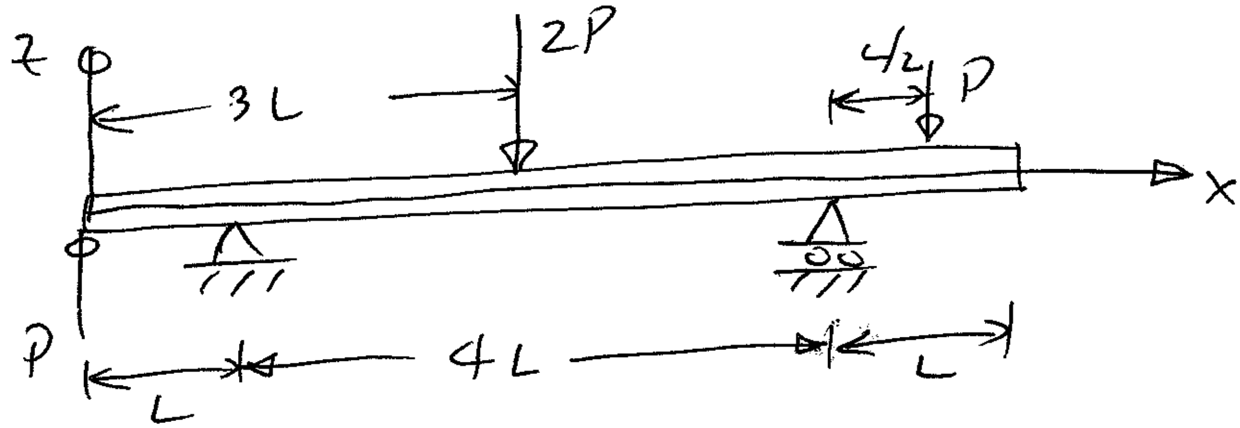
PAL
2/14/09

Unified Engineering Problem Set 2

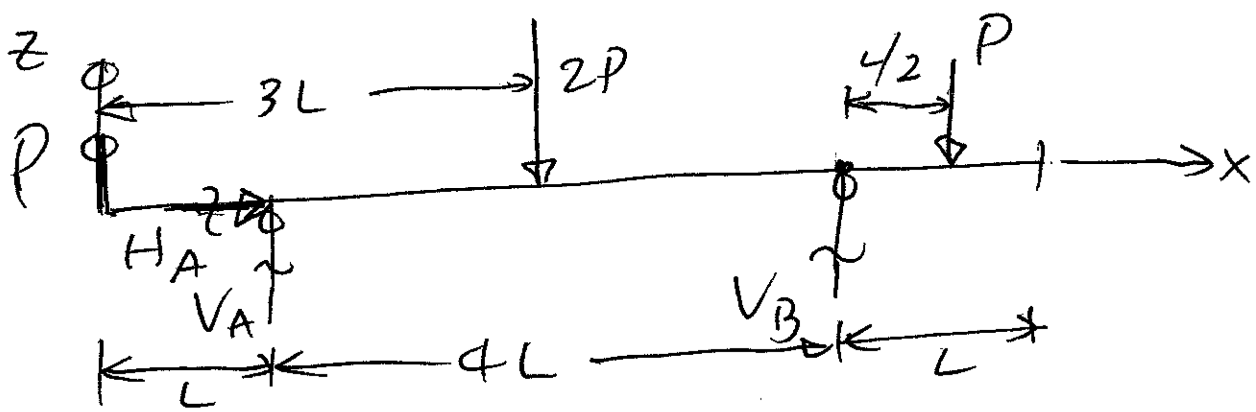
Week 3 Spring, 2009

SOLUTIONS

M4 (M3.1)



(a) Draw the Free Body Diagram (FBD):



Take equilibrium:

$$\sum F_x = 0 \ \xrightarrow{+} \ \Rightarrow \ H_A = 0$$

$$\sum F_z = 0 \ \uparrow + \ \Rightarrow \ P + V_A - 2P + V_B - P = 0$$

giving: $V_A + V_B - 2P = 0 \quad (1)$

$$\sum M_A = 0 \ \circlearrowleft + \ \Rightarrow \ -(P)(L) - (2P)(2L) + (V_B)(4L) - (P)\left(\frac{9L}{2}\right) = 0$$

giving: $4V_B = P + 4P + \frac{9}{2}P$

$$\Rightarrow V_B = \frac{19}{8}P$$

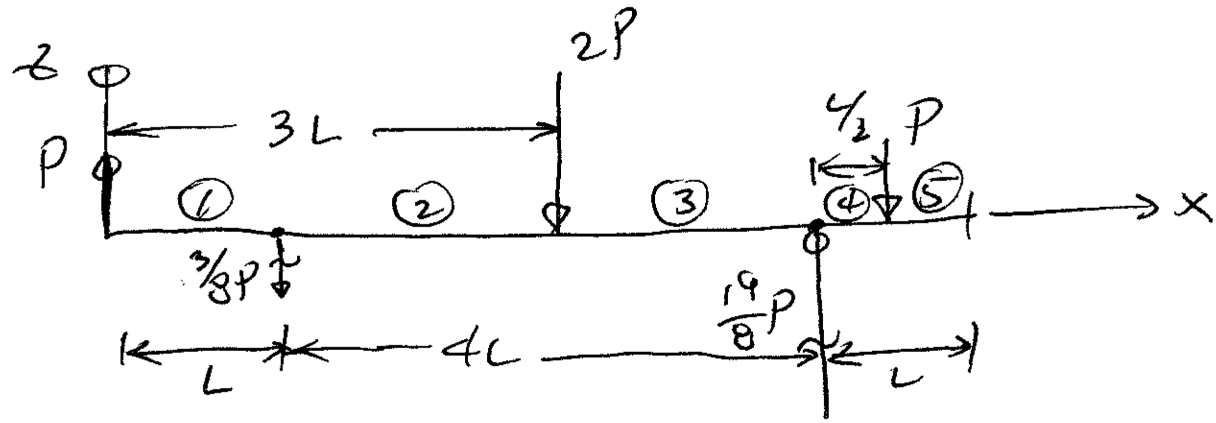
Apply equation (1):

$$\Rightarrow V_A = -\frac{3}{8}P$$

Summarizing:

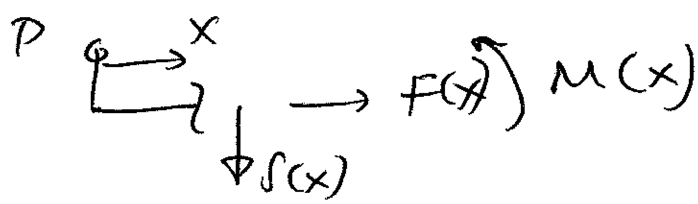
$$\boxed{\begin{aligned} V_A &= -\frac{3}{8}P \\ V_B &= \frac{19}{8}P \\ H_A &= 0 \end{aligned}}$$

(b) First redraw the Free Body Diagram:



Now take cuts in each of the five sections (as labeled) and apply equilibrium. The sections are defined by where loading changes.

→ Section ①: $0 < x < L$



$\sum F_x = 0 \xrightarrow{+} \Rightarrow F(x) = 0$

$\sum F_z = 0 \uparrow + \Rightarrow P - S(x) = 0$

$\Rightarrow S(x) = P$

$\sum M_x = 0 \circlearrowleft + \Rightarrow -Px + M(x) = 0$

$\Rightarrow M(x) = Px$

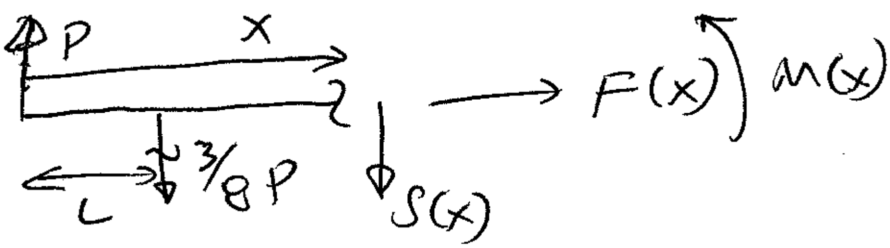
So for:

$0 < x < L$

$$\boxed{\begin{aligned} S(x) &= P \\ M(x) &= Px \end{aligned}}$$

with $F(x) = 0$

→ Section ② : $L < x < 3L$



$\Sigma F_x = 0 \xrightarrow{+} \;\Rightarrow\; F(x) = 0$

$\Sigma F_z = 0 \uparrow + \;\Rightarrow\; P - \frac{3}{8}P - S(x) = 0 \;\Rightarrow\; S(x) = \frac{5}{8}P$

$\Sigma M_x = 0 \circlearrowleft + \;\Rightarrow\; -Px + \frac{3}{8}P(x-L) + M(x) = 0$

$\Rightarrow M(x) = \frac{P}{8}(5x + 3L)$

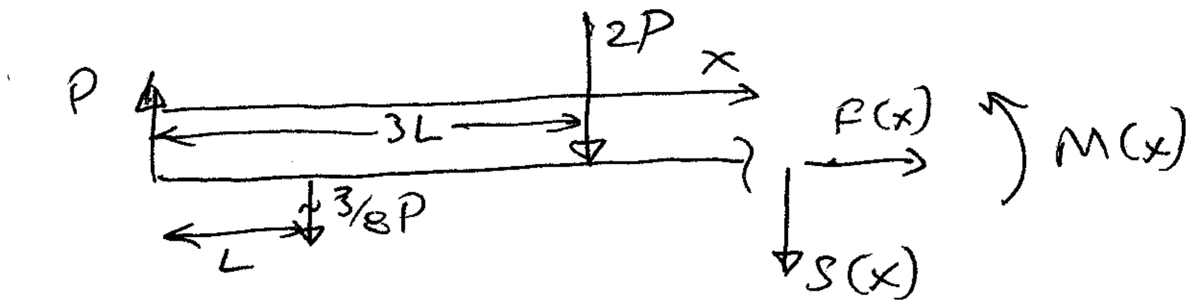
So for:

$L < x < 3L$

$$\boxed{\begin{aligned} S(x) &= \frac{5}{8}P \\ M(x) &= \frac{P}{8}(5x+3L) \end{aligned}}$$

with $F(x) = 0$

→ Section ③ : $3L < x < 5L$



$$\sum F_x = 0 \quad \xrightarrow{+} \quad \Rightarrow \quad F(x) = 0$$

$$\sum F_z = 0 \quad \uparrow + \quad \Rightarrow P - \frac{3}{8}P - 2P - S(x) = 0$$

giving $S(x) = -\frac{11}{8}P$

$$\sum M_x = 0 \quad \circlearrowleft + \quad \Rightarrow -Px + \frac{3}{8}P(x-L) + 2P(x-3L) + M(x) = 0$$

workthrough:

$$M(x) = P\left(x - \frac{3}{8}x - 2x\right) + P\left(\frac{3}{8}L + 6L\right)$$

$$\Rightarrow M(x) = \frac{P}{8}(-11x + 51L)$$

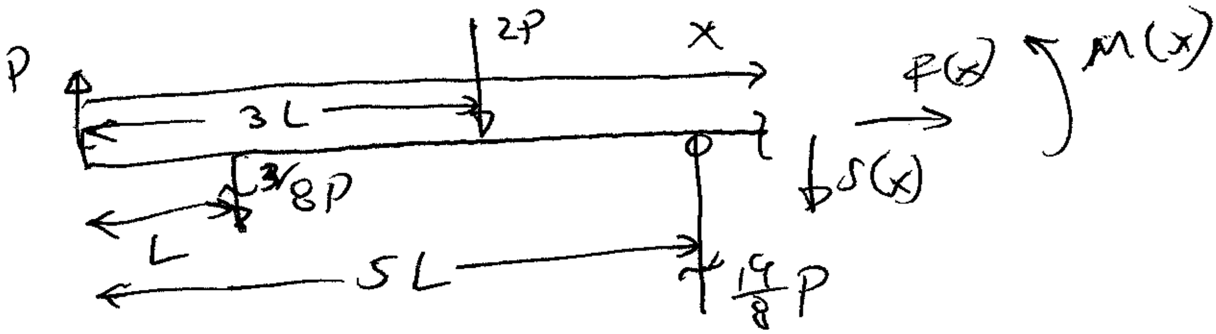
so for:

$3L < x < 5L$

$$\boxed{\begin{aligned} S(x) &= -\frac{11}{8}P \\ M(x) &= \frac{P}{8}(-11x + 51L) \end{aligned}}$$

with $F(x) = 0$

$\rightarrow$ Section (4) : $5L < x < \frac{11}{2}L$



(NOTE: Could also consider – face cut and consider less applied loads....)

$\sum F_x = 0 \quad \xrightarrow{+} \quad \Rightarrow F(x) = 0$

$\sum F_z = 0 \quad \uparrow + \quad \Rightarrow P - \frac{3}{8}P - 2P + \frac{19}{8}P - S(x) = 0$

giving: $S(x) = P$

$\sum M_x = 0 \quad \circlearrowleft + \quad \Rightarrow -Px + \frac{3}{8}P(x-L) + 2P(x-3L)$

$- \frac{19}{8}P(x-5L) + M(x) = 0$

work through:

$M(x) = P(x - \frac{3}{8}x - 2x + \frac{19}{8}x) + P(\frac{3}{8}L + 6L - \frac{95}{8}L$

$\Rightarrow M(x) = P(x - \frac{44}{8}L) = P(x - \frac{11}{2}L)$

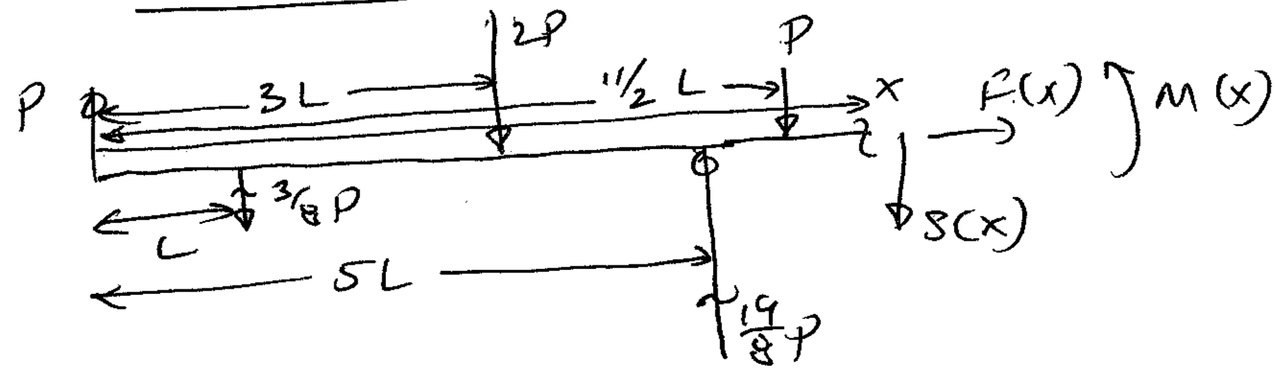
so for:

$5L < x < \frac{11}{2}L$

$$\boxed{\begin{aligned} S(x) &= P \\ M(x) &= P(x - \tfrac{11}{2}L) \end{aligned}}$$

with $F(x) = 0$

→ Section 5 : $\frac{11}{2}L < x < 6L$



(Same NOTE as before....)

$\sum \vec{F}_x = 0 \quad \xrightarrow{+} \Rightarrow F(x) = 0$

$\sum F_z = 0 \quad \uparrow + \Rightarrow P - \frac{3}{8}P - 2P + \frac{19}{8}P - P - S(x) = 0$

giving $S(x) = 0$

$\sum M_x = 0 \quad \circlearrowleft + \Rightarrow -Px + \frac{3}{8}P(x-L) + 2P(x-3L)$
$- \frac{19}{8}P(x-5L) + P(x - \frac{11}{2}L) + M(x) = 0$

workthrough:

$M(x) = P(x - \frac{3}{8}x - 2x + \frac{19}{8}x - x) + P(\frac{3}{8}L + 6L - \frac{95}{8}L + \frac{11}{2}L)$

$\Rightarrow M(x) = 0$

so for: $\frac{11}{2}L < x < 6L$

$\boxed{\begin{array}{l} S(x) = 0 \\ M(x) = 0 \end{array}}$

with $F(x) = 0$

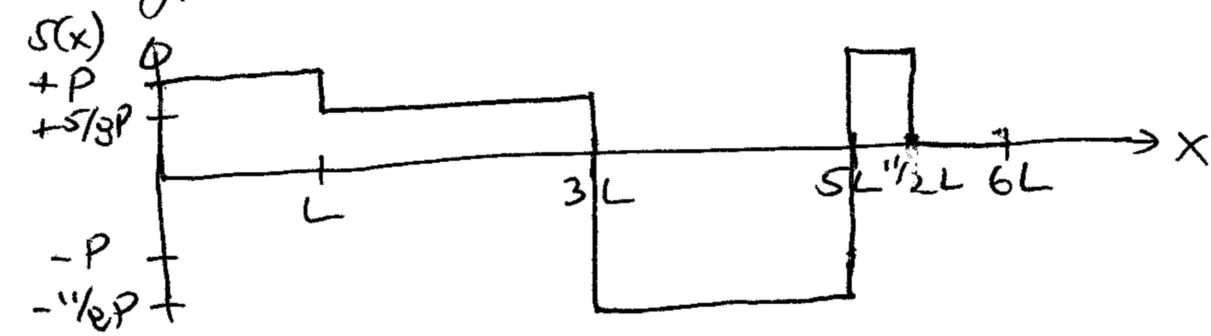
→ NOTE: ways to check this:

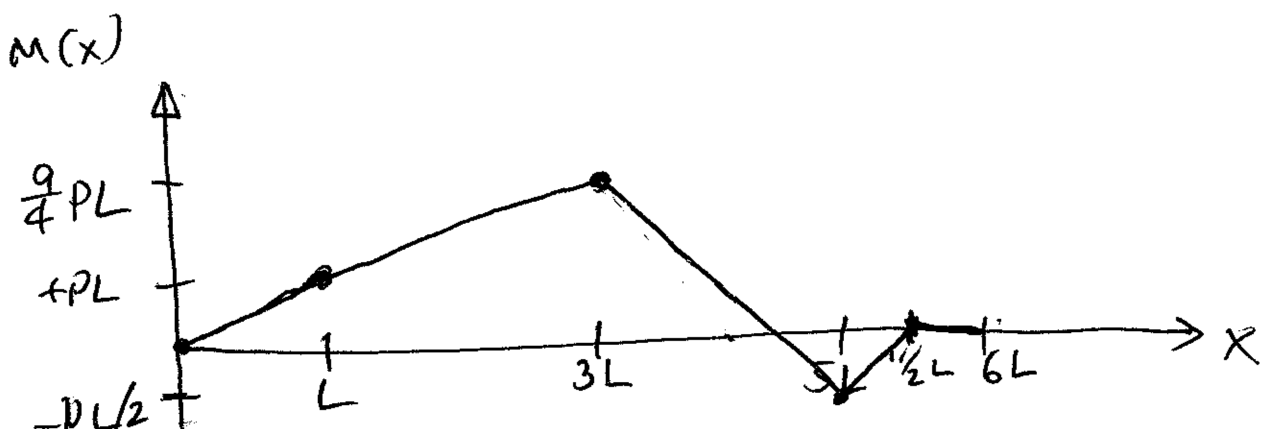
1. At junctions of sections, the moments must be the same. They are in all cases.

2. At junctions of sections, the shears must change by the magnitude and direction of any point load applied at that point. They do in all cases

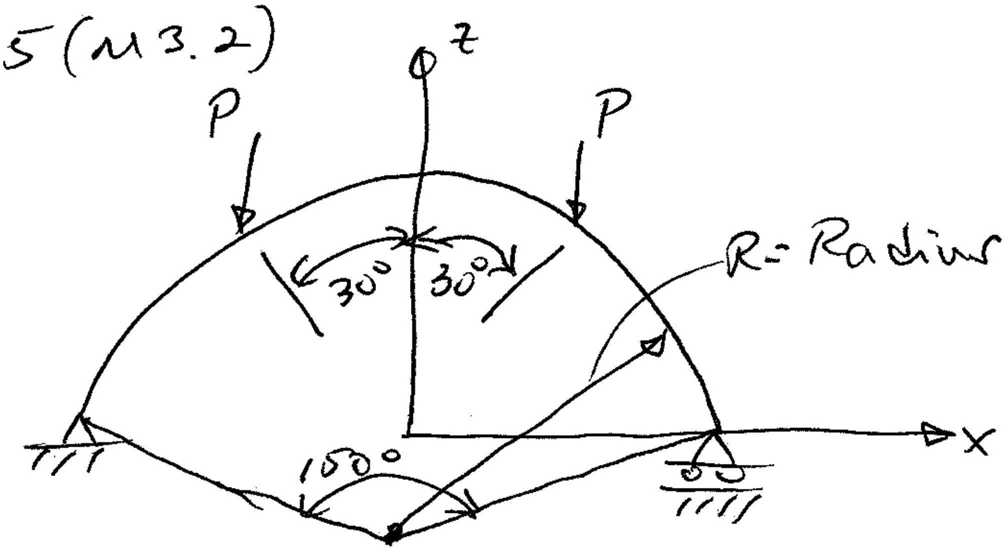
3. At the unloaded tip, the shear and moment must be zero. They are.

Finally, draw the diagrams:



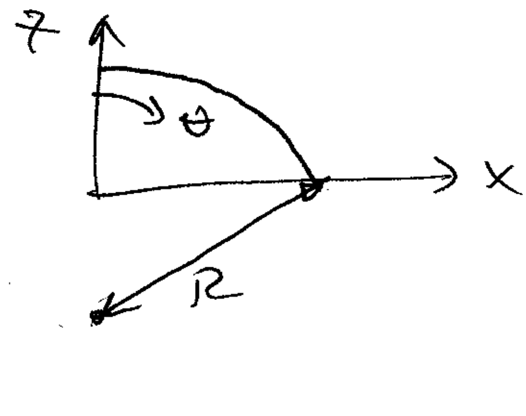


M5 (M3.2)



