Unit M4.5
Solutions For Various Beams

Readings:

CDL 7.6, 8.1-8.5

16.003/004 -- "Unified Engineering"
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LEARNING OBJECTIVES FOR UNIT **M4.5**

Through participation in the lectures, recitations, and work associated with Unit M4.5, it is intended that you will be able to………

- ….**determine/calculate** beam parameters for various beam configurations and **examine** the importance of the various parts of the configurations
- ….**analyze** the behavior and specific aspects of various beam configurations
- ….**model** beam configurations with variations in two dimensions by extending Simple Beam Theory and its associated assumptions

Now that we have defined the model and arrived at the governing equations for the model, we can explore how this can be used for various configurations. We note that the geometry of the cross-section plays an important role (I and Q), so we first look at…

Section Properties

There are two cross-section properties which are important in bending:

I - moment of inertia
Q - first moment of area

Let's take them one at a time to look at the effect of geometry on bending

--> Moment of Inertia, I

General expression is

$$I = \int z^2 b(z)\, dz$$

Note that the width, b, can be a function of z.

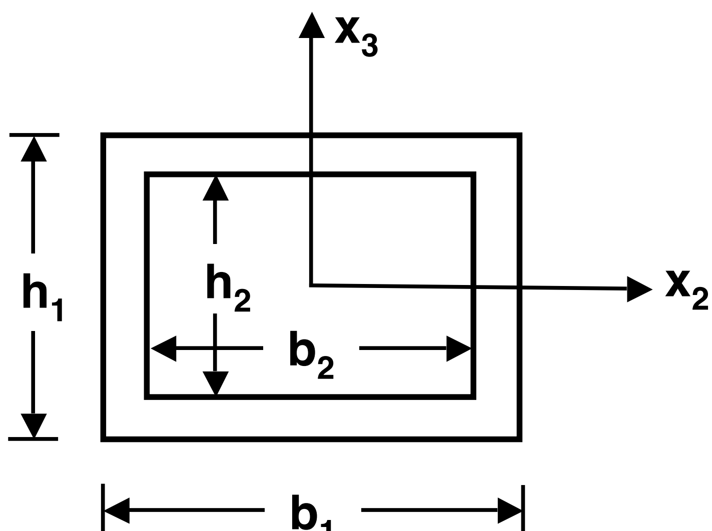
--> For solid, symmetric (in y and z) sections, this integral can be done relatively easily.

See handout M-9 for some common section properties

--> Note that for rings (including thick rings), can use superposition

Example: Rectangular ring

Figure M4.5-1 **Geometry of rectangular box/ring**



$$I_{ring} = I_1 - I_2$$ moment of inertia "removed" from overall rectangle

$$I_{ring} = \frac{b_1 h_1^3}{12} - \frac{b_2 h_2^3}{12}$$

just like area: $A_{ring} = A_1 - A_2 = b_1 h_1 - b_2 h_2$

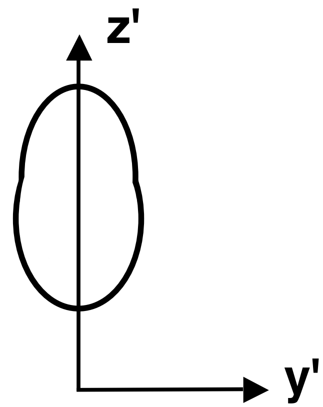
$\Rightarrow$ linear

Limitations are that shapes must be the same and must be concentric

--> For a more general section with bending about one axis (Simple Beam Theory), can relax condition of symmetry about y-axis.

Must have:
- reference axes at ***centroid***
- symmetry in z

Figure M4.5-2 **Geometry of general cross-section with symmetry in z**



Find location of ***centroid*** with reference to arbitrary axis y′ to place axis y:

$$z_{centroid} = \frac{\iint z\,dA}{\iint dA}$$

Definition: ***centroid*** is the center of area
(like center of mass)

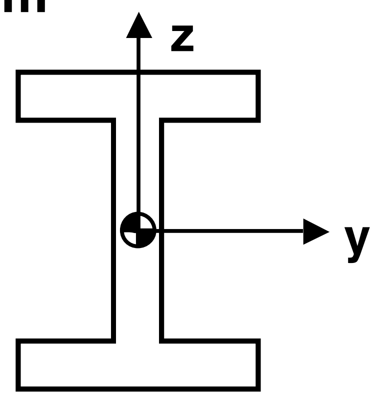
So relocate axes:

Figure M4.5-3 **General cross-section with axes recounted at centroid**

--> same equations and solutions apply
--> get I via normal integration procedures

