PAL
4/16/09

Unified Engineering Problem Set 10

Week 12 Spring 2009

SOLUTIONS

M18 (M12.1)

Condition A:

$\sigma_{11} = 8q$ $\sigma_{12} = 4q$
$\sigma_{22} = 2q$ $\sigma_{13} = 0$
$\sigma_{33} = q$ $\sigma_{23} = 0$

Condition B:

$\sigma_{11} = -2q$ $\sigma_{12} = 0$
$\sigma_{22} = -2q$ $\sigma_{13} = 0$
$\sigma_{33} = -2q$ $\sigma_{23} = 0$

Condition C:

$\sigma_{11} = q$ $\sigma_{12} = 0$
$\sigma_{22} = 0.5q$ $\sigma_{13} = 0$
$\sigma_{33} = 2q$ $\sigma_{23} = 0$

Condition D:

$\sigma_{11} = -q$ $\sigma_{12} = 0$
$\sigma_{22} = 2q$ $\sigma_{13} = 0$
$\sigma_{33} = 0.5q$ $\sigma_{23} = 0$

(a) Application of the Tresca condition requires knowledge of the principal stresses.

For Conditions B, C and D, there are no applied shear stresses, so the applied normal stresses are the principal stresses.

→ Put these in appropriate order based on magnitude:

Condition B	Condition C	Condition D
$\sigma_I = \sigma_{11} = -2q$	$\sigma_I = \sigma_{33} = 2q$	$\sigma_I = \sigma_{22} = 2q$
$\sigma_{II} = \sigma_{22} = -2q$	$\sigma_{II} = \sigma_{11} = q$	$\sigma_{II} = \sigma_{11} = -q$
$\sigma_{III} = \sigma_{33} = -2q$	$\sigma_{III} = \sigma_{22} = 0.5q$	$\sigma_{III} = \sigma_{33} = 0.5q$

→ for <u>Condition A</u>, there is no applied shear stress in the 3-axis since $\sigma_{13} = 0$ and $\sigma_{23} = 0$. Thus, σ_{33} is a principal stress.
However, σ_{12} is nonzero, so the principal stresses in the 1-2 plane need to be determined.

Call $\sigma_{33} = \sigma_{III}$ for now and label the two in the 1-2 plane as σ_I and σ_{II}. Actual "order" will be determined after determining principal stresses in 1-2 plane.

→ For the case of planar stress, the principal stresses are the roots (τ) of the equation:

$$\tau^2 - \tau(\sigma_{11} + \sigma_{22}) + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0$$

Applies here since have principal stress along x_3.
Using the values for Condition A:

$$\tau^2 - \tau(8q + 2q) + [(8q)(2q) - (4q)^2] = 0$$

$$\Rightarrow \tau^2 - 10q\tau + [16q^2 - 16q^2] = 0$$

$$\tau(\tau - 10q) = 0$$

$$\Rightarrow \tau = \sigma_I = 10q$$

$$\tau = \sigma_{II} = 0$$

Finally, in order for Condition A:

$$\sigma_I = 10q$$

$$\sigma_{II} = q \quad (\text{from } \sigma_{33})$$

$$\sigma_{III} = 0$$

→ Now apply the Tresca criterion where yield occurs if:

$$|\sigma_{I} - \sigma_{II}| = \sigma_Y$$

or

$$|\sigma_{II} - \sigma_{III}| = \sigma_Y$$

or

$$|\sigma_{III} - \sigma_{I}| = \sigma_Y$$

In addition, the directionality associated with this is that yielding occurs via shear on the plane of maximum shear stress corresponding to the difference in those two principal stresses

→ For the titanium under consideration,

$$\sigma_Y = 98.0 \text{ ksi}$$

→ Apply each condition....

