PAL
2/5/09

Unified Engineering Problem Set 1

Week 2 Spring, 2009

SOLUTIONS

M1 (M2.1) (a) There are three key sets of equations:

- Equilibrium Equations $\left[\frac{\partial \sigma_{mn}}{\partial x_n} + f_m = 0\right]$

gives 3 equations:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 = 0$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0$$

→ These are based on the fundamental of $\boxed{\text{equilibrium}}$

- Strain-Displacement

$$\left[\epsilon_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right)\right]$$

gives $\underline{\underline{6}}$ equations:

$$\epsilon_{11} = \partial u_1/\partial x_1$$
$$\epsilon_{22} = \partial u_2/\partial x_2$$
$$\epsilon_{33} = \partial u_3/\partial x_3$$
$$\epsilon_{21} = \epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$
$$\epsilon_{31} = \epsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$
$$\epsilon_{32} = \epsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

→ These are based on **geometrical relationships**

→ These have the key assumption that <u>strains are small</u> such that angular changes are small. This can be measured/expressed as: $\cos\theta \approx 1$; $\sin\theta \approx \theta$

• Stress-Strain $\left[\sigma_{mn} = E_{mnpq}\, \epsilon_{pq}\right]$

gives $\underline{\underline{6}}$ equations:

$$\sigma_{11} = E_{1111}\epsilon_{11} + E_{1122}\epsilon_{22} + E_{1133}\epsilon_{33} + 2E_{1123}\epsilon_{23} + 2E_{1113}\epsilon_{13} + 2E_{1112}\epsilon_{12}$$

$$\sigma_{22} = E_{1122}\epsilon_{11} + E_{2222}\epsilon_{22} + E_{2233}\epsilon_{33} + 2E_{2223}\epsilon_{23} + 2E_{2213}\epsilon_{13} + 2E_{2212}\epsilon_{12}$$

$$\sigma_{33} = E_{1133}\epsilon_{11} + E_{2233}\epsilon_{22} + E_{3333}\epsilon_{33} + 2E_{3323}\epsilon_{23} + 2E_{3313}\epsilon_{13} + 2E_{3312}\epsilon_{12}$$

$$\sigma_{23} = E_{1123}\epsilon_{11} + E_{2223}\epsilon_{22} + E_{3323}\epsilon_{33} + 2E_{2323}\epsilon_{23} + 2E_{1323}\epsilon_{13} + 2E_{1223}\epsilon_{12}$$

$$\sigma_{13} = E_{1113}\epsilon_{11} + E_{2213}\epsilon_{22} + E_{3313}\epsilon_{33} + 2E_{2313}\epsilon_{23} + 2E_{1313}\epsilon_{13} + 2E_{1213}\epsilon_{13}$$

$$\sigma_{12} = E_{1112}\epsilon_{11} + E_{2212}\epsilon_{22} + E_{3323}\epsilon_{33} + 2E_{2312}\epsilon_{23} + 2E_{1312}\epsilon_{13} + 2E_{1212}\epsilon_{12}$$

→ These are based only on linear relationship between stress and strain, and they are **constitutive**

(b) Compatibility equations come from geometrical restrictions as manifested in the strain-displacement equations. Displacements must be continuous functions of the defined space (x_1, x_2, and x_3) as the material body is continuous. Thus, with three such functions, the six strains cannot be independent as they derive from three independent functions. The compatibility equations relate the strain fields to be compatible with the continuity of the displacements.

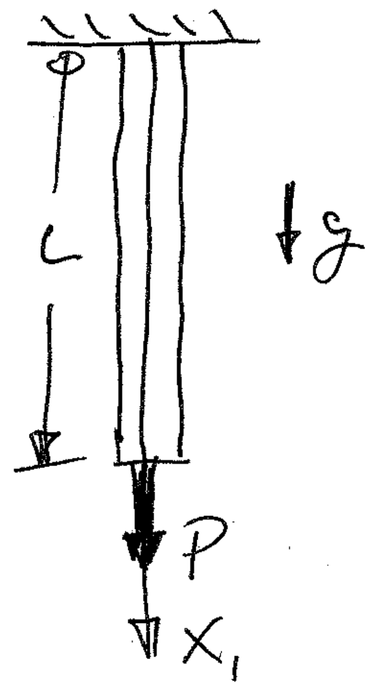
There are 3 independent displacement functions and 6 strain-displacement relations. Thus, there must be $(6-3=)3$ compatibility conditions. They are derived by using the strain-displacement equations, taking "cross" derivatives and equating these.