--> Consider cross-sections with discontinuous parts like <u>the</u> <u>I-beam</u>

Figure M4.5-4 **Geometry of I-beam**



1. Find centroid
2. Divide cross-section into convenient sub-sections
3. Find I of each sub-section about centroid (I_{yy})
4. Add up contributions of all sub-sections (integrating piecewise)

In order to do this we need the…

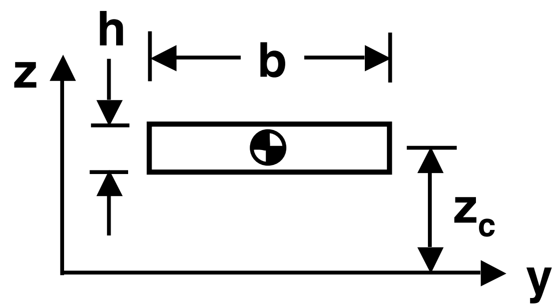
<u>Parallel</u> <u>Axis</u> <u>Theorem</u>:

Moment of inertia of a body about any axis is the moment of inertia of the body about its centroid (I_o) plus its area times the square of the distance from the centroid to the axis.

$$\boxed{I_{yy} = I_o + A z_c^2}$$

Example: Rectangular cross-section

Figure M4.5-5 **Configuration of rectangular cross-section above reference axis y**



$$I_o = \frac{bh^3}{12}$$

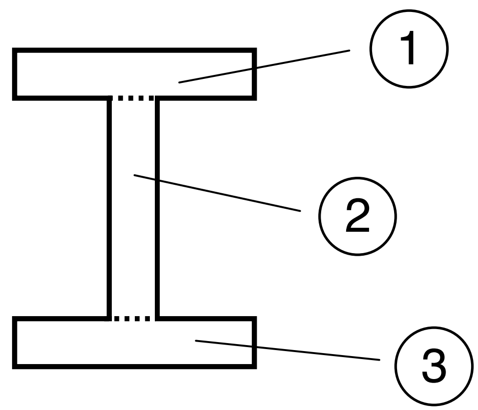
$$A = bh$$

$$\Rightarrow I_{yy} = \frac{bh^3}{12} + bhz_c^2$$

--> Return to I-beam

- Divide up:

Figure M4.5-6 **Geometry of dividing up I-beam into rectangular cross-sections**



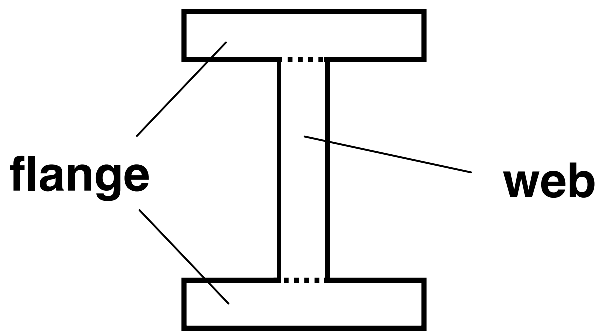
- Find centroid

--> Make table

Section	A	z_c	Az_c	Az_c^2	I_o
1					
2					
3					

$$I_{total} = \sum_{\substack{\text{\# of}\\ \text{sections}}} I_o + \sum_{\substack{\text{\# of}\\ \text{sections}}} A z_c^2$$

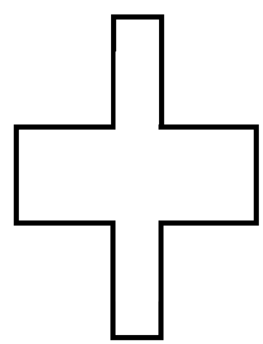
--> <u>Terminology</u> for I-beam



<u>flanges</u> account for most of bending stiffness (greatest contribution to I - *will see in recitation and problem set*)

<u>web</u> connects flanges and provides shear continuity

I-beam more efficient than:



Same area <u>but</u> lower I.

--> get as much area as "far away" from centroid as possible. Why?

- Parallel Axis Theorem: $I = I_o + Az_c^2$

- Definition of I: $I = \int z^2 b(z)\, dz$

increase z (distance) ⇒ increase **I**

--> <u>Efficiency</u> of section measured via I/A:

I/A = moment of inertia per cross-section weight

Maximize this.

