BAL
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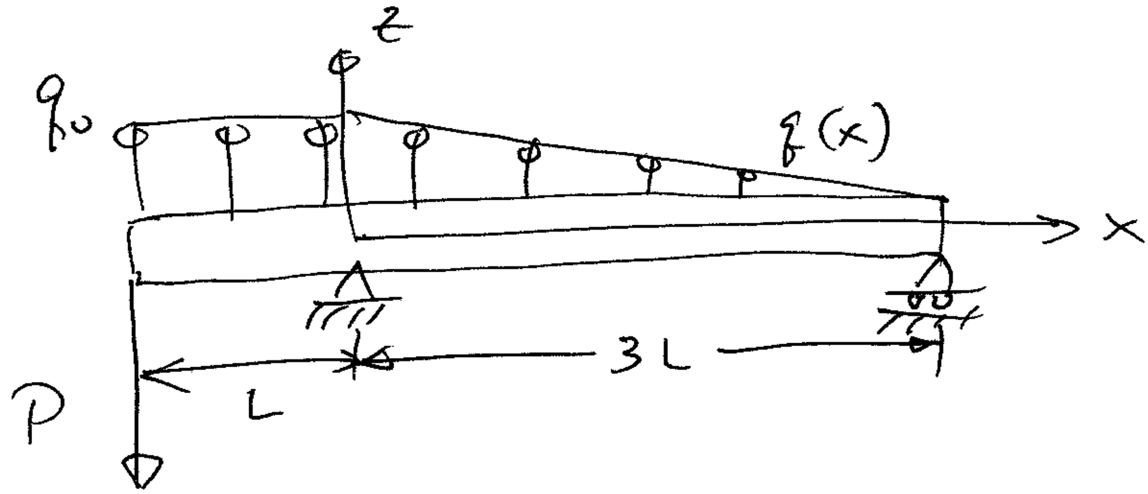
Unified Engineering Problem Set 3

week 4 Spring, 2009

SOLUTIONS

M6 (M4.1)



First determine the expression for $q(x)$ and the relative magnitude of q_0 in terms of P.

Known facts for $q(x)$ $(0 < x < 3L)$

① $q(x) = q_0$ @ $x = 0$

② $q(x) = 0$ @ $x = 3L$

③ $q(x)$ varies linearly

General solution:

$$q(x) = ax + b \quad (\text{from } ③)$$

use ① and ② in this expression:

$$q_0 = b \quad (\text{from } ①)$$

$$0 = a(3L) + b \quad (\text{from } ②)$$

So $b = q_0$ and then $a = -\frac{b}{3L} = -\frac{q_0}{3L}$

Finally: $q(x) = q_0\left(-\frac{x}{3L} + 1\right)$ for $0 < x < 3L$

check conditions ① and ②

① $x = 0$, $q(x) = q_0$ ✓

② $x = 3L$, $q(x) = 0$ ✓

Now to determine q_0, we know

$\int$ distributed loading $dx = P$

And $q(x) = q_0 \quad -L < x < 0$

So:

$$\int_{-L}^{0} q_0 \, dx + \int_{0}^{3L} q_0\left(-\frac{x}{3L} + 1\right) dx = P$$

proceeding:

$$q_0 x\Big]_{-L}^{0} + \left(-\frac{q_0 x^2}{6L} + q_0 x\right)\Big]_{0}^{3L} = P$$

$$\Rightarrow \quad q_0 L - q_0 \frac{9L^2}{6L} + 3 q_0 L = P$$

$$\frac{5}{2} q_0 L = P$$

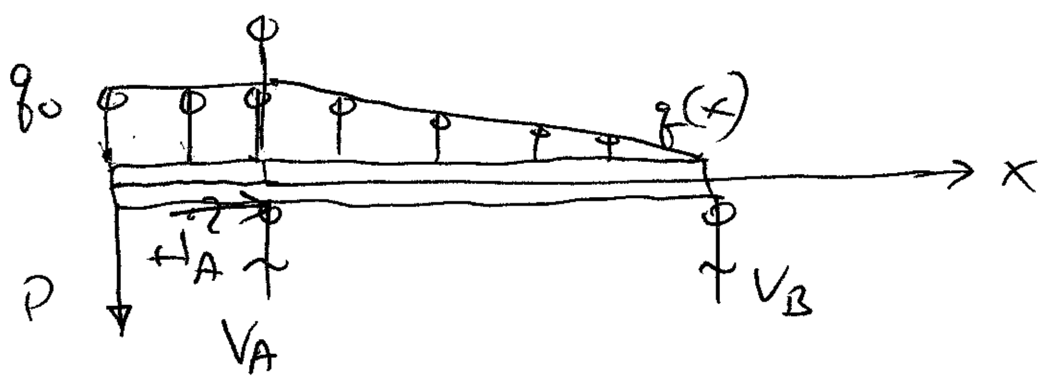
Finally: $q_0 = \dfrac{2P}{5L}$

giving:

$$q(x) = \begin{cases} \dfrac{2P}{5L} & -L < x < 0 \\ \dfrac{2P}{5L}\left(-\dfrac{x}{3L} + 1\right) & 0 < x < 3L \end{cases}$$

Proceed to:

(a) First step is to draw the free body Diagram:



Use equilibrium:

$$\sum F_x = 0 \quad \xrightarrow{+} \quad \Rightarrow H_A = 0$$

$$\sum F_z = 0 \quad \uparrow + \quad \Rightarrow V_A + V_B - P + \int_{-L}^{0} q_0 \, dx$$

$$+ \int_0^{3L} q_0 \left(-\frac{x}{3L} + 1\right) dx = 0$$

Recall that the latter two integrals sum to $+P$....