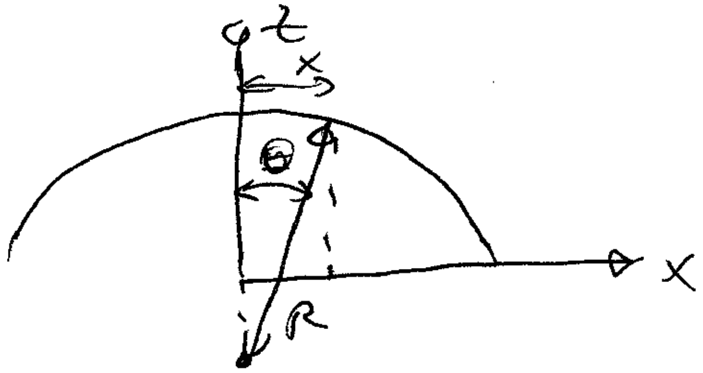
arched beam

(a) Before drawing the free body diagram, it helps to do a bit of geometry in order to express any point on the beam in terms of the x-z system as related to the angle, θ, in the polar coordinate system of the circle with θ measured +cw from the z-axis:



This will also allow us to determine the distance to each support and the locations of the applied loads P.

So:



The distance in x is: $x = R\sin\theta$

So:

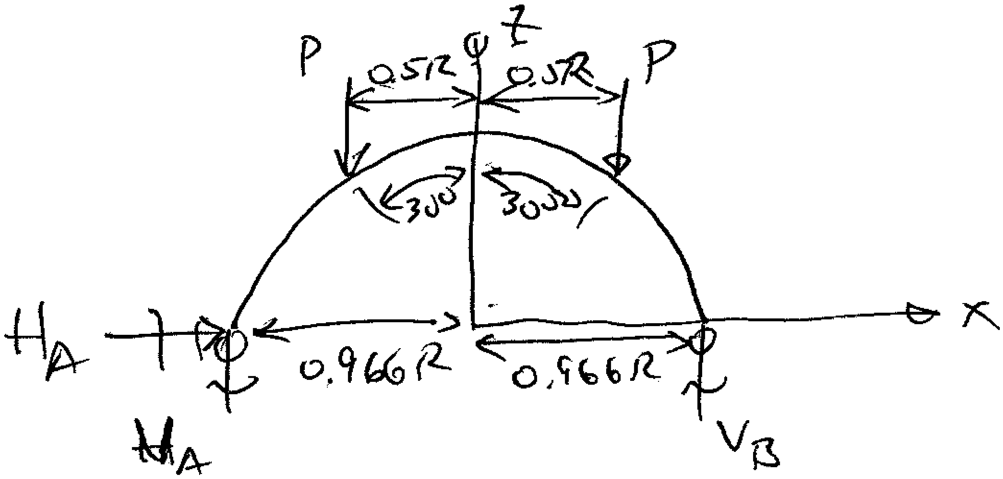
The pin is at $\theta = -75° \Rightarrow x = -0.966R$

The first load is at $\theta = -30° \Rightarrow x = -0.5R$

The second load is at $\theta = +30° \Rightarrow x = 0.5R$

The roller is at $\theta = +75° \Rightarrow x = 0.966R$

Now Draw the Free Body Diagram:



Use equilibrium:

$\sum F_x = 0 \ \xrightarrow{+} \ \Rightarrow H_A = 0$

$\sum F_z = 0 \ \uparrow + \ \Rightarrow V_A - P - P + V_B = 0$

$\sum M_A = 0 \ \circlearrowleft + \ \Rightarrow -P(0.466R) - P(1.466R)$
$+ V_B(1.932R)$

$\Rightarrow V_B = P$

Using this in the previous gives:
$V_A = P$

→ This makes sense since the configuration and its loading is symmetric about z, so the reactions must also be symmetric.

Summarizing:

$$\boxed{\begin{aligned} H_A &= 0 \\ V_A &= P \\ V_B &= P \end{aligned}}$$

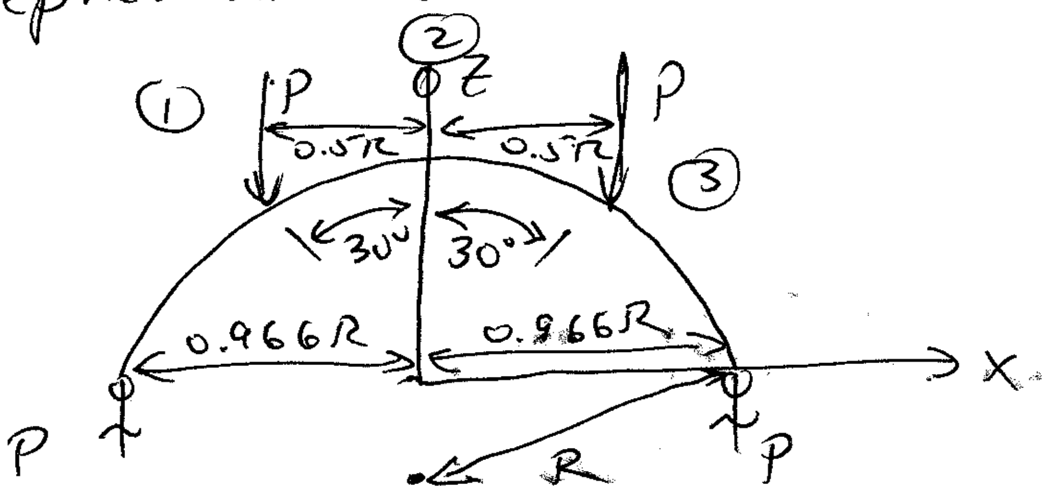
(b) This needs to be analyzed in 3 sections:

① prior to the first load ($-0.966R < x < 0.5R$)

② between the two loads ($-0.5R < x < 0.5R$)

③ after the second load ($0.5R < x < 0.966R$)