→ These equations express geometrical restrictions

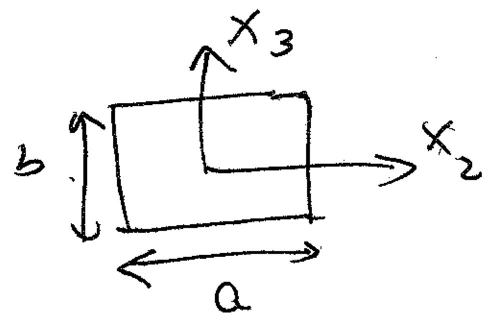
(c) In using engineering notation equations the <u>form</u> of the equations change (e.g. σ_x rather than σ_{11}) but the underlying fundamentals and associated assumptions stay the same and the equations represent <u>the</u> <u>same</u> <u>thing</u>. Only the notation changes.

→ One key change due to definitions is that <u>engineering shear strain</u> is <u>2</u> <u>times</u> <u>tensorial shear strain</u>, so this factor of 2 must be incorporated in all equations with engineering shear strain.

M2 (M2-2)



CROSS-SECTION



→ ignore mass of rod for this problem

(a) Boundary Conditions

@ $x_1 = 0$, rod is fixed to overhead support

⇒ displacements are zero

$$\boxed{@\ x_1 = 0 :\quad u_1,\ u_2,\ u_3 = 0}$$

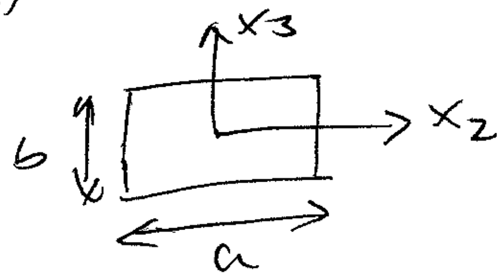
@ $x_1 = L$, overall force of P is applied over area: $A = ab$

so:

$$\boxed{\begin{array}{l} \sigma_{11} = P/A = P/ab \quad @\ x_1 = L \\ \sigma_{13},\ \sigma_{12} = 0 \end{array}}$$

(NOTE: Must include direction of applied load in all considerations. Thus +/– is incorporated into "P" to be included in applying value depending upon whether applied load is tensile (+) or compressive (–).)

→ Along all other surfaces, all stresses on surfaces are zero. Boundary surface is at:



$x_2 = \pm a/2$

$x_3 = \pm b/2$

So:

$$\boxed{@\ x_2 = \pm a/2, \quad x_3 = \pm b/2 \qquad \sigma_{22}, \sigma_{33}, \sigma_{23} = 0}$$

(b) In neglecting the mass of the rod ⇒ no body forces. The only nonzero stress is σ_{11}, so applying the equilibrium equations gives:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \cancelto{=0}{f_1} = 0$$

integrating this gives:

$$\sigma_{11} = \text{constant}$$

Apply the B.C.: @ $x_1 = L$: $\sigma_{11} = \frac{P}{ab}$

$$\Rightarrow \boxed{\sigma_{11} = \frac{P}{ab} \text{ throughout}}$$

all other stresses are zero everywhere

→ To determine the strains, use the stress-strain relations. For an isotropic material, need the longitudinal (Young's) modulus, E, and the Poisson's ratio, ν.