... yielding: $V_A + V_B = 0 \qquad (1)$

$$\sum M_A = 0 \quad \circlearrowleft + \quad \Rightarrow PL + V_B(3L) - \int_{-L}^{0} q_0 |x| \, dx$$

$$+ \int_0^{3L} q_0 \left(-\frac{x}{3L} + 1\right) x \, dx = 0$$

$$\Rightarrow PL + 3V_B L - q_0 \left[\frac{x^2}{2}\right]_{-L}^{0} + \left[\left(-\frac{q_0 x^3}{9L} + \frac{q_0 x^2}{2}\right)\right]_0^{3L} = 0$$

continuing:

$$PL + 3V_B L - \frac{q_0 L^2}{2} - \frac{27 q_0 L^3}{9L} + \frac{9 q_0 L^2}{2} = 0$$

$$P + 3V_B - 3 q_0 L + 4 q_0 L = 0$$

with the value for q_0:

$$P + 3V_B + \frac{2}{5} P = 0$$

Finally:

$$V_B = -\frac{7}{15} P$$

using (1): $V_A = +\frac{7}{15} P$

Summarizing the reactions are:

$$\boxed{\begin{aligned} H_A &= 0 \\ V_A &= +\frac{7}{15} P \\ V_B &= -\frac{7}{15} P \end{aligned}}$$

(b) This needs to be done in parts since there is a point load (reaction) along the beam at $x = 0$ and there is also a change in $q(x)$ at that point.

→ So, for $\underline{-L < 0 < x}$:

There is no loading in $x \Rightarrow f(x) = 0$

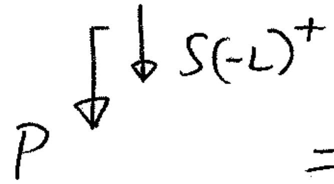
$$q(x) = q_0 = \frac{2P}{5L}$$

Use: $\frac{dS}{dx} = q(x)$

$$\Rightarrow S(x) = \int q(x)\,dx = \int \frac{2P}{5L}\,dx$$

$$\Rightarrow S(x) = \frac{2P}{5L} x + C_1$$

Use a boundary condition to get the constant of integration. At $x = -L^{(+)}$:



So: $\Sigma F_z = 0 \uparrow + \Rightarrow -P - S(-L)^+ = 0$

$$\Rightarrow S(-L)^+ = -P$$

using in the equation:

$$S(-L)^+ = \frac{2P}{5L}(-L) + C_1 = -P$$

$$\Rightarrow -\frac{2P}{5} + C_1 = -P$$

giving: $C_1 = -\frac{3}{5}P$

So: $$S(x) = \frac{P}{5}\left(2\frac{x}{L} - 3\right)$$

Proceeding to the moment...

$$\frac{dM}{dx} = S$$

$$\Rightarrow M(x) = \int S(x)\,dx = \int \frac{P}{5}\left(2\frac{x}{L} - 3\right)dx$$

So: $$M(x) = \frac{2Px^2}{10L} - \frac{3Px}{5} + C_2$$

Again, use a boundary condition. At the tip, $x = -L$, there is no applied moment.

So: $$M(-L) = 0 = \frac{2P(-L)^2}{10L} - \frac{3P}{5}(-L) + C_2$$

going to:

$$\frac{PL}{5} + \frac{3PL}{5} + C_2 = 0$$

$$\Rightarrow C_2 = -\frac{4PL}{5}$$

$$\Rightarrow M(x) = \frac{P}{5}\left(\frac{x^2}{L} - 3x - 4L\right)$$

Summarizing for $-L < x < 0$:

$$\boxed{\begin{aligned} q(x) &= 2P/5L \\ F(x) &= 0 \\ S(x) &= \frac{P}{5}\left(\frac{2x}{L} - 3\right) \\ M(x) &= \frac{P}{5}\left(\frac{x^2}{L} - 3x - 4L\right) \\ &\text{or} \\ &\frac{PL}{5}\left\{\left(\frac{x}{L}\right)^2 - 3\left(\frac{x}{L}\right) - 4\right\} \end{aligned}}$$

$\rightarrow$ Move on to $\underline{0 < x < 3L}$:

There is still no loading in x, so $F(x) = 0$

$$q(x) = \frac{2P}{5L}\left(-\frac{x}{3L} + 1\right)$$

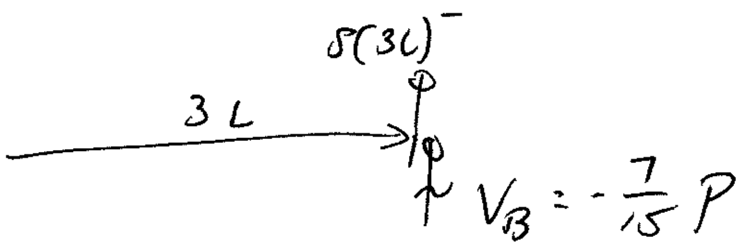
use: $\frac{dS}{dx} = q(x)$

$$\Rightarrow S(x) = \int \frac{2P}{5L}\left(-\frac{x}{3L} + 1\right) dx$$

gives:

$$S(x) = \frac{2P}{5L}\left(-\frac{x^2}{6L} + x\right) + C_3$$

There are two ways to fit a boundary condition -- at the tip or at the pin. The tip $(x = 3L)$ is easier as the only load is the reaction:



$$\Sigma F_z = 0 \;\uparrow + \Rightarrow S(3L)^- - \frac{7}{15}P = 0$$

$$\Rightarrow S(3L)^- = \frac{7}{15}P$$

So:

$$S(3L)^- = \frac{7}{15}P = \frac{2P}{5L}\left(-\frac{(3L)^2}{6L} + 3L\right) + C_3$$

$$\frac{7}{15}P = \frac{2P}{5L}\left(-\frac{3}{2}L + 3L\right) + C_3$$

$$\frac{7}{15}P = \frac{3}{5}P + C_3$$

finally: $C_3 = -\frac{2}{15}P$

yielding: $S(x) = \frac{2P}{5}\left\{-\frac{1}{6}\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right) - \frac{1}{3}\right\}$