• Condition A : $|\sigma_{I} - \sigma_{II}| = |10q - q| = \sigma_Y$

$$\Rightarrow |9q| = 98.0 \text{ ksi}$$

$$\Rightarrow q = \pm 10.9 \text{ ksi}$$

$$|\sigma_{II} - \sigma_{III}| = |q - 0| = \sigma_Y$$

$$\Rightarrow q = \pm 98.0 \text{ ksi}$$

$$|\sigma_{III} - \sigma_{I}| = \sigma_y$$

$$\Rightarrow \quad |0 - 10q| = 98.0 \text{ ksi}$$

$$\Rightarrow \quad q = \pm 9.8 \text{ ksi}$$

→ Critical case is the last one

so, for A:

$$\boxed{\begin{array}{l}\text{yielding at } q = \pm 9.8 \text{ ksi} \\ \text{on plane at } 45^\circ \text{ to direction of} \\ \text{principal stress in 1-2 plane}^* \\ \text{and } \sigma_{33}\end{array}}$$

NOTE: Find angle in 1-2 plane by using:

$$\theta_p = \frac{1}{2} \tan^{-1}\left(\frac{2\sigma_{12}}{\sigma_{11} - \sigma_{22}}\right)$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{8q}{8q - 2q}\right)$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{8}{6}\right)$$

$$\Rightarrow \theta_p = \frac{1}{2}(53.1^\circ)$$

giving $\boxed{\theta_p = 26.6^\circ}$

- <u>Condition B</u>: No shear stresses and all equal extensional stresses
  $\Rightarrow$ hydrostatic stress state
  ~~giving~~ all differences = 0

  so, for (B): $\boxed{\text{No yielding}}$

- <u>Condition C</u>:

$$|\sigma_I - \sigma_{II}| = |2q - q| = |q| = \sigma_Y$$

$$\Rightarrow q = \pm 98.0 \text{ ksi}$$

$$|\sigma_{II} - \sigma_{III}| = |q - 0.5q| = |0.5q| = \sigma_Y$$

$$\Rightarrow q = \pm 196 \text{ ksi}$$

$$|\sigma_{III} - \sigma_I| = |0.5q - 2q| = |1.5q| = \sigma_Y$$

$$\Rightarrow q = \pm 65.3 \text{ ksi}$$

$\rightarrow$ critical case is the last one.

So, for C:

$$\boxed{\begin{array}{c}\text{yielding at } q = \pm 65.3\,\text{ksi} \\ \text{on plane at } 45^\circ \text{ between} \\ \sigma_{22} \text{ and } \sigma_{33}\end{array}}$$

• Condition D:

$$|\sigma_{I} - \sigma_{II}| = |2q - (-q)| = \sigma_Y$$
$$\Rightarrow |3q| = 98.0\,\text{ksi}$$
$$\Rightarrow q = \pm 32.7\,\text{ksi}$$

$$|\sigma_{II} - \sigma_{III}| = |-q - 0.5q| = \sigma_Y$$
$$\Rightarrow |1.5q| = 98.0\,\text{ksi}$$
$$\Rightarrow q = \pm 65.3\,\text{ksi}$$

$$|\sigma_{III} - \sigma_{I}| = |0.5q - 2q| = \sigma_Y$$
$$\Rightarrow |1.5q| = 98.0\,\text{ksi}$$
$$\Rightarrow q = \pm 65.3\,\text{ksi}$$

→ critical case is the first

So, for D:

$$\boxed{\text{yielding at } q = \pm 32.7\text{ksi on plane at } 45^\circ \text{ between } \sigma_{11} \text{ and } \sigma_{22}}$$

(b) The von Mises criterion is:

$$(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_{III} - \sigma_I)^2 = 2\sigma_Y^2$$

→ Look at each condition again.

- Condition A:

$$(10q - q)^2 + (q - 0)^2 + (0 - 10q)^2 = 2\sigma_Y^2$$

$$\Rightarrow \quad 81q^2 + q^2 + 100q^2 = 2\sigma_Y^2$$

$$182 q^2 = 2\sigma_Y^2$$

$$\Rightarrow \quad q = \pm\sqrt{\frac{1}{91}}\,\sigma_Y$$

$$\Rightarrow \quad \text{for A: } \boxed{q = \pm 10.3 \text{ ksi}}$$

- Condition B:

All differences equal zero.