With σ_{11} the only nonzero stress:

$$\boxed{\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0}$$ (no shear strains)

and:

$$\epsilon_{11} = \sigma_{11}/E$$

$$\epsilon_{22} = -\nu/E \, \sigma_{11}$$

$$\epsilon_{33} = -\nu/E \, \sigma_{11}$$

$$\Rightarrow \boxed{\epsilon_{11} = \frac{P}{Eab} \qquad \epsilon_{22} = \epsilon_{33} = -\frac{\nu P}{Eab}}$$

throughout

(c) To find the displacements apply the strain-displacement relations. The only primary consideration is u_1, so use:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = \frac{P}{Eab}$$

integrating gives:

$$u_1 = \frac{P}{Eab} x_1 + C$$

constant of integration since no variation in x_2 and x_3

To find the constant, apply the B.C.:

$$@ \; x_1 = 0, \; u_1 = 0 \quad \Rightarrow \quad C = 0$$

This results in:

$$\boxed{u_1 = \frac{P}{Eab} x_1}$$

→ By definition of the model, u_2 and u_3 are 0. However, note the slight inconsistency since ϵ_{22} and ϵ_{33} are nonzero. Using:

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \epsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

and integrating gives:

$$u_2 = -\frac{\nu P}{Eab} x_2 + C_2$$

$$u_3 = \frac{-\nu P}{Eab} x_3 + C_3$$

Define the midpoint of the cross-section as the point of zero displacement (NOTE: can define any point as the point of reference) to give $C_2 = 0$, $C_3 = 0$

and thus:

$$\boxed{\begin{aligned} u_2 &= \frac{-\nu P}{Eab} x_2 \\ u_3 &= \frac{-\nu P}{Eab} x_3 \end{aligned}}$$

vs. $= 0$ by model

(d) There are always inconsistencies in the model with regard to the u_2 and u_3 displacements. This is built into the model. One must use the St. Venant's Principle in the vicinity of the attachment to the support. "Away" from this region, the model is valid. "Near" the region, σ_{11} may vary with x_2 and x_3 and other stresses may be present.

This will also be affected by the "part" of the problem being considered as the variability of the primary parameters in x_1 changes and this will relate to the importance of the inconsistencies (quantitatively). However, the way to address these do not change.

(e) This is simply an extension of the model to allow area to vary with x_1. In the general case, we replace the constant area ($A = ab$) with a general functional relationship in x_1:

$$A = A(x_1)$$

We have some particular restrictions here:

$$\frac{a}{b} = 2 \Rightarrow b = \frac{a}{2}$$

and

$$A(x_1 = L) = 2A(x_1 = 0)$$

→ However, the key lies in the functional relationship of area to x_1: $A(x_1)$.

Use this in the equations developed herein.

For the stress:

$$\sigma_{11} = \frac{P}{A(x_1)}$$

$$\Rightarrow \sigma_{11} \text{ varies with } x_1$$

Similarly:

$$\epsilon_{11} = \frac{P}{EA(x_1)} \Rightarrow \epsilon_{11} \text{ varies with } x_1$$

as do ϵ_{22} and ϵ_{33}:

$$\epsilon_{22} = \epsilon_{33} = \frac{-\nu P}{EA(x_1)}$$

The expression for the displacement u_1 becomes more involved:

$$u_1 = \int \epsilon_{11}\, dx_1 = \int \frac{P}{EA(x_1)}\, dx_1$$

So an expression for $A(x_1)$ is needed to be specific. However, that is not necessary to explain this generically.
Similarly, u_2 and u_3 vary with x_1. This may make the inconsistencies more important as this depends upon the rate that the area varies with x_1: $\partial A/\partial x_1$, as area controls all the results with it being in the denominator for all the key item.