Proceed again to $\frac{dM}{dx} = S(x)$

$$\Rightarrow M(x) = \int S(x)\,dx = \frac{2P}{5}\int\left\{-\frac{1}{6}\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right) - \frac{1}{3}\right\}dx$$

giving:

$$M(x) = \frac{2P}{5}\left\{-\frac{1}{18}\frac{x^3}{L^2} + \frac{x^2}{2L} - \frac{1}{3}x\right\} + C_4$$

Again, going to the tip $(x = 3L)$, there is no applied or reactionary moment. So:

$$M(3L)^- = 0$$

$$\Rightarrow 0 = \frac{2P}{5}\left\{-\frac{27}{18}L + \frac{9}{2}L - L\right\} + C_4$$

$$C_4 = -\frac{2P}{5}(+2L) \Rightarrow C_4 = -\frac{4PL}{5}$$

Finally:

$$M(x) = \frac{2P}{5}\left\{-\frac{1}{18}\frac{x^3}{L^2} + \frac{x^2}{2L} - \frac{1}{3}x - 2L\right\}$$

Summarizing: for $0 < x < 3L$

$$\boxed{\begin{aligned} q(x) &= \frac{2P}{5L}\left(-\frac{x}{3L} + 1\right) \\ F(x) &= 0 \\ S(x) &= \frac{2P}{5}\left\{-\frac{1}{6}\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right) - \frac{1}{3}\right\} \\ M(x) &= \frac{2P}{5}\left\{-\frac{1}{18}\frac{x^3}{L^2} + \frac{x^2}{2L} - \frac{1}{3}x - 2L\right\} \\ &\text{or} \\ &\frac{2PL}{5}\left\{-\frac{1}{18}\left(\frac{x}{L}\right)^3 + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{3}\left(\frac{x}{L}\right) - 2\right\} \end{aligned}}$$

→ NOTE: A way to check -- there are no point moments applied, so the solutions for $M(x)$ for the two segments must be equal where they meet ($x=0$):

for: $-L<x<0$? $0<x<3L$

$M(x) = -4PL/5 \stackrel{?}{=} -4PL/5$ ✓ YES

→ Now draw these. In sketching, use the relations of the derivatives to get a shape. Calculate endpoint values to begin. And recall that point loads cause equal jumps in shear (account for proper direction and sign).

$F(x) = 0$ everywhere ... no need to plot

to next page --.

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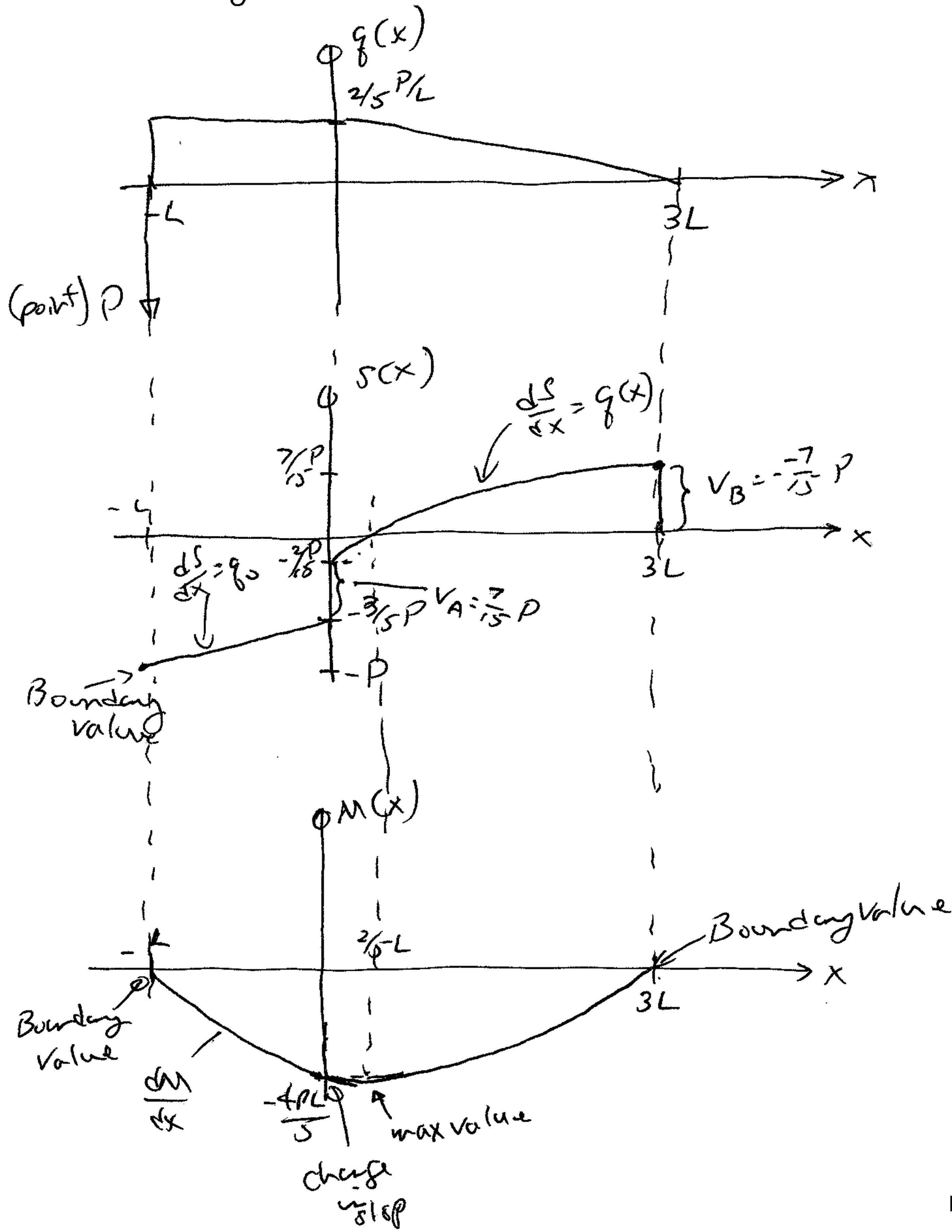
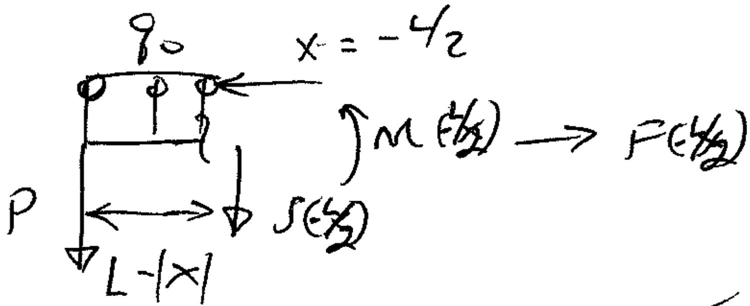