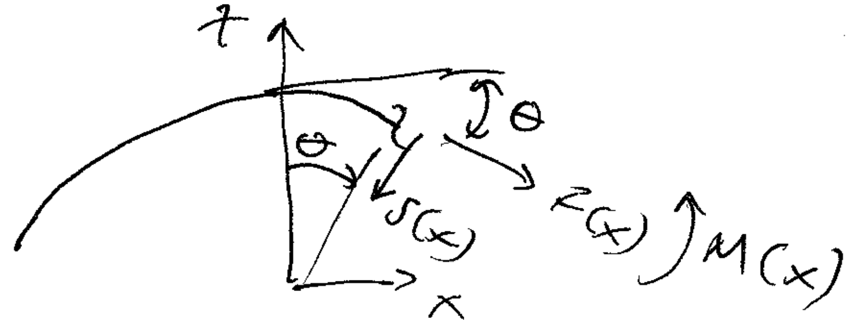
Redraw the Free Body Diagram to represent this:



As the arched beam is cut, it is important to note that the internal resultant forces have an angle to them since they act parallel and perpendicular to the tangent line to the beam and this is not parallel to the x-axis except at the peak.

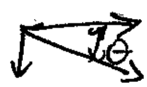
→ The first task is thus to draw these resultants and resolve them in the x-z system.

For $x > 0$, the signs of the angle (θ) make things relatively easy:



for $F(x)$:

$F_x(x) = F(x)\cos\theta$



$F(x)$

$F_z(x) = F(x)\sin\theta$

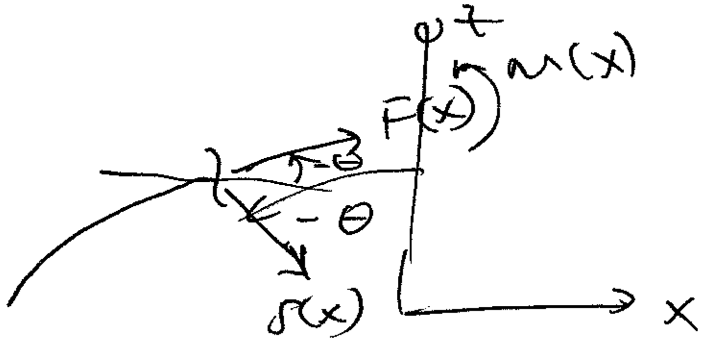
for $S(x)$:

$S_x(x) = S(x)\sin\theta$

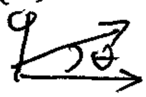
$S(x)$ $\quad S_z(x) = S(x)\cos\theta$

There is nothing to do for $M(x)$ since the same moment acts irrespective of rotation about the point.

For $x<0$, the negative sign of θ must be considered



for $F(x)$: $F_z(x) = F(x)\sin|\theta|$



$F_x(x) = F(x)\cos|\theta|$

since $\theta = -\theta$

$F_x(x)$ continues to point in $+x$

but $F_z(x) = F(x)\sin(-\theta) = -F(x)\sin|\theta|$

so the + expression for $F_z(x)$ points in $-z$ as for $x>0$ $\Rightarrow$ use throughout as above in $+z$ with $|\theta|$ accounting for the $-$

for $S(x)$: $S_x(x) = S(x)\sin|\theta|$

$S_z(x) = S(x)\cos|\theta|$

since $\theta = -\theta$ $S_z(x)$ continues to point in $-z$

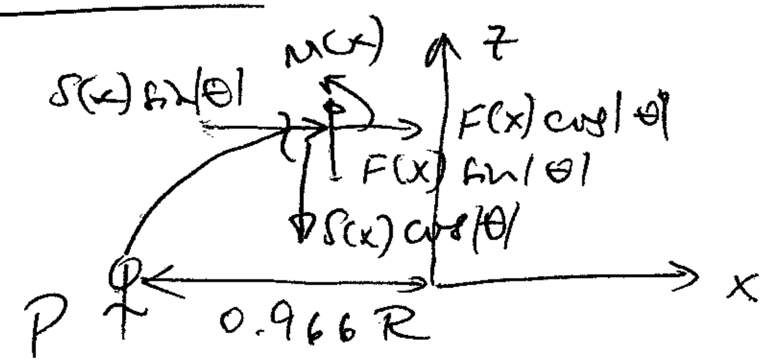
but $S_x(x) = S(x)\sin(-\theta) = -S(x)\sin|\theta|$

so the + expression for $S_x(x)$ points in $-x$ as for $x>0$ $\Rightarrow$ use throughout as above in $+x$ with $|\theta|$ to account for $-$

Finally, nothing has to be done to $M(x)$.

→ Now work each of the sections

→ <u>Section 1</u> : $-0.966R < x < -0.5R$



Apply equilibrium:

$$\Sigma F_x = 0 \xrightarrow{+} \Rightarrow +S(x)\sin|\theta| + F(x)\cos|\theta| = 0$$

$$\Rightarrow F(x) = -S(x)\tan|\theta| \quad (1)$$

$$\Sigma F_z = 0 \uparrow + \Rightarrow P - S(x)\cos|\theta| + F(x)\sin|\theta| = 0$$

Use equation (1) in this:

$$P - S(x)\cos|\theta| - S(x)\frac{\sin^2\theta}{\cos\theta} = 0$$

working through:

$$P\cos\theta - S(x)\left[\cos^2\theta + \sin^2\theta\right] = 0$$

$$\Rightarrow S(x) = P\cos\theta$$

using in (1): $F(x) = -P \sin|\theta|$

recall that $x = R\sin\theta$
finding: $\sin\theta = \frac{x}{R} \Rightarrow F(x) = -P\frac{|x|}{R}$

thus: $\cos\theta = \sqrt{1-(\frac{x}{R})^2} \Rightarrow S(x) = P\sqrt{1-(\frac{x}{R})^2}$