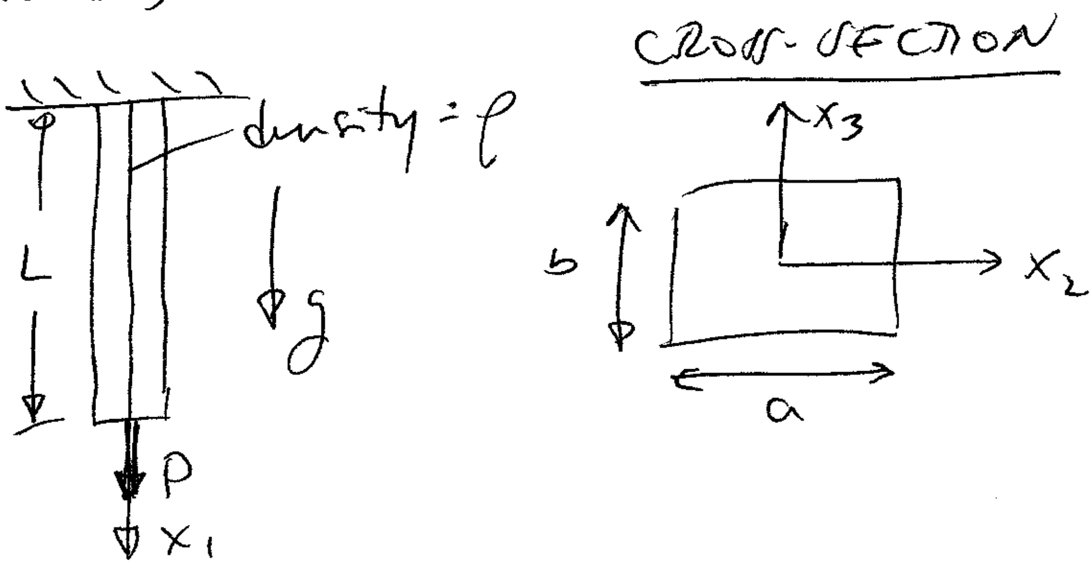
→ Finally, also note that if σ_{11} is a function of x_1, then $\partial\sigma_{11}/\partial x_1$ is nonzero. The first stress equilibrium equation is:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0$$

Since the first term is nonzero one or both other terms <u>must</u> exist to satisfy equilibrium so σ_{12} and/or σ_{13} <u>exist</u>! ⇒ the model further falls apart

→ So the model extension can be used but is less applicable and must be more carefully checked for consistency, etc.

M3 (M2.3)



→ Now include the effects of the mass of the rod

(a) The Boundary conditions $\boxed{\text{do not change}}$ if the body force is taken into account. The body force, f_1, is internal and does not contribute to the surface/boundary conditions.

So repeat:

$$\boxed{\begin{array}{l} @\ x_1 = 0: \quad u_1, u_2, u_3 = 0 \\ @\ x_1 = L: \quad \sigma_{11} = P/ab, \quad \sigma_{12}, \sigma_{13} = 0 \\ @\ \text{all surfaces of} \\ \quad x_2 = \pm a/2: \quad \sigma_{22}, \sigma_{23} = 0 \\ \quad x_3 = \pm b/2: \quad \sigma_{33}, \sigma_{23} = 0 \end{array}}$$

(b) We must now include the body force:

$$f_1 = \frac{\rho g \text{ volume}}{\text{volume}}$$

in the equilibrium equation. So:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \rho g = 0$$

Integrating this gives:

$$\sigma_{11} = -\rho g x_1 + C$$

Again, apply the B.C.: @ $x_1 = L$, $\sigma_{11} = \frac{P}{ab}$

$$\Rightarrow \frac{P}{ab} = -\rho g L + C$$

giving: $C = \frac{P}{ab} + \rho g L$

This results in:

$$\boxed{\sigma_{11} = \frac{P}{ab} + \underbrace{\rho g (L - x_1)}_{\text{new term}}}$$

→ the strains are related through the same equations as in the previous case (basic stress-strain equations do not change). This gives:

$$\boxed{\begin{aligned} \epsilon_{11} &= \frac{P}{Eab} + \underbrace{\frac{\rho g}{E}(L - x_1)} \\ \epsilon_{22} = \epsilon_{33} &= \frac{-\nu P}{Eab} - \underbrace{\frac{\nu \rho g}{E}(L - x_1)} \end{aligned}}$$

new terms

and all shear strains ($\epsilon_{12}, \epsilon_{13}, \epsilon_{23}$) continue to be zero.

(c) The displacement becomes more complicated as ϵ_{11} is a function of x_1. So:

$$u_1 = \int \left\{ \frac{P}{Eab} + \frac{\rho g}{E}(L - x_1) \right\} dx_1$$

$$\Rightarrow u_1 = \frac{P}{Eab} x_1 + \frac{\rho g}{E} L x_1 - \frac{\rho g}{2E} x_1^2 + c_1$$

Again the B.C. of $u_1 = 0$ @ $x_1 = 0$ gives:

$$c_1 = 0$$

yielding:

$$\boxed{u_1 = \frac{P}{Eab} x_1 + \underbrace{\frac{\rho g x_1}{E}\left(L - \frac{x_1}{2}\right)}_{\text{new term}}}$$

Again, the model says u_2 and u_3 are zero, though there is slight inconsistency since ϵ_{22} and ϵ_{33} are nonzero.