In limit…remove flanges to ∞ with infinitesimally thin web, <u>but</u> other restrictions prevent this:

- shearing
- other loads
- size limits

(shearing, other loads: beyond Simple Beam Theory)

Now let's turn to the…

--> Moment of Area, Q

General expression is:

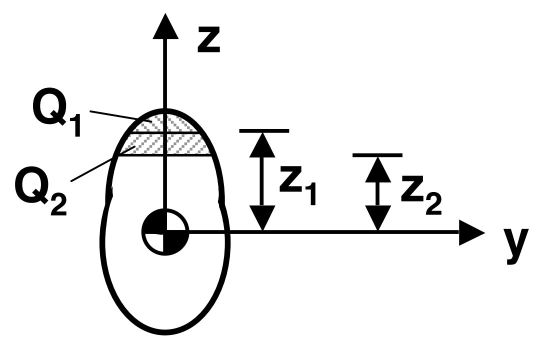
$$Q(z) = \int_{z}^{z_{top}} z' \, b(z') \, dz'$$

defined at each location z

Note: If have discontinuous section (e.g., I-beam), can divide up cross-section and add up (integrating piecewise)

As one progresses in z, are basically adding effect of next section.

Figure M4.5-7 **Illustration of adding contributions of sections for moment of area**



$$Q(z_2) = Q_1 + Q_2$$

since:

$$\int_{z_2}^{z_{top}} f(z) \, dz = \int_{z_1}^{z_{top}} f(z) \, dz + \int_{z_2}^{z_1} f(z) \, dz$$

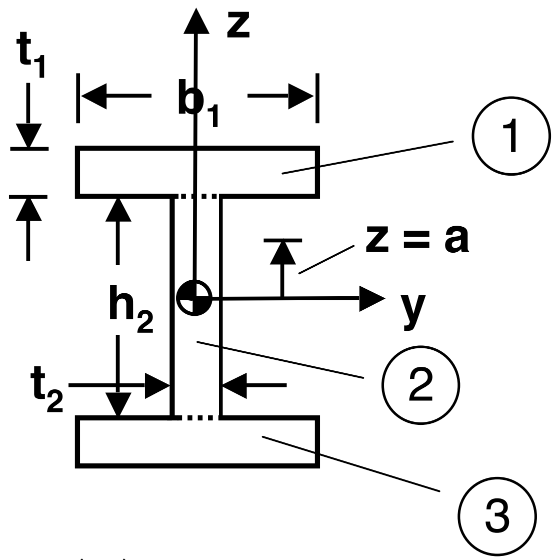
So can divide up into many sections and add:

$$Q = \sum_{\substack{\#\text{ of} \\ \text{subsections}}} Q_i$$

<u>Example</u>: I-beam

Take Q at some point in web

Figure M4.5-8 **Detailed geometry of I-beam cross-section**



$$Q(a) = Q_1 + Q_2(a)$$

$$= \int_{h_2/2}^{h_2/2+t_1} z b_1 dz + \int_{a}^{h_2/2} z t_2 dz \qquad \text{etc.}$$

Note: Generally divide up into subsections of constant width so that b (width) is not a function of z.

Final Note: At “bottom” of cross-section, must get Q = 0 since moment of area above and below centroid is equal in magnitude and opposite in sign.

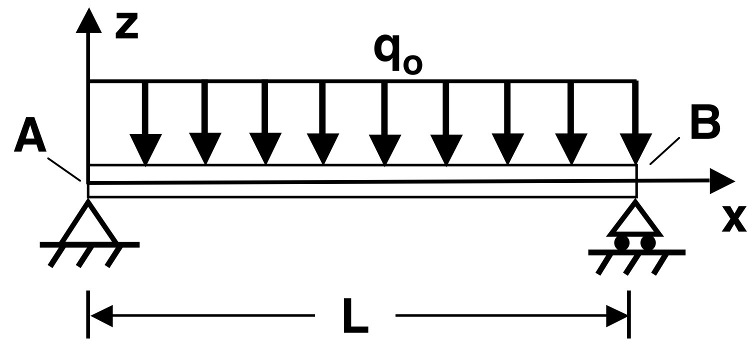
Now that we have a general feeling for beam section properties, we can look non specifically at solutions for various configurations. First, let’s consider the…