So, again for B:

$$\boxed{\text{No yielding}}$$

· Condition C:

$$(2q-q)^2+(q-0.5q)^2+(0.5q-2q)^2=2\sigma_Y^2$$

$$\Rightarrow q^2+0.25q^2+2.25q^2=2\sigma_Y^2$$

$$3.5q^2=2\sigma_Y^2$$

giving: $q=\pm\sqrt{\frac{2}{3.5}}\,\sigma_Y$

$$\Rightarrow \text{for C: } \boxed{q=\pm 74.1\text{ ksi}}$$

· Condition D:

$$(2q-(-q))^2+(-q-0.5q)^2+(0.5q-2q)^2=2\sigma_Y^2$$

$$\Rightarrow 9q^2+2.25q^2+2.25q^2=2\sigma_Y^2$$

$$13.5q^2=2\sigma_Y^2$$

giving $q=\pm\sqrt{\frac{2}{13.5}}\,\sigma_Y$

$$\Rightarrow \text{for D: } \boxed{q=\pm 37.7\text{ ksi}}$$

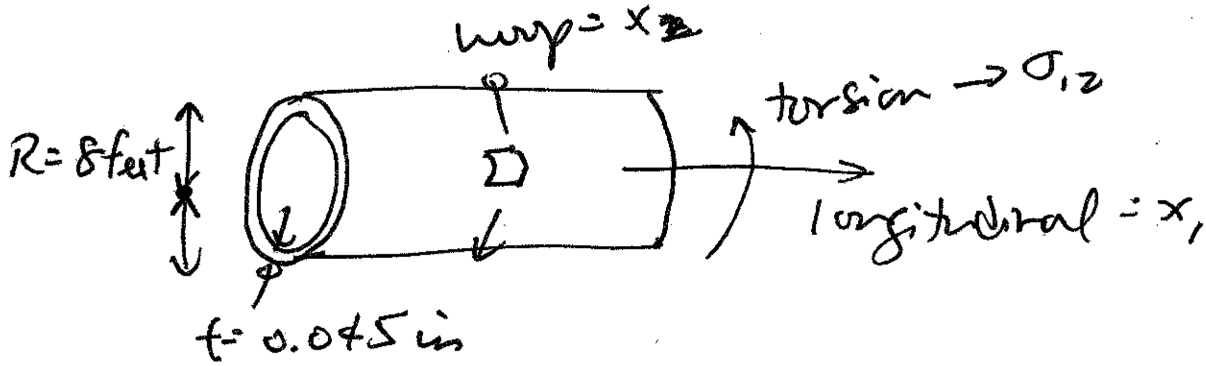
(c) First summarize the overall results:

Condition	Critical $\|q\|$, [ksi]	
	Tresca	von Mises
A	9.8	10.3
B	—	—
C	65.3	74.1
D	32.7	37.7

- For each condition, the Tresca Criterion gives a more conservative estimate of the yielding load characteristic q, except for the hydrostatic case (Condition B) where both criteria predict no yielding as this is a fundamental basic for each

- The Tresca Criterion considers yielding on a single plane and thus only the two principal stresses acting on that plane. In contrast the von Mises Criterion involves and interacts all the applied stresses and thus slightly higher values.

M19 (M12.2) Airplane fuselage



At limit, $p = 10$ psi (pressure differential)
pressure causes stress of:

$$\sigma_{hoop} = \sigma_{22} = \frac{pR}{t}$$

$$\sigma_{long} = \sigma_{11} = \frac{pR}{2t}$$

(a) Stress in skin of fuselage is sum of stress due to pressure differential and stress from empennage loads accounting for 40% load-carrying factor of skin.