q(x)
2/5 P/L
-L
3L
x
(point) P
S(x)
dS/dx = q(x)
7/15 P
V_B = -7/15 P
-L
x
3L
dS/dx = q_0
-2/15 P
V_A = 7/15 P
-8/15 P
-P
Boundary value
M(x)
2/5 L
Boundary value
x
3L
Boundary value
dM/dx
-4PL/5
max value
change of slope

(c) Cut the beam . . .

→ first at $x = -L/2$:



$$\sum F_x = 0 \xrightarrow{+} \Rightarrow F(-L/2) = 0 \checkmark$$

$$\sum F_z = 0 \uparrow + \Rightarrow -P + q_0 (L/2) - S(-L/2) = 0$$

with $q_0 = \frac{2}{5}\frac{P}{L}$

$$\Rightarrow S(-L/2) = -\frac{4}{5}P$$

check:

$$S(x) = \frac{P}{5}\left(\frac{2x}{L} - 3\right) = \frac{P}{5}\left(\frac{2}{L}\left(-\frac{L}{2}\right) - 3\right)$$

$$= -\frac{4}{5}P \checkmark$$

Finally:

$$\sum M_{(-L/2)} = 0 \circlearrowleft + \Rightarrow P(L/2) - \int_{-L}^{-L/2} q_0 (L - |x|)\,dx + M(-L/2) = 0$$

$$\Rightarrow \frac{PL}{2} - \left[q_0 L x - \frac{q_0 |x^2|}{2}\right]_{-L}^{-L/2} + M(-L/2) = 0$$

$$M(-L/2) = -\frac{PL}{2} + \left[q_0 \frac{L^2}{2} - \frac{3 q_0 L^2}{8}\right]$$

$$\Rightarrow M(-L/2) = -\frac{PL}{2} + \frac{2}{5}\frac{P}{L}\left(-\frac{L^2}{8}\right) = -\frac{9}{20} PL$$

check:

$$M(x) = \frac{P}{5}\left(\frac{x^2}{L} - 3x - 4L\right)$$

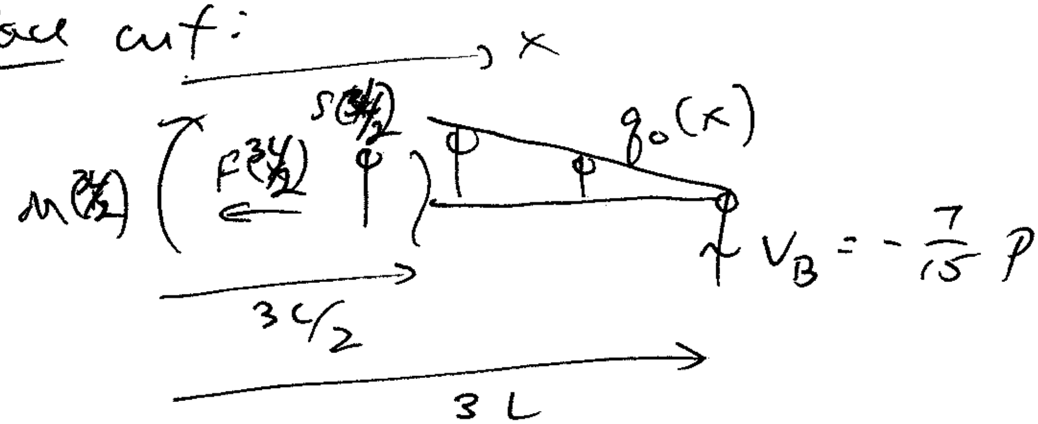
at $x = -L/2$

$$M(-L/2) = \frac{P}{5}\left(\frac{L}{4} + \frac{3L}{2} - 4L\right)$$

$$\Rightarrow M(-L/2) = -\frac{9PL}{20} \quad ✓$$

→ Now cut the beam at $x = \frac{3L}{2}$

Make it simpler by taking a <u>negative face</u> cut:



$$\sum F_x = 0 \xrightarrow{+} \Rightarrow F(3L/2) = 0 \quad ✓$$

$$\sum F_z = 0 \quad \uparrow + \Rightarrow S\left(\frac{3L}{2}\right) - \frac{7}{15}P + \int_{3L/2}^{3L} \frac{2P}{5L}\left(-\frac{x}{3L} + 1\right)dx = 0$$

$$\Rightarrow S\left(\frac{3L}{2}\right) = \frac{7}{15}P + \frac{2P}{5L}\left(\frac{x^2}{6L} - x\right)\Bigg]_{3L/2}^{3L}$$