General Case: Statically Determinate Beams

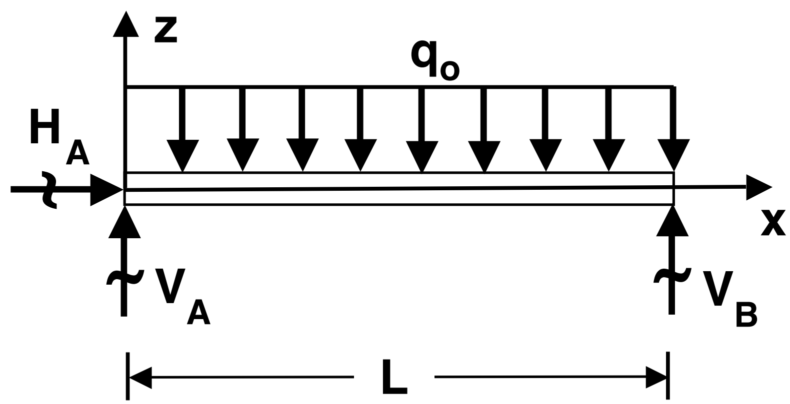
We proceed as indicated in handout M-8. We’ll illustrate the generic procedure for a statically determinate case via an

Example: simply-supported beam with uniform load

Figure M4.5-9 Geometry of simply-supported beam with uniform load



1. Draw Free Body Diagram



2. Get Reactions:

Normal procedure:

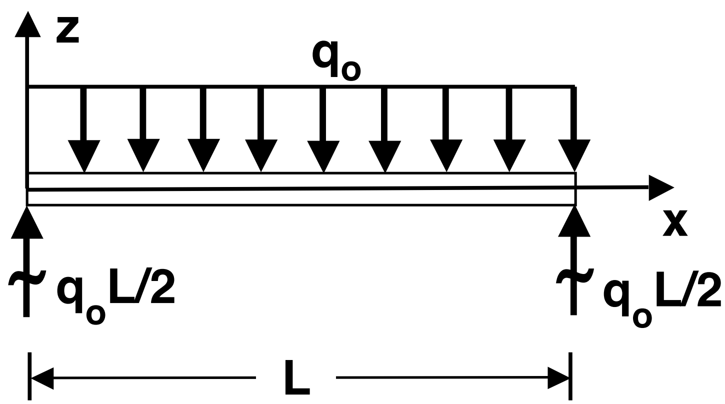
$$\sum F_H = 0 \quad \Rightarrow H_A = 0$$

$$\sum F_V = 0 \quad \Rightarrow V_A + V_B - q_o L = 0$$

$$\sum M_A = 0 \quad \Rightarrow V_B L - \int_o^L q_o x dx = 0$$

$$\Rightarrow V_B = \frac{q_o L}{2} \text{ gives.... } V_A = \frac{q_o L}{2}$$

So:



3. Use relations between q, S, and M

$q(x) = -q_o$ (a constant)

include direction

$$S = \int q(x)dx \Rightarrow S(x) = -q_o x + C$$

use Boundary Conditions:

$$@\ x = 0,\ \ S = \frac{q_o L}{2} \Rightarrow C = \frac{q_o L}{2}$$

$$\Rightarrow S(x) = q_o \left(\frac{L}{2} - x\right)$$

then:

$$M = \int S(x)dx \Rightarrow M(x) = -\int q_o \left(x - \frac{L}{2}\right) dx$$

$$= -q_o \left(\frac{x^2}{2} - \frac{Lx}{2}\right) + C_1$$

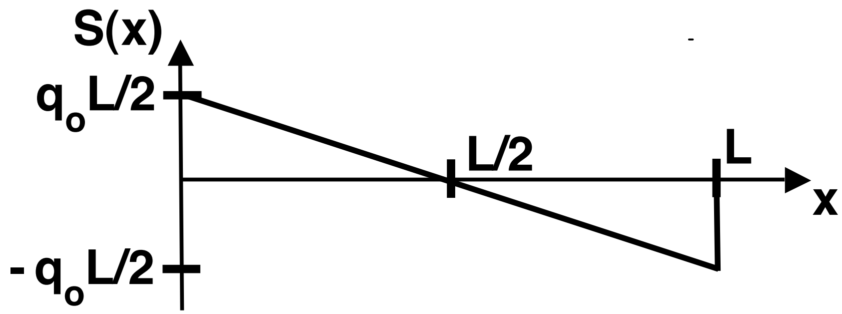
use Boundary Conditions:

$$@\ x = 0,\ \ M = 0 \Rightarrow C_1 = 0$$

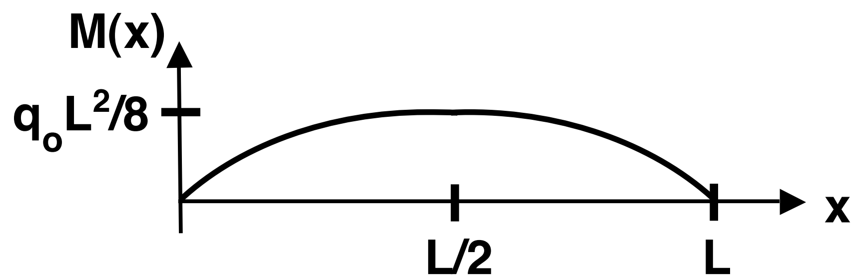
$$\Rightarrow M(x) = \frac{q_o x}{2} (L - x)$$