Lastly: $\sum M_x = 0$ ⟲+ $\Rightarrow -P(0.966R + \underset{\varphi}{x}) + M(x) = 0$

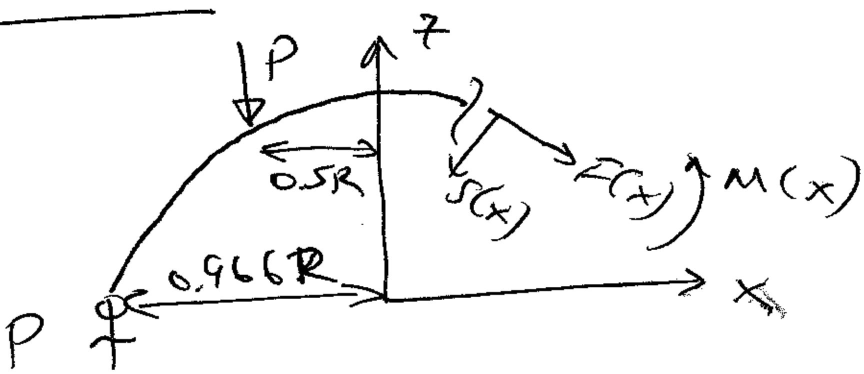
since $x<0$

$\Rightarrow M(x) = P(0.966R + x)$
or $= P(0.966R - |x|)$

So, summarizing for:
$-0.966R < x < -0.5R$

$$\boxed{\begin{aligned} F(x) &= -P\frac{|x|}{R} = +\frac{Px}{R} \\ S(x) &= P\sqrt{1-(x/R)^2} \\ M(x) &= P(0.966R - |x|) = P(0.966R + x) \end{aligned}}$$

→ Section 2: $-0.5R < x < 0.5R$



again apply equilibrium including the resolution of $S(x)$ and $F(x)$ in $x-z$ coordinates:

$$\sum F_x = 0 \xrightarrow{+} \Rightarrow -S(x)\sin\theta + F(x)\cos\theta = 0$$

$$\Rightarrow F(x) = S(x)\tan\theta$$

$$\sum F_z = 0 \uparrow + \Rightarrow +P - P - S(x)\cos\theta - F(x)\sin\theta = 0$$

$$\Rightarrow -F(x)\tan\theta = S(x)$$

These two can only be true simultaneously if $F(x) = S(x) = 0$

then

$$\sum M_x = 0 \Rightarrow -P(0.966R + x) + P(0.5R + x) + M(x) = 0$$

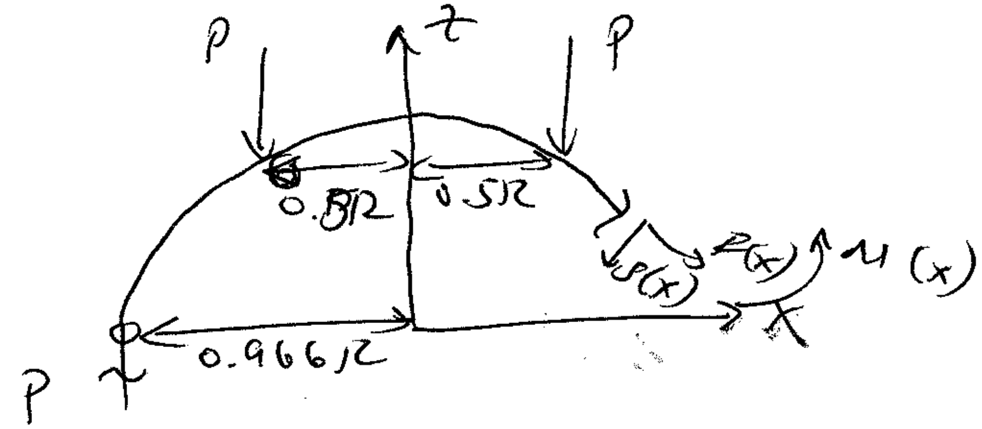
$$\Rightarrow M(x) = P(0.466R)$$

So, summarizing for:

$-0.5R < x < 0.5R$

$$\boxed{\begin{aligned} F(x) &= 0 \\ S(x) &= 0 \\ M(x) &= 0.466PR \end{aligned}}$$

→ Section 3 : $0.5R < x < 0.966R$



Again apply equilibrium and account for x-z resolution of F(x) and S(x):

$$\sum F_x = 0 \quad \Rightarrow \quad -S(x)\sin\theta + F(x)\cos\theta = 0$$

$$\Rightarrow F(x) = S(x)\tan\theta$$

$$\sum F_z = 0 \quad \Rightarrow \quad P - P - P - S(x)\cos\theta - F(x)\sin\theta = 0$$

$$\Rightarrow -P - S(x)\cos\theta - F(x)\sin\theta = 0$$

using the previous result gives:

$$-P - S(x)\cos\theta - S(x)\frac{\sin^2\theta}{\cos\theta} = 0$$

working yields:

$$-P\cos\theta - S(x)\left(\cos^2\theta + \sin^2\theta\right) = 0$$

$$\Rightarrow S(x) = -P\cos\theta = -P\sqrt{1-\left(\frac{x}{R}\right)^2}$$

Using in the $\sum F_x = 0$ result:

$$F(x) = -P\sin\theta = -P\frac{x}{R}$$

and then:

$$\sum M_x = 0 \ \circlearrowleft + \Rightarrow -P(0.966R+x) + P(0.5R+x) + P(x-0.5R) + M(x) = 0$$