$$= \frac{7}{15}P + \frac{2P}{5L}\left(\frac{9L^2 - \frac{9}{4}L^2}{6L} - (3L - 3L/2)\right)$$

continuing:

$$\delta\left(\frac{3L}{2}\right) = \frac{7}{15}P + \frac{2P}{5L}\left(\frac{27}{24}L - \frac{3L}{2}\right)$$

$$= \frac{7}{15}P + \frac{2P}{5L}\left(-\frac{3}{8}L\right)$$

$$= \frac{28}{60}P - \frac{9P}{60}$$

$$\Rightarrow \quad \delta\left(\frac{3L}{2}\right) = \frac{19P}{60}$$

check:

$$\delta(x) = \frac{2P}{5}\left\{-\frac{1}{6}\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right) - \frac{1}{3}\right\}$$

at $x = \frac{3L}{2}$

$$\delta\left(\frac{3L}{2}\right) = \frac{2P}{5}\left\{-\frac{9}{24} + \frac{3}{2} - \frac{1}{3}\right\}$$

$$= \frac{2P}{5}\left\{-\frac{9}{24} + \frac{36}{24} - \frac{8}{24}\right\}$$

$$\Rightarrow \delta\left(\frac{3L}{2}\right) = \frac{19}{60}P \quad \checkmark$$

finally:

$$\sum M_{(3L/2)} = 0 \quad \circlearrowleft + \Rightarrow -M\left(\frac{3L}{2}\right) + V_B\left(\frac{3L}{2}\right) + \int_{3L/2}^{3L} q_0(x)\left(x - \frac{3L}{2}\right)dx = 0$$

write out:

$$M\left(\frac{3L}{2}\right) = -\frac{7}{10}PL + \int_{3L/2}^{3L} \frac{2P}{5L}\left(-\frac{x}{3L} + 1\right)\left(x - \frac{3L}{2}\right)dx$$

$$M\left(\frac{3L}{2}\right) = -\frac{7}{10}PL + \frac{2P}{5L}\int_{3L/2}^{3L}\left(-\frac{x^2}{3L} + \frac{3x}{2} - \frac{3L}{2}\right)dx$$

$$= -\frac{7}{10}PL + \frac{2P}{5L}\left[-\frac{x^3}{9L} + \frac{3x^2}{4} - \frac{3Lx}{2}\right]_{3L/2}^{3L}$$

$$= -\frac{7}{10}PL + \frac{2P}{5L}\left[-\frac{\left(27-\frac{27}{8}\right)}{9}L^2 + \frac{3}{4}\left(9-\frac{9}{4}\right)L^2 - \frac{9L^2}{4}\right]$$

$$= -\frac{7}{10}PL + \frac{2P}{5L}\left[-\frac{21}{8}L^2 + \frac{81}{16}L^2 - \frac{9L^2}{4}\right]$$

$$= -\frac{7}{10}PL + \frac{2}{5}PL\left[\frac{3}{16}\right]$$

$$\Rightarrow M\left(\frac{3L}{2}\right) = -\frac{28}{40}PL + \frac{3}{40}PL = -\frac{25}{40}PL = -\frac{5}{8}PL$$

Check:

$$M(x) = \frac{2PL}{5}\left\{-\frac{1}{18}\left(\frac{x}{L}\right)^3 + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{3}\left(\frac{x}{L}\right) - 2\right\}$$

$$\Rightarrow M\left(\frac{3L}{2}\right) = \frac{2PL}{5}\left\{-\frac{1}{18}\left(\frac{27}{8}\right) + \frac{1}{2}\left(\frac{9}{4}\right) - \frac{1}{3}\left(\frac{3}{2}\right) - 2\right\}$$

$$= \frac{2PL}{5}\left\{-\frac{3}{16} + \frac{9}{8} - \frac{1}{2} - 2\right\}$$

$$= \frac{2PL}{5}\left\{-\frac{3}{16} + \frac{18}{16} - \frac{8}{16} - \frac{32}{16}\right\}$$

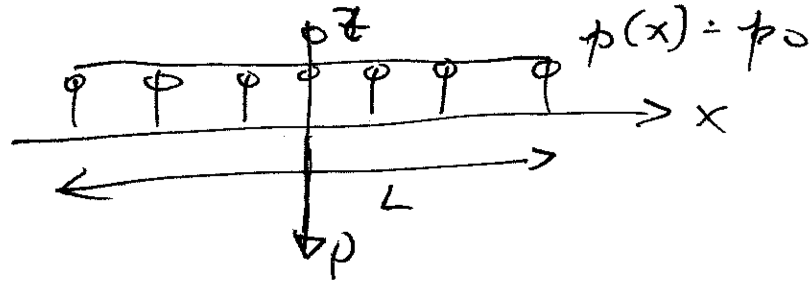
$$= \frac{2PL}{5}\left\{\frac{25}{16}\right\}$$

$$\Rightarrow M\left(\frac{3L}{2}\right) = -\frac{5PL}{8} \quad \checkmark \, !$$

M7 (M4.2)

(a) Note that this can be done irrespective of the model used. Consider the reasons....

The model is the free body diagram. (shown for Case 1 of constant along span)



There are $\boxed{\text{no reaction forces}}$ since the wing has no internal support that can carry load.

In level flight: $\int p(x) = P$

Furthermore, the lift must be symmetric so that there is no net moment about the fuselage such that the plane does not roll.

(b) Now consider this case by case.