Sketch these:

- Shear Force Diagram



- Moment Diagram



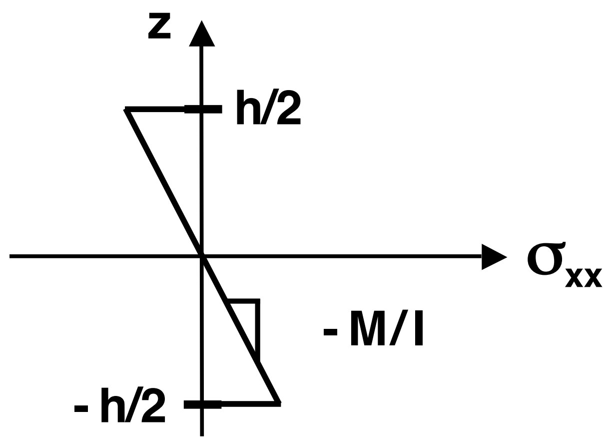
<u>Note</u>: Can check at any point by taking a "cut" and using equilibrium

4. Find stresses and strains

$$\sigma_{xx}(x,z) = -\frac{Mz}{I}$$

--> At any location x, distribution of σ_{11} through thickness is linear

Figure M4.5-10 Distribution of axial stress through thickness



Note: stress (and strain) is zero at midline (in symmetric beams) or more generally at centroid; equal and opposite (tension and compression) on top and bottom, or vice versa

Strain also varies linearly through thickness:

$$\varepsilon_{xx}(x,\ z) = \frac{\sigma_{xx}}{E_x} = -\frac{M(x)z}{EI}$$

$\Rightarrow$ maximum values occur for extremes of z

Q: where does maximum value of σ_{xx} occur in beam?

$$\sigma_{xx} = -\frac{M(x)z}{I}$$

if I does not vary with x, then it occurs at maximum value of M(x).

--> find via $\frac{dM(x)}{dx} = 0$

determine x-location, then plug back in [also check Boundary Conditions if no internal min/max]

σ_{xx}(x) looks the same as M(x) (for constant cross-section)

--> Now look at shear stress:

$$\sigma_{xz}(x,\ z) = -\frac{S(x)Q}{Ib}$$

Maximum value in x occurs at maximum/minimum value of shear:

--> find via $\frac{dS(x)}{dx} = 0$

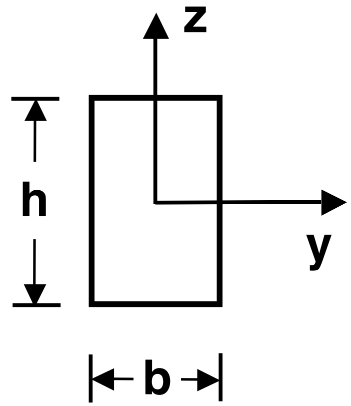
determine x-location, then plug back in [again check Boundary Conditions if no internal min/max]

Q: what about variation in z?

Depends on variation of Q [Q (z)].

$\Rightarrow$ depends on cross-section geometry.