$$\Rightarrow M(x) = P(0.966R - x)$$

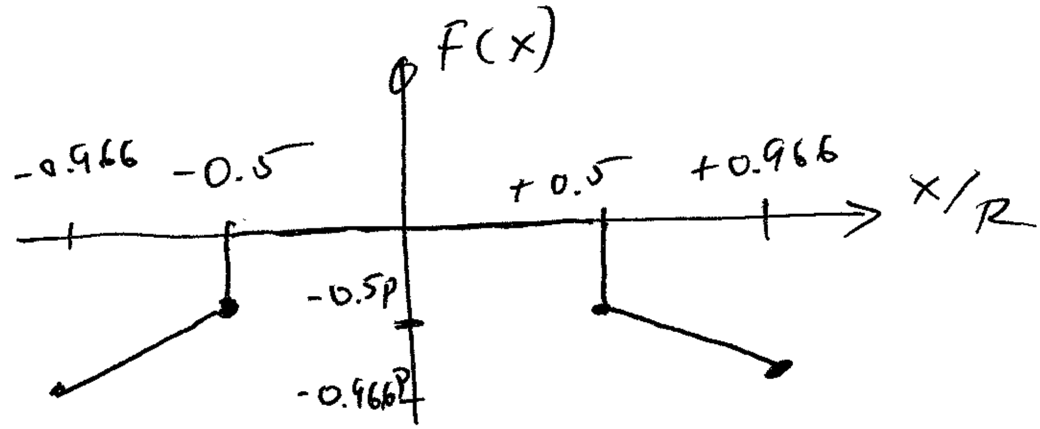
So, summarizing for:

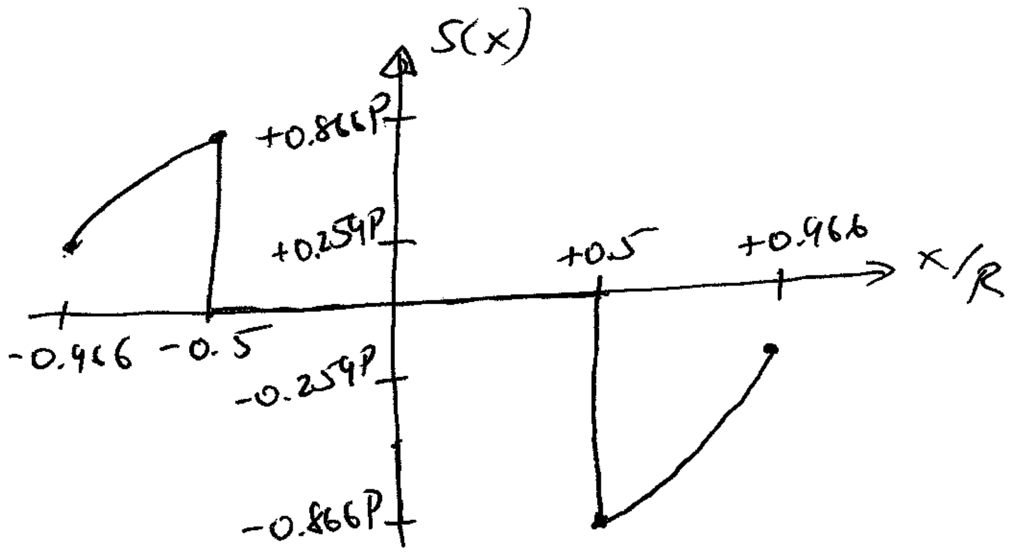
$$0.5R < x < 0.966R$$

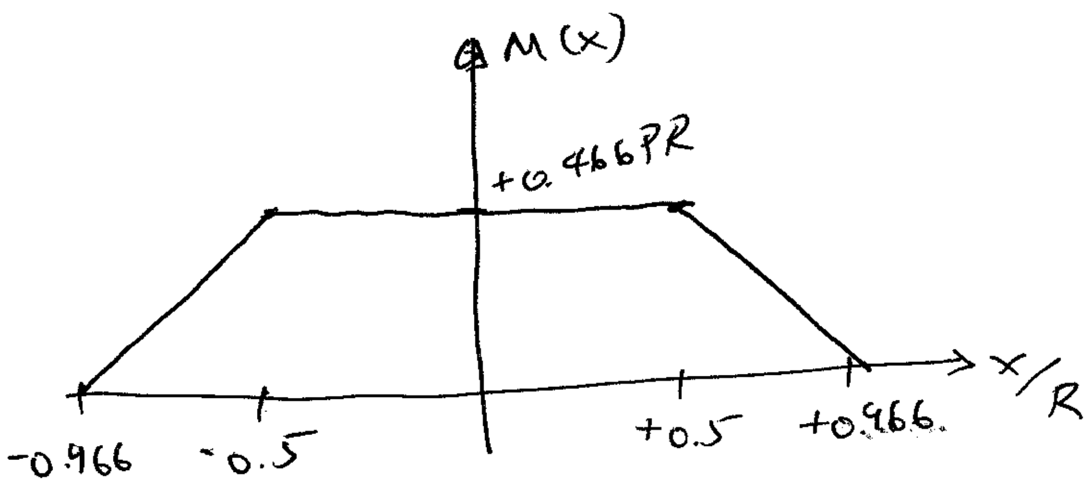
$$\boxed{\begin{aligned} F(x) &= -P\frac{x}{R} \\ S(x) &= -P\sqrt{1-\left(-\frac{x}{R}\right)^2} \\ M(x) &= P(0.966R - x) \end{aligned}}$$

These are symmetric (accounting for the proper signs) consistent with the configuration.

Now draw sketches in x. x is related to R, so nondimensionalize and draw relative to $\left(\frac{x}{R}\right)$


F(x)
-0.966
-0.5
+0.5
+0.966
x/R
-0.5P
-0.966P


S(x)
+0.866P
+0.259P
+0.5
+0.966
x/R
-0.966
-0.5
-0.259P
-0.866P


M(x)
+0.466PR
x/R
-0.966
-0.5
+0.5
+0.966.