→ Case 1 -- lift constant along span.

$$\Rightarrow \int p_0 \, dx = P \Rightarrow p_0 = \frac{P}{L} = q(x)$$

There are no axial forces, so $F(x) = 0$

We have two sections of the wing:

$$-L/2 < \cancel{x} < 0$$
$$0 < x < L/2$$

There is symmetry, so our result should be the same, but let's be sure:

For: $-L/2 < x < 0$

$$q(x) = p_0$$

Use: $\frac{dS}{dx} = q(x) \Rightarrow S(x) = \int p_0 \, dx$

$$= \frac{P}{L} x + C_1$$

Go to the tip and see $S = 0$ at $x = -L/2$

$$\Rightarrow S(-L/2) = 0 = \frac{P}{L}(-L/2) + C_1$$

$$\Rightarrow C_1 = \frac{P}{2}$$

$$\Rightarrow S(x) = P\left(\frac{x}{L} + \frac{1}{2}\right)$$

Progress to: $\dfrac{dM}{dx} = S$

$$\Rightarrow M(x) = \int P\left(\frac{x}{L} + \frac{1}{2}\right) dx$$

$$= P\left(\frac{x^2}{2L} + \frac{x}{2}\right) + C_2$$

Again, at the tip: $M = 0$ at $x = -L/2$

$$\text{so: } M(-L/2) = 0 = P\left(\frac{L}{8} - \frac{L}{4}\right) + C_2$$

$$\Rightarrow C_2 = \frac{PL}{8}$$

resulting in: $M(x) = PL\left\{\frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{1}{2}\left(\frac{x}{L}\right) + \frac{1}{8}\right\}$

Now for: $0 < x < L/2$

again $q(x) = p_0 = \dfrac{P}{L}$

$$\frac{dS}{dx} = q(x) \Rightarrow S(x) = \int \frac{P}{L} dx = \frac{Px}{L} + C_3$$

again go to the tip where $S = 0$ (at $x = L/2$)

$$\Rightarrow S(L/2) = 0 = \frac{P}{2} + C_3 \Rightarrow C_3 = -\frac{P}{2}$$

giving: $S(x) = P\left(\dfrac{x}{L} - \dfrac{1}{2}\right)$

Progress to: $\dfrac{dM}{dx} = S$

$$\Rightarrow M(x) = \int P\left(\frac{x}{L} - \frac{1}{2}\right) dx$$

giving: $M(x) = \frac{Px^2}{2L} - \frac{P}{2}x + C_4$

again to the tip where there is no moment $(x = L/2)$

$$\Rightarrow M(L/2) = 0 = \frac{PL}{8} - \frac{PL}{4} + C_4$$

giving: $C_4 = \frac{PL}{8}$

giving a final expression:

$$M(x) = PL\left\{\frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{2}\left(\frac{x}{L}\right) + \frac{1}{8}\right\}$$

Summarizing for Case 1

$$\boxed{\begin{aligned} F(x) &= 0 \quad \text{everywhere} \\ S(x) &= P\left(\frac{x}{L} + \frac{1}{2}\right) \quad -L/2 < x < 0 \\ &= P\left(\frac{x}{L} - \frac{1}{2}\right) \quad 0 < x < L/2 \\ M(x) &= \frac{PL}{2}\left\{\left(\frac{x}{L}\right)^2 + \left(\frac{x}{L}\right) + \frac{1}{4}\right\} \quad -L/2 < x < 0 \\ &= \frac{PL}{2}\left\{\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right) + \frac{1}{4}\right\} \quad 0 < x < L/2 \end{aligned}}$$

NOTE: consider the symmetry

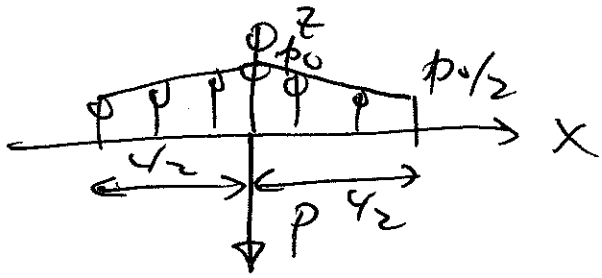
- $F(x) = 0$ everywhere, so satisfied

- Changing $S(x)$ by P at the fuselage ($+P$ if proceeding from $+x$ to $-x$) and changing the sign of even powers of x
- Changing odd powers of x in the moment expression $M(x)$ in their sign

Thus, for subsequent cases, calculate for one wing and use their symmetry.

→ <u>Case 2</u>: lift linear variation along span with maximum value at root down to half this value at tip.

From last term:



$$p(x) = p_0\left(1-\frac{x}{l}\right) \qquad 0<x<l/2$$

$$= p_0\left(1+\frac{x}{l}\right) \qquad -l/2<x<0$$

and $\int p(x) = P \Rightarrow p_0\int_{-l/2}^{0}\left(1+\frac{x}{l}\right)dx + p_0\int_{0}^{l/2}\left(1-\frac{x}{l}\right)dx = P$

$$\Rightarrow p_0\left[x+\frac{x^2}{2L}\right]_{-L/2}^{0} + p_0\left[x-\frac{x^2}{2L}\right]_{0}^{L/2} = P$$

$$p_0\left[\frac{L}{2}-\frac{L}{8}\right] + p_0\left[\frac{L}{2}-\frac{L}{8}\right] = P$$

$$\Rightarrow 2p_0\left[\frac{3L}{8}\right] = P$$

$$\Rightarrow p_0 = \frac{4}{3}\frac{P}{L}$$

Consider for $0 < x < L/2$

$$\Rightarrow q(x) = \frac{4}{3}\frac{P}{L}\left(1-\frac{x}{L}\right)$$

Proceeding to $\frac{dV}{dx} = q(x) \Rightarrow V(x) = \int q(x)$

$$\Rightarrow V(x) = \frac{4}{3}\frac{P}{L}\int\left(1-\frac{x}{L}\right)dx$$

$$= \frac{4}{3}\frac{P}{L}\left(x-\frac{x^2}{2L}\right) + C_5$$

to the tip $(x = L/2)$ where $V = 0$

$$\Rightarrow V(L/2) = 0 = \frac{4}{3}\frac{P}{L}\left(\frac{L}{2}-\frac{L}{8}\right) + C_5$$

$$\Rightarrow C_5 = -\frac{P}{2}$$

$$\Rightarrow V(x) = \frac{4}{3}P\left\{\frac{x}{L}-\frac{1}{2}\left(\frac{x}{L}\right)^2-\frac{3}{8}\right\}$$