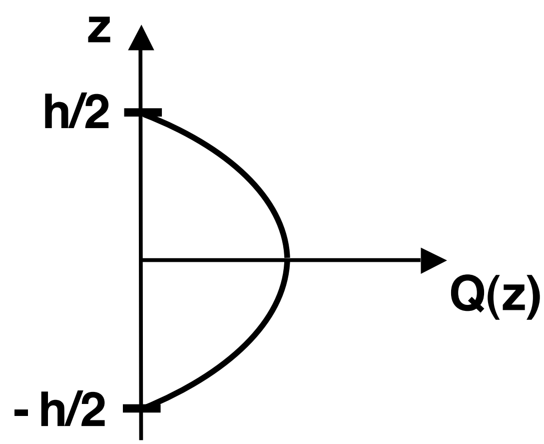
For rectangular cross-section



$$Q = \int_z^{h/2} z' b dz'$$

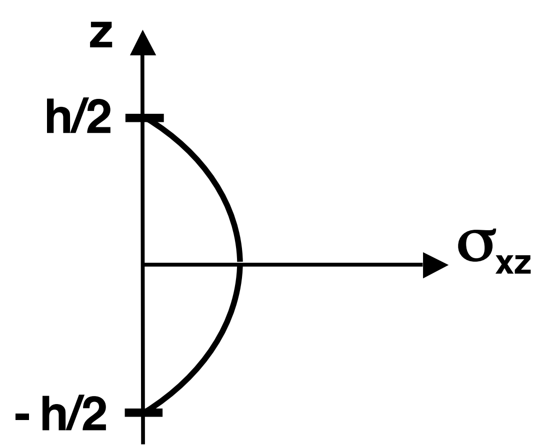
$$= \frac{bz'}{2}\Big]_z^{h/2} = \frac{bh^2}{8} - \frac{bz^2}{2} = \frac{b}{2}\left[\frac{h^2}{4} - z^2\right]$$

Figure M4.5-11 **Variation of moment of area in thickness direction for rectangular cross-section**



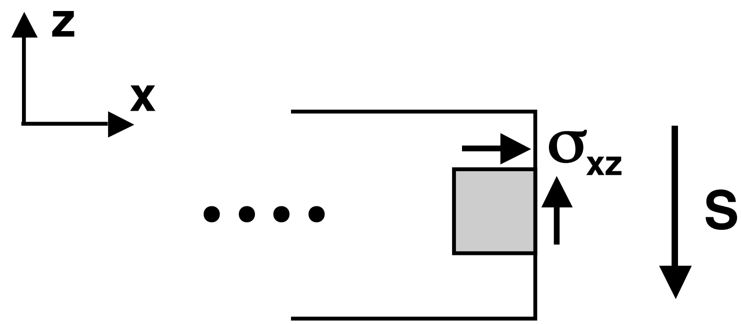
So we have the same for σ_{xz}:

Figure M4.5-12 **Variation of shear stress in the thickness direction for rectangular cross-section**



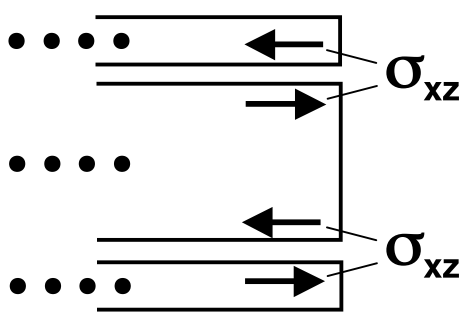
Notes:
- parabolic in shape
- maximum at center line (for symmetric sections) or more generally at centroid
- careful if b(z) (i.e., b is a function of z)
- σ_{xz} generally considerably smaller than σ_{xx} but it "holds beam together"

Figure M4.5-13 **Illustration of beam shear stress**

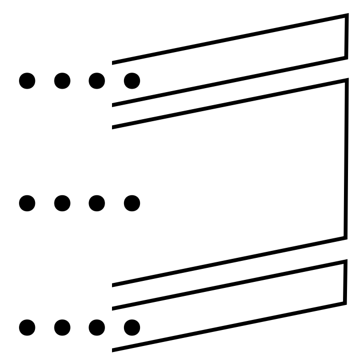


If cut beam through thickness…

Figure M4.5-14 **Illustration of how shear stress keeps beam together**



Keeps beam together, otherwise slippage (like deck of cards)



--> shear stresses are very important for sandwiches, thin webs, etc.
(bonded structure)

5. Find deflections (bending!)

Go to Moment-Curvature relation:

$$EI \frac{d^2w}{dx^2} = M(x)$$

$$\Rightarrow \frac{d^2w}{dx^2} = \frac{M(x)}{EI}$$

$$\Rightarrow w = \iint \frac{M(x)}{EI}$$

For E and I constant with x, this becomes

$$w(x) = \frac{1}{EI} \iint M(x)$$

for our case of simply-supported beam with uniform load:

$$M(x) = -\frac{q_o x}{2}(x - L)$$

$$\Rightarrow \frac{dw(x)}{dx} = -\frac{1}{EI}\int \frac{q_o x}{2}(x - L)\,dx$$

$$= -\frac{q_o}{2}\left[\frac{x^3}{3} - \frac{Lx^2}{2}\right] + C_1$$

So:

$$w(x) = -\frac{1}{EI}\int\left(\frac{q_o}{2}\left[\frac{x^3}{3} - \frac{Lx^2}{2}\right] + C_1\right)dx$$

$$\Rightarrow w(x) = -\frac{q_o}{2EI}\left[\frac{x^4}{12} - \frac{Lx^3}{6} + C_1 x + C_2\right]$$