So:

$$\sigma_{11} = 0.4\left(\sigma_{11\,(\text{due to } p)} + \sigma_{11\,(\text{applied long.})}\right) \quad (1)$$

$$\sigma_{22} = 0.4\left(\sigma_{22\,(\text{due to } p)}\right) \quad (2)$$

$$\sigma_{12} = 0.4\left(\sigma_{12\,(\text{applied torsion})}\right) \quad (3)$$

→ Using the pressure equations at the limit condition:

$$\sigma_{11}\left(\text{due to } p\right) = \frac{(10\,psi)(8\,feet)(12\,in/ft)}{2(0.045\,in)}$$

$$= 10{,}667\,psi = 10.7\,ksi$$

$$\sigma_{22}\left(\text{due to } p\right) = \frac{(10\,psi)(8\,feet)(12\,in/ft)}{2(0.045\,in)}$$

$$= 21{,}333\,psi = 21.3\,ksi$$

Using in the above of (1), (2), (3):
All in [ksi]

$$\sigma_{11} = 4.3 + 0.4\,\sigma_{11}\,(\text{applied lung}) \qquad (1')$$

$$\sigma_{22} = 8.5 \qquad (2')$$

$$\sigma_{12} = 0.4\,\sigma_{12}\,(\text{applied torsion}) \qquad (3')$$

→ Now use the Tresca Criterion:

$$|\sigma_{I} - \sigma_{II}| = \sigma_Y$$

or

$$|\sigma_{II} - \sigma_{III}| = \sigma_Y$$

or

$$|\sigma_{III} - \sigma_{I}| = \sigma_Y$$

The case is plane stress with $\sigma_{III} = 0$.

→ It is necessary to find the in-plane principal stresses σ_{I} and σ_{II}. Use:

$$\tau^2 - \tau(\sigma_{11} + \sigma_{22}) + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0$$

and find roots. Do so in this form with the quadratic solution:

$$\tau = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \sigma_{I}, \sigma_{II} = \frac{1}{2}\left\{(\sigma_{11} + \sigma_{22}) \pm \left[(\sigma_{11} + \sigma_{22})^2 - 4(\sigma_{11}\sigma_{22} - \sigma_{12}^2)\right]^{1/2}\right\}$$

working through gives:

$$= \frac{1}{2}(\sigma_{11} + \sigma_{22}) \pm \frac{1}{2}\left[\sigma_{11}^2 + 2\sigma_{11}\sigma_{22} + \sigma_{22}^2 - 4\sigma_{11}\sigma_{22} + 4\sigma_{12}^2\right]^{1/2}$$

$$= \frac{1}{2}(\sigma_{11} + \sigma_{22}) \pm \frac{1}{2}\left[\sigma_{11} - 2\sigma_{11}\sigma_{22} + \sigma_{22}^2 + 4\sigma_{12}^2\right]^{1/2}$$

$$= \frac{1}{2}(\sigma_{11} + \sigma_{22}) \pm \frac{1}{2}\left[4\sigma_{12}^2 + (\sigma_{11} - \sigma_{22})^2\right]^{1/2}$$

finally:

$$\sigma_{I} = \left(\frac{\sigma_{11} + \sigma_{22}}{2}\right) + \sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2}$$

$$\sigma_{II} = \left(\frac{\sigma_{11} + \sigma_{22}}{2}\right) - \sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2}$$

Rewrite the Tresca equation using these equations and $\sigma_{III} = 0$ and $\sigma_{Y} = 50$ ksi

$$50\,ksi = |\sigma_I - \sigma_{II}| = \left|2\sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2}\right| \quad (4)$$

$$50\,ksi = |\sigma_I| = \left|\left(\frac{\sigma_{11}+\sigma_{22}}{2}\right) + \sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2}\right| \quad (5)$$

$$50\,ksi = |\sigma_{II}| = \left|\left(\frac{\sigma_{11}+\sigma_{22}}{2}\right) - \sqrt{\sigma_{12}^2 + \left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2}\right| \quad (6)$$

→ Use the specific expressions of (1'), (2') and (3') to express these for the particular loading.