Progressing to: $\frac{dM}{dx} = S$

$$\Rightarrow M(x) = \frac{4}{3} P \int \left\{ \frac{x}{L} - \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{3}{8} \right\} dx$$

$$= \frac{4}{3} P \left\{ \frac{x^2}{2L} - \frac{x^3}{6L^2} - \frac{3}{8}x + C_6 \right\}$$

again at the tip ($x = L/2$) the moment is zero:

$$M(L/2) = 0 = \frac{\cancel{L}^2}{8L} - \frac{\cancel{L}^3}{48L^2} + \frac{3L}{16} + C_6$$

$$\Rightarrow L\left(\frac{1}{8} - \frac{1}{48} + \frac{3}{16}\right) = -C_6$$

giving $C_6 = -L\left(\frac{6}{48} - \frac{1}{48} - \frac{9}{48}\right) = \frac{L}{12}$

$$\Rightarrow M(x) = \frac{4}{3} PL \left\{ \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{6}\left(\frac{x}{L}\right)^3 - \frac{3}{8}\left(\frac{x}{L}\right) + \frac{1}{12} \right\}$$

Summarizing for <u>Case</u> 2

$F(x) = 0$ everywhere

$$S(x) = \frac{4}{3} P \left\{ \frac{x}{L} - \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{3}{8} \right\} \quad 0 < x < L/2$$

$$= \frac{4}{3} P \left\{ \frac{x}{L} + \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{3}{8} \right\} + P \quad -L/2 < x < 0$$

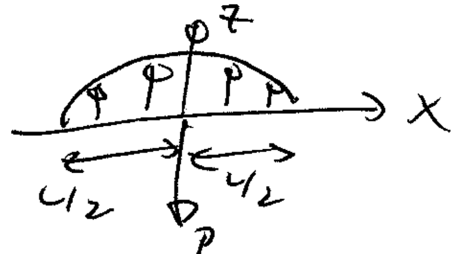
$$M(x) = \frac{4}{3} PL \left\{ \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{6}\left(\frac{x}{L}\right)^3 - \frac{3}{8}\left(\frac{x}{L}\right) + \frac{1}{12} \right\} \quad 0 < x < L/2$$

$$= \frac{4}{3} PL \left\{ \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{3}{8}\left(\frac{x}{L}\right) + \frac{1}{12} \right\} \quad -L/2 < x < 0$$

PAL

→ Case 3: lift quadratic along span with maximum at root to zero at tip

From last term:



$$p(x) = p_0\left(1 - \frac{4x^2}{L^2}\right)$$

valid throughout

and $\int p(x) = P$ or $\int_0^{L/2} p(x) = \frac{P}{2}$

$$\Rightarrow p_0 \int_0^{L/2} \left(1 - \frac{4x^2}{L^2}\right) dx = \frac{P}{2}$$

$$p_0 \left(x - \frac{4x^3}{3L^2}\right)\Bigg]_0^{L/2} = \frac{P}{2}$$

$$\Rightarrow p_0 \left(L/2 - \frac{4L^3}{24L^2}\right) = P/2$$

$$p_0 \left(L/3 \right) = P/2 \quad \text{giving: } p_0 = \frac{3P}{2L}$$

So: $p(x) = \frac{3P}{2L}\left(1 - \frac{4x^2}{L^2}\right) = q(x)$

Then with $\frac{dS}{dx} = q(x)$

$$\Rightarrow S(x) = \frac{3P}{2L}\int\left(1 - \frac{4x^2}{L^2}\right)dx$$

$$= \frac{3P}{2L}\left(x - \frac{4x^3}{3L^2} + C_7\right)$$

to the tip $(x = L/2)$ for shear of zero:

$$S(L/2) = 0 \Rightarrow \frac{L}{2} - \frac{4L^3}{24L^2} + C_7 = 0$$

$$\Rightarrow \quad C_7 = -\frac{L}{3}$$

So:

$$S(x) = \frac{3P}{2}\left\{\frac{x}{L} - \frac{4}{3}\left(\frac{x}{L}\right)^3 - \frac{1}{3}\right\}$$

and then to: $\frac{dM}{dx} = S$

$$\Rightarrow M(x) = \frac{3P}{2}\int\left\{\frac{x}{L} - \frac{4}{3}\left(\frac{x}{L}\right)^3 - \frac{1}{3}\right\}dx$$

$$= \frac{3P}{2}\left(\frac{x^2}{2L} - \frac{4x^4}{12L^3} - \frac{1}{3}x + C_8\right)$$

and to the tip $(x = L/2)$ where $M = 0$:

$$M(L/2) = 0 = \left(\frac{L^2}{8L} - \frac{4L^4}{(12)(16L^3)} - \frac{L}{6} + C_8\right)$$

$$= \left(\frac{6L}{48} - \frac{L}{48} - \frac{8L}{48} + C_8\right)$$

$$\Rightarrow C_8 = \frac{L}{16}$$

giving:

$$M(x) = \frac{3PL}{2}\left\{\left(\frac{x}{L}\right)^2 - \frac{1}{3}\left(\frac{x}{L}\right)^4 - \frac{1}{3}\left(\frac{x}{L}\right) + \frac{1}{16}\right\}$$

