
\frac{(\gamma - 1)^2 h_o^2}{u_1^2 u_2^2} = \left( \frac{\gamma + 1}{2} \right)^2 \tag{9}
\]

Since \( h_o = h_{o1} = h_{o2} \), we can write

\[
(\gamma - 1)^2 h_o^2 = (\gamma - 1)h_{o1}(\gamma - 1)h_{o2} = a_1^2 \left( 1 + \frac{\gamma - 1}{2} M_1^2 \right) a_2^2 \left( 1 + \frac{\gamma - 1}{2} M_2^2 \right)
\]

and using this to eliminate \( h_o^2 \) from equation (9), and solving for \( M_2 \), yields the desired \( M_2(M_1) \) function. This is shown plotted for \( \gamma = 1.4 \).

\[
M_2^2 = \frac{1 + \frac{\gamma - 1}{2} M_1^2}{\gamma M_1^2 - \frac{\gamma - 1}{2}} \tag{10}
\]

The \( M_1 \to 1^+, M_2 \to 1^- \) limit corresponds to infinitesimal shock, or a sound wave. The \( M_2(M_1) \) function is not shown for \( M_1 < 1 \), since this would correspond to an “expansion shock” which is physically impossible based on irreversibility considerations.

**Static jump relations**

The jumps in the static flow variables are now readily determined as ratios using the known \( M_2 \). From the mass equation (3) we have

\[
\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{u_1^2}{u_1 u_2}
\]

From the Mach definition we have

\[
u_1^2 = M_1^2 a_1^2 = M_1^2 \frac{(\gamma - 1)h_o}{1 + \frac{\gamma - 1}{2} M_1^2}
\]
and from equation (8) we have
\[
\frac{1}{u_1 u_2} = \frac{1}{(\gamma-1) h_o} \frac{\gamma+1}{2}
\]
Combining these gives the shock density ratio in terms of \(M_1\) alone.
\[
\frac{\rho_2}{\rho_1} = \frac{\left(\gamma+1\right) M_1^2}{2 + (\gamma-1) M_1^2}
\] (11)
The combination of the momentum equation (4) and mass equation (3) gives
\[
p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right) = \rho_1 u_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right)
\]
which can be further simplified by using the general relation \(\rho u^2 = \gamma p M^2\), dividing by \(p_1\), and then using (11) to eliminate \(\rho_2/\rho_1\) in terms of \(M_1\). The final result for the shock static pressure ratio is
\[
\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1} \left(M_1^2 - 1\right)
\] (12)
The static temperature or enthalpy ratio is now readily obtained from the pressure and density ratios via the state equation.
\[
\frac{T_2}{T_1} = \frac{p_2}{p_1} \frac{\rho_1}{\rho_2}
\]
The result is
\[
\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma+1} \left(M_1^2 - 1\right)\right] \frac{2 + (\gamma-1) M_1^2}{(\gamma+1) M_1^2}
\] (13)
The three static quantity ratios (11), (12), (13), are shown plotted versus \(M_1\).