Integration through $\epsilon_{22} = \frac{\partial u_2}{\partial x_2}$ and $\epsilon_{33} = \frac{\partial u_3}{\partial x_3}$ yields:

$$\boxed{\begin{aligned} u_2 &= -\frac{\nu P}{Eab} x_2 - \underbrace{\frac{\nu \rho g}{E}(L-x_1)x_2} + C_2 \\ u_3 &= -\frac{\nu P}{Eab} x_3 - \underbrace{\frac{\nu \rho g}{E}(L-x_1)x_3}_{\text{new term}} + C_3 \end{aligned}}$$

Again, $C_2 = C_3 = 0$ by placing the point of reference at the midpoint of the cross-section.

(d) The same inconsistencies exist as in the previous case. However, their importance may be affected by the additional term and therefore the applicability of the model may be affected. This will depend on the specific values of the various parameters

(e) For all the key variables, there are two contributions: one from the tip load (proportional to P/ab) [NOTE: same as in the previous problem], and a new term due to the density of the rod (proportional to $\rho g(L-x_1)$).

Thus, the mass of the rod becomes important when it is of similar magnitude as that of the tip load divided by the area. To do a rough comparison, one can look at the maximum value for the rod contribution and compare it ~~to~~ the other for the applied tip load:

$$\boxed{\rho g L \text{ vs. } \frac{P}{a_b}}$$

[NOTE: check consistency of units:

$$\left[\frac{M}{L^3}\right]\cdot\left[\frac{L}{T^2}\right]\cdot[L] \quad \text{vs.} \quad \Bigg[\underbrace{\frac{M\cdot L}{T^2}}^{\text{Force}}\Bigg]\cdot\left[\frac{1}{L^2}\right]$$

$$= \left[\frac{M}{L\cdot T^2}\right] \quad \text{vs.} \quad \left[\frac{M}{L\cdot T^2}\right] \checkmark$$

]

→ Comparing these two terms allows one to ascertain a % contribution of the rod mass and thus do a quantitative comparison.

→ The direction of the tip load determines whether the stress due to the rod mass will alleviate (P is compressive) or add to (P is tensile) the stress and the ~~tass~~associated items.

(f) To design the rod to be a "structural fuse" at the overhead attachment, the maximum stress magnitude must occur at this point and it must be designed to reach σ_{ult} (+ or −) for the desired load, P_{fuse}. This is not just making a change in cross-sectional area that allows the model to still be applied (as discussed in (e) of the previous problem) and making this the point of minimum area, but this must be done in such a way as to maximize the magnitude of σ_{11}. This is due to the fact that the stress due to the mass ~~of~~ of the rod can act opposite to the stress due to the applied load if that load is compressive.

→ Nevertheless, applying the equations as developed herein and finding an area that maximizes the stress at this point will yield the desired result.

→ the operational equation is:

$$\sigma_{11} = \frac{P}{ab} + \rho g (L - x_1)$$

to determine where the stress is maximum, look at the two ends (since it varies linearly)

$$@\ x_1 = L : \quad \sigma_{11} = \frac{P}{ab}$$

$$@\ x_1 = 0 : \quad \sigma_{11} = \frac{P}{ab} + \rho g L$$

Consider the 2 cases of tensile and compressive tip load:

→ $P > 0$ ⇒ maximum stress always at $x_1 = 0$.

Thus, failure occurs there. Make a slight decrease in area to promote that.

→ $P < 0$ ⇒ mass alleviates the stress due to the tip load. Thus, need an extra decrease of area at $x_1 = 0$ to get stress maximum here.

change (ab) @ $x_1 = 0$ so that:

$$\sigma_{11}(x_1 = 0) = \frac{-|P|}{(ab)_{local}} + \rho g L < \underbrace{\frac{-|P|}{ab}}_{\text{max value of stress without area changed}}$$

<u>NOTE</u>: This would change if the mass of the rod were to <u>double</u> the stress from the tip load and again the maximum occur at $x_1 = 0$ without a change in area:

$$\sigma_{11} = \frac{-|P|}{ab} + \underbrace{\rho g}_{\text{doubles other factor}} L$$