Need two boundary Conditions

$$@\ x = 0,\ w = 0 \Rightarrow C_2 = 0$$

$$@\ x = L,\ w = 0$$

latter $\Rightarrow 0 = \dfrac{L^4}{12} - \dfrac{L^4}{6} + C_1 L \Rightarrow C_1 = \dfrac{L^3}{12}$

Thus: $w(x) = -\frac{q_o}{24EI}\left[x^4 - 2Lx^3 + L^3 x\right]$

--> Find maximum by determining point where: $\frac{dw(x)}{dx} = 0$

here:

$$\frac{dw(x)}{dx} = 0 \Rightarrow 4x^3 - 6Lx^2 + L^3 = 0$$

find "maximum" at $x = \frac{L}{2}$ (plug back into equation)

Makes sense due to symmetry

This has been the case when the loading was continuous and thus all variables were continuous functions of x. But what if we have…

Discontinuous Loading

As we saw in Unit M4.3, the initial procedure (to get reactions and S and M) is the same except we need to do for each different section of loading. Can include point loads as well.

⇒ stresses and strains have “discontinuities” in x-direction like related shear forces and moment

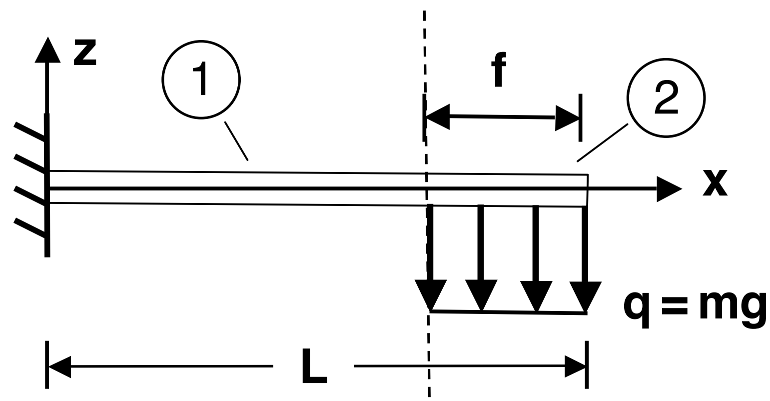
Q: But what about deflection?

Must now do in a sequential fashion and use results at end of previous section as Boundary Conditions for the next section.

Illustrate this via an….

Example of a Cantilevered Flag (recall as done in Unit M4-3)

Figure M4.5-15 **Geometry of cantilevered flag**



Region 1: $0 < x < (L - f)$

Region 2: $(L - f) < x < L$

From before get $M_1(x)$, $S_1(x)$, $M_2(x)$, $S_2(x)$

Now use Moment-Curvature relation in sequential fashion.

--> First in section 1:

$$w_1(x) = \frac{1}{EI} \iint M_1(x)$$

Need 2 Boundary Conditions because it is a double integral

$$@\ x = 0: \quad w = 0, \quad \frac{dw}{dx} = 0$$

--> Now go to junction of sections 1 and 2 and into section 2

$$w_2(x) = \frac{1}{EI} \iint M_2(x)$$

need 2 Boundary Conditions

Match deflection and slope at junction:

$$w_1(L - f) = w_2(L - f)$$

$$\frac{dw_1}{dx}(L - f) = \frac{dw_2}{dx}(L - f)$$

also write: $w_1\big|_{x=L-f} = w_2\big|_{x=L-f}$, etc.

Can extend this to as many junctions/sections that exist

Note: Must check each section to find overall maximum/minimum values of stress, etc.

Let's now again go back and consider more….

General Cross-Sections

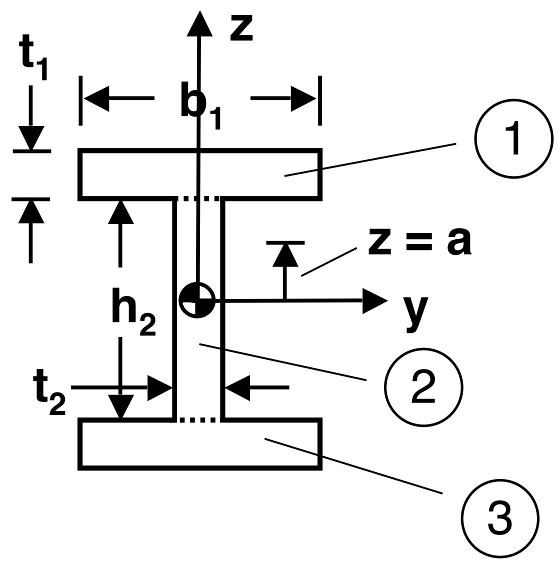
There are two ways that cross-sections can vary: in x and in z.