NOTE: ALL IN [ksi]

from (4):
$$50 = \left|2\sqrt{0.16\,\sigma_{A.T.}^2 + \left(\frac{4.3+0.4\sigma_{A.L.}-8.5}{2}\right)^2}\right|$$

$$\Rightarrow \boxed{25 = \left|\sqrt{0.16\,\sigma_{A.T.}^2 + (0.2\sigma_{A.L.} - 2.1)^2}\right|} \quad (4')$$

from (5):
$$50 = \left|\frac{1}{2}(4.3+0.4\sigma_{A.L.}+8.5) + \sqrt{(0.2\sigma_{A.L.} - 2.1)^2 + 0.16\,\sigma_{A.T.}^2}\right|$$

$$\Rightarrow \boxed{50 = \left|(6.4+0.2\sigma_{A.L.}) + \sqrt{0.16\,\sigma_{A.T.}^2 + (0.2\sigma_{A.L.}-2.1)^2}\right|} \quad (5')$$

from (6):
$$\boxed{50 = \left|(6.4+0.2\sigma_{AL}) - \sqrt{0.16\,\sigma_{AT}^2 + (0.2\sigma_{AL}-2.1)^2}\right|} \quad (6')$$

with $\sigma_{AL} = \sigma_{11}$ (applied longitudinal); $\sigma_{AT} = \sigma_{12}$ (applied torsion)

→ Now apply different values of σ_{AL} for each case and determine the values of σ_{AT} that cause failure. Be sure to catch cases of each equal to zero. Then plot these.

Work the equations to use them:
All in [ksi]

from (4'):

$$625 = 0.16\,\sigma_{AT}^2 + (0.2\sigma_{AL} - 2.1)^2$$

$$\Rightarrow \sigma_{AT} = 2.5\sqrt{625 - (0.2\sigma_{AL} - 2.1)^2} \quad (4'')$$

from (5'):

$$(43.6 - 0.2\sigma_{AL})^2 = 0.16\,\sigma_{AT}^2 + (0.2\sigma_{AL} - 2.1)^2$$

or

$$(-56.4 - 0.2\sigma_{AL})^2 = 0.16\,\sigma_{AT}^2 + (0.2\sigma_{AL} - 2.1)^2$$

giving:

$$\sigma_{AT} = 2.5\sqrt{(43.6 - 0.2\sigma_{AL})^2 - (0.2\sigma_{AL} - 2.1)^2} \quad (a)$$

or

$$= 2.5\sqrt{(-56.4 - 0.2\sigma_{AL})^2 - (0.2\sigma_{AL} - 2.1)^2} \quad (b)$$

(5'')

from (6'):

$$0.16\,\sigma_{AT}^2 + (0.2\sigma_{AL} - 2.1)^2 = (-43.6 + 0.2\sigma_{AL})^2$$

or

$$= (56.4 + 0.2\sigma_{AL})^2$$

giving:

$$\sigma_{AT} = 2.5\sqrt{(-43.6 + 0.2\sigma_{AL})^2 - (0.2\sigma_{AL} - 2.1)^2} \quad (a)$$

or

$$= 2.5\sqrt{(56.4 + 0.2\sigma_{AL})^2 - (0.2\sigma_{AL} - 2.1)^2} \quad (b)$$

(6'')

→ work these with all values in [ksi]

- from (4'')

$\sigma_{11\,A.C.}$	$\sigma_{12\,A.T.}$
0	± 62.3
+10.5, −	± 62.5, −
+30, −30	±61.7, ±59.1
+60, −60	±57.4, ±51.6
+90, −90	±48.2, ±37.2
+120, −	±30.1, −
+135, −114	0

- from (5'')