Summarizing for Case 3:

$$F(x) = 0 \quad \text{everywhere}$$

$$S(x) = \frac{3P}{2}\left\{\frac{x}{L} - \frac{4}{3}\left(\frac{x}{L}\right)^3 - \frac{1}{3}\right\} \qquad 0 < x < L/2$$

$$= \frac{3P}{2}\left\{\frac{x}{L} - \frac{4}{3}\left(\frac{x}{L}\right)^3 - \frac{1}{3}\right\} + P \qquad -L/2 < x < 0$$

$$M(x) = \frac{3PL}{2}\left\{\frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{3}\left(\frac{x}{L}\right)^4 - \frac{1}{3}\left(\frac{x}{L}\right) + \frac{1}{16}\right\} \qquad 0 < x < L/2$$

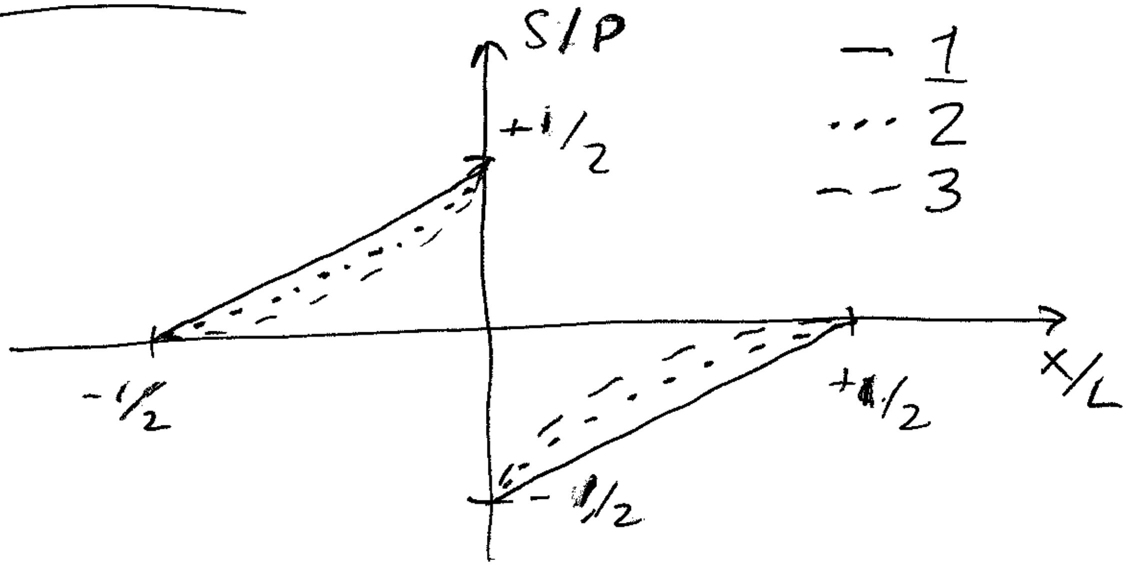
$$= \frac{3PL}{2}\left\{\frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{3}\left(\frac{x}{L}\right)^4 + \frac{1}{3}\left(\frac{x}{L}\right) + \frac{1}{16}\right\} \qquad -L/2 < x < 0$$

(c) Now compare via plotting.

NOTE: all values are normalized
- $\left(\frac{x}{L}\right)$ for all distances
- P for shear
- PL for moment

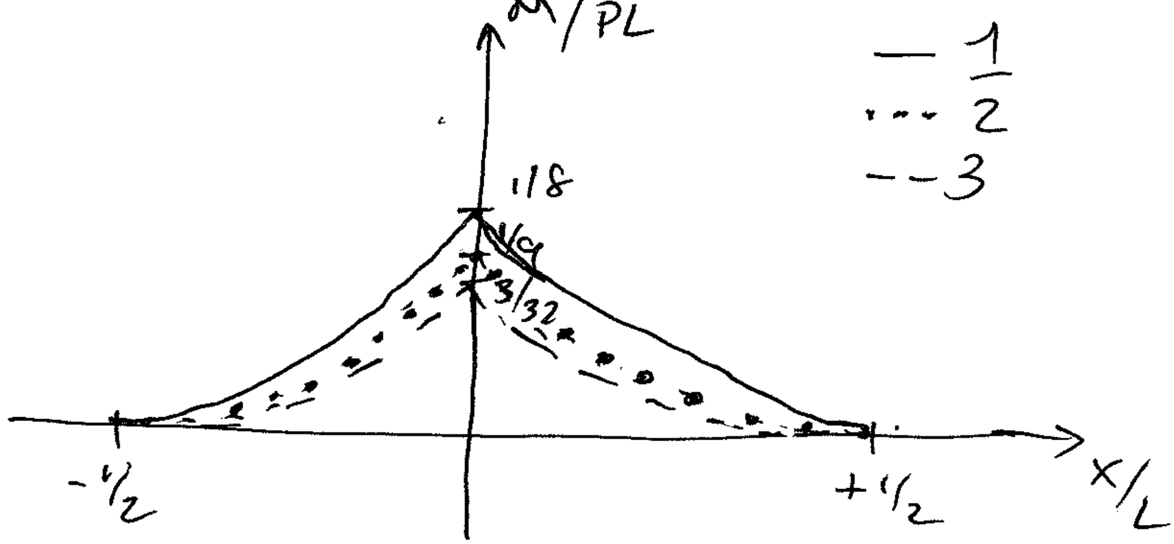
Axial Force -- zero everywhere in all cases (no need to plot)

Shear Force



NOTES: All models have the same value at the root. Reason -- each wing carries the same integrated lift of half the weight (P/2).
All models change by the concentrated weight, P, at the fuselage

Moment



NOTE: The moment at the root is maximum in all cases, but varies a great deal in value. Thus, the lift distribution plays a considerable roll in the moment carried by the beam.

(d) The highest moment is at the root, so that is the location of greatest loading as is shear for all cases. The load is transferred at the attachment to the fuselage.