--> we have looked at the effect of variation in z on the section properties. With regard to the overall solution:

- S, M, σ_{xx}, and w are not affected except by the overall value of I
- σ_{xz} depends on Q and b: $\sigma_{xz} = -\dfrac{SQ}{Ib}$

Q(z) varies “smoothly”, but b(z) can take a sudden “jump” such as in an I-beam

--> Consider the I-beam and the junction between the flange and web ($z = h_2/2$)



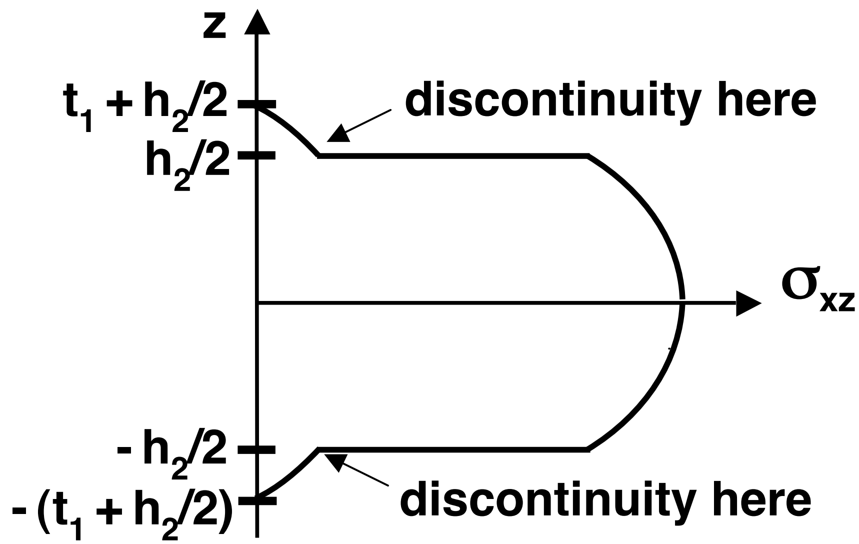
@ $z = h_2/2$, look at case:

in flange $b(z) = b_1$

in web $b(z) = t_2$

So shape of σ_{xz} becomes:

Figure M4.5-16 **Variation of shear stress in thickness direction in I-beam**



$\Rightarrow$ shear tends to be carried in narrow webs

--> Q: What about variation of the cross-section in x?

In that case I and Q are functions of x

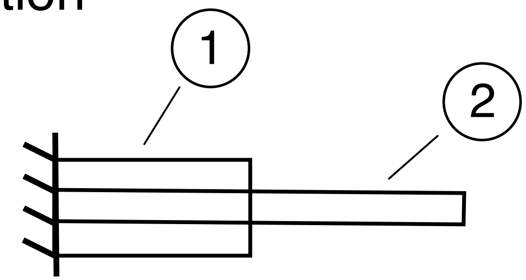
$\Rightarrow$ I(x) and Q(x)

Formally, the model is 1-D and this is not "allowed". **But** slight relaxation of this does not dramatically change limitations.

--> use same equations, but keep I(x) and Q(x) inside any integral

Note: can even have discontinuities in the cross-section

for example:



Q: Could modulus (E) vary with x?

Using the same reasoning, this is not formally allowed, but the same procedure can be used.

Finally let's consider the case of…

Statically Indeterminate Beams

The difference is that in the statically determinate case, the equations are solved sequentially since do not need constitutive relation (Moment-Curvature here) to get S and M.

In the statically indeterminate case, need constitutive relation and must solve equations simultaneously

So procedure becomes:

1. Draw Free Body Diagram
2. Determine equations relating reactions
3. Express S(x) and M(x) in terms of reactions and loading [q(x)]
4. Express w(x) in terms of reactions and loading [q(x)] via expression for M(x)
5. Solve equations from 2, 3, and 4 simultaneously
6. Go back and put it all together, get stresses, etc.

--> Fundamentally we are **superposing** all pieces of the puzzle

FINAL NOTE: Can "pile on" all complications we have talked about and still use *Simple Beam Theory*. Key is to check assumptions and limitations and see if results are "good enough".

(i.e. *consistency*)

Next: Consider a long, slender structural member under yet another type of loading.

Unit M4.5 (New) Nomenclature

I_o -- Moment of inertia of body about its centroid
I_{yy} -- Moment of inertia about y-axis
I/A -- bending efficiency of cross-section
z_c, $z_{centroid}$ -- distance of centroid from reference axis
z_{top} -- distance to top of cross-section from reference axis