$\sigma_{11\,A.C.}$	$\sigma_{12\,A.T.}$ (a)	$\sigma_{12\,A.T.}$ (b)
0	± 109	±140
+30, −30	±93.5, ±122	±156, ±124
+60, −60	±75.0, ±134	±169, ±105
+90, −90	±50.1, ±146	±181, ± 81.8
+114, −	0, −	higher, −
−, −136	−, higher	−, 0

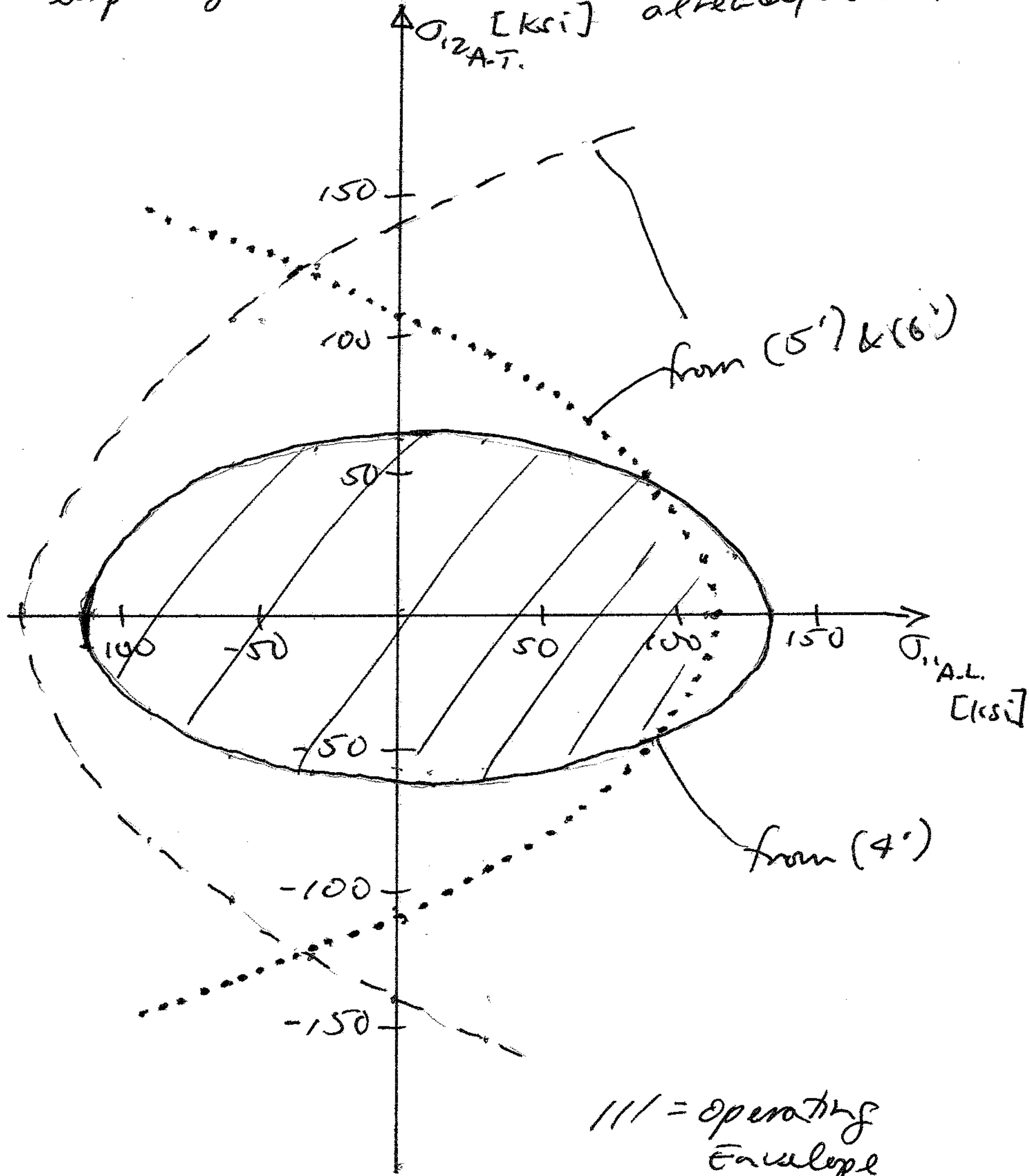
• from (6'')

σ_{11} A.L.	σ_{12} A.T. (a)	σ_{12} A.T. (b)
0	±109	±140
+30, −30	±93.5, ±122	±156, ±124
+60, −60	±75.0, ±134	±169, ±105
+90, −90	±50.1, ±146	±181, ±81.8
+114, −	0 , −	higher, −
− , −136	− , higher	− , 0

Note that (5'') and (6'') produce the same results.

→ Plot these using the lower values of the (a) and (b) to fit the final envelope

→ "Operating stresses envelope" for fuselage via Tresca condition for empennage stresses with limit pressure already accounted



(b) With the "damage tolerant" approach use the basic fracture mechanics equation:

$$\sigma_f = \frac{K_{IC}}{\sqrt{\pi a}}$$

Here: $2a = 0.30$ in $\Rightarrow a = 0.150$ in

$K_{IC} = 31$ ksi$\sqrt{\text{in}}$ for the 2024 aluminum

$$\Rightarrow \sigma_f = \frac{31 \text{ ksi}/\sqrt{\text{in}}}{\sqrt{\pi (0.150 \text{ in})}}$$

$$\Rightarrow \sigma_f = 45.2 \text{ ksi}$$

Thus, if the stress perpendicular to the crack exceeds 45.2, there is failure. The crack can be oriented in any direction so one must find the principal stresses (i.e. the maximum extensional stresses) and then the related direction for the worst case.

→ The principal stresses were found in part (a)

$$\sigma_I = 6.4 + 0.2\sigma_{AL} + \sqrt{0.16\sigma_{AT}^2 + (0.2\sigma_{AL} - 2.1)^2}$$

$$\sigma_{II} = 6.4 + 0.2\sigma_{AL} - \sqrt{0.16\sigma_{AT}^2 + (0.2\sigma_{AL} - 2.1)^2}$$

VALUES IN [ksi]

→ In the case of fracture mechanics, only tensile values are considered and only the largest value needs to be considered.

Thus, use σ_I and set this to the determined value of σ_F VALUES IN [ksi]

$$\Rightarrow \boxed{45.2 = 6.4 + 0.2\sigma_{AL} + \sqrt{0.16\sigma_{AT}^2 + (0.2\sigma_{AL} - 2.1)^2}}$$

working:

$$(38.8 - 0.2\sigma_{AL})^2 = 0.16\sigma_{AT}^2 + (0.2\sigma_{AL} - 2.1)^2$$

→ Again, apply different values of σ_{AL} and get resulting values of σ_{AT}. Then plot these.

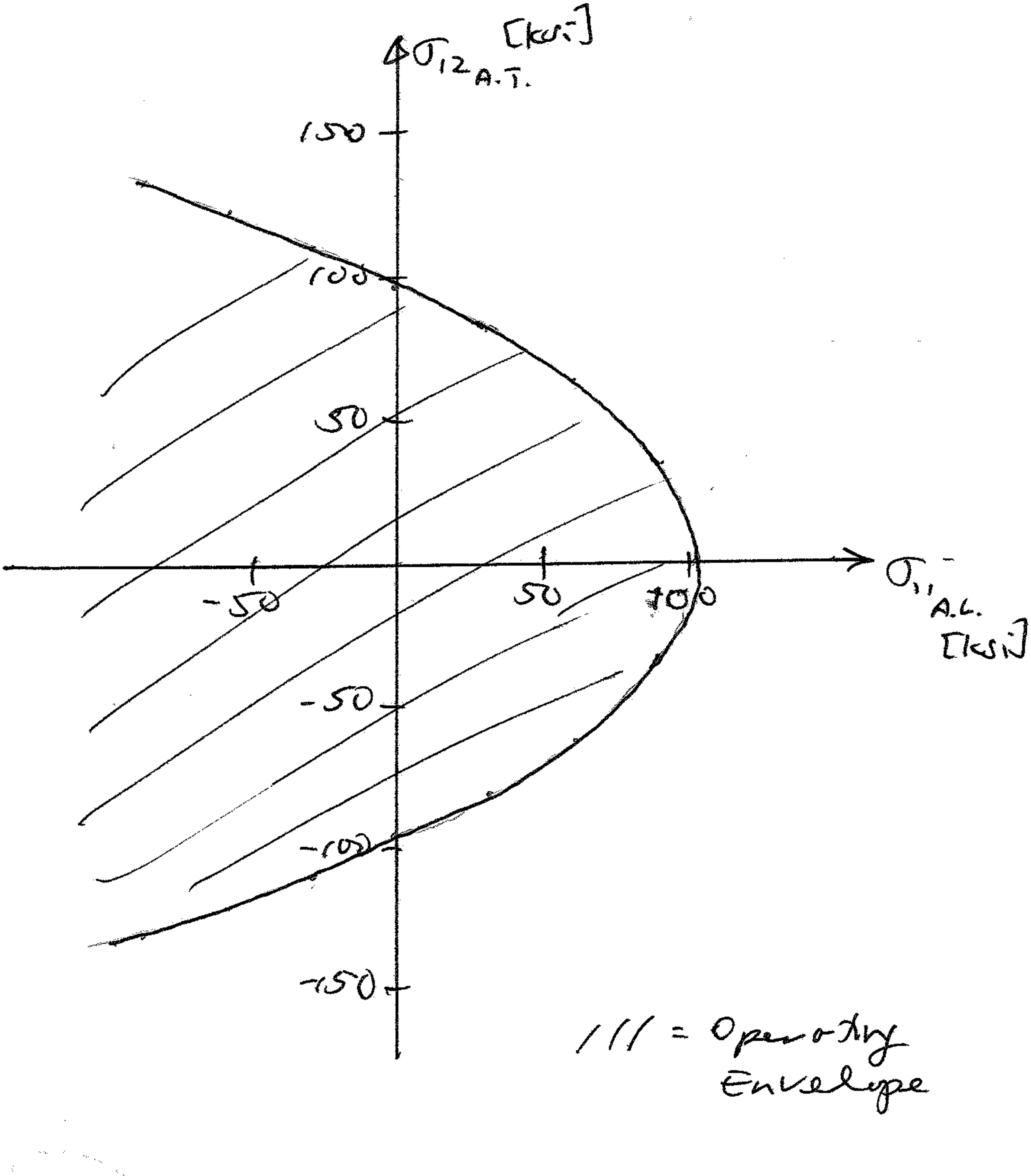
$$\sigma_{AT} = \pm 2.5\sqrt{(38.8 - 0.2\sigma_{AL})^2 - (0.2\sigma_{AL} - 2.1)^2}$$

all values in [ksi]

$\sigma_{11\,A.L.}$	$\sigma_{12\,A.T.}$
0	± 96.9
+30, −30	± 81.4, ±110
+60, −60	± 62.3, ±122
+90, −90	± 33.5, ±133
+102	0

→ Plot these as before

→ "Operating stress envelope" for fuselage via damage tolerant approach with limit pressure already accounted



(c) Each of these approaches are different criteria and the plots do look substantially different as may well be expected. The Tresca Criterion gives the yield point, while the damage tolerant approach gives the stress at which a crack will critically propagate. This latter case occurs only for tensile stresses and thus has a large expanding area for a compressive longitudinal stress. The Tresca Criterion considers compressive stress conditions as well and thus closes off the operating envelope for that consideration. In the tensile regime, the operating stresses begin in a similar area as may be expected for a good design.