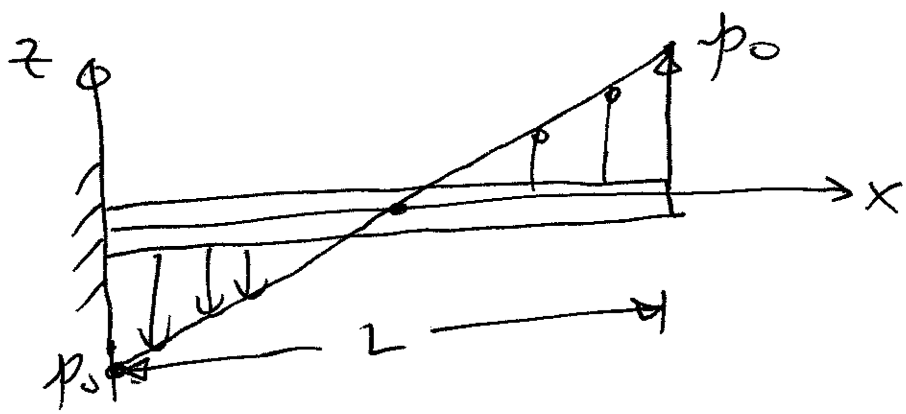
PAL
2/26/09

Unified Engineering Problem Set 4

Week 5 Spring, 2009

SOLUTIONS

M8 (M5.1)



(a) First determine an expression for the loading as a function of x: $q(x)$

- loading is linear in x

$$\Rightarrow q(x) = mx + b$$

- at $x = 0$, loading acts downward with magnitude p_0

$$\Rightarrow q(0) = -p_0$$

giving: $-p_0 = m(0) + b$

$$\Rightarrow b = -p_0$$

- at $x = L$, loading acts upward with magnitude p_0

$$\Rightarrow q(L) = +p_0$$

giving: $+p_0 = m(L) + b$

and $b = -p_0$

$$\Rightarrow m = +\frac{2p_0}{L}$$

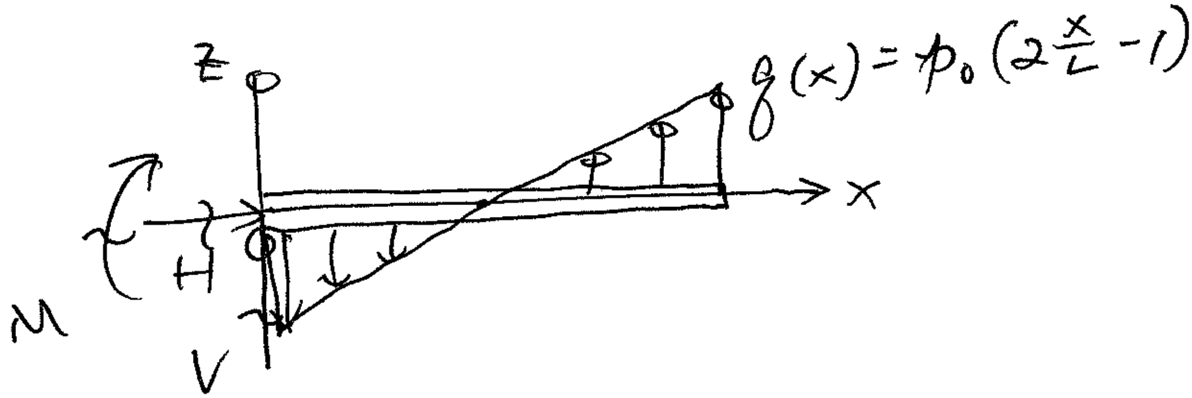
So: $q(x) = \frac{2p_0}{L}x - p_0 = p_0\left(\frac{2x}{L} - 1\right)$

- Do a check... should crossover and equal 0 at midpoint $(x = \frac{L}{2})$

$$q\left(\frac{L}{2}\right) \overset{?}{=} 0 = p_0\left(\frac{2}{2} - 1\right) \checkmark \; \underline{\text{yes}}$$

$\rightarrow$ Move forward to ....

Step 1 - Free Body Diagram



→ Apply Equilibrium to get reactions

(NOTE: Value of $q(x)$ accounts for direction of that loading so use $q(x)$ generically as in + - direction)

$$\Sigma F_x = 0 \quad \xrightarrow{+} \quad \Rightarrow H = 0$$

$$\Sigma F_z = 0 \quad \uparrow + \quad \Rightarrow V + \int_0^L q(x)\,dx = 0$$

working: $V + \int_0^L p_0 \left(2\frac{x}{L} - 1\right) dx = 0$

$$V = -p_0 \left(\frac{x^2}{L} - x\right)\Big]_0^L$$

$$= -p_0 \left(\frac{L^2}{L} - L\right) = 0 \Rightarrow V = 0$$

(makes sense since net force of $q(x)$ loading is 0)

$$\Sigma M_o = 0 \quad \circlearrowleft + \quad \Rightarrow -M + \int_0^L q(x)\,x\,dx = 0$$

working:

$$M = \int_0^L p_0\left(2\frac{x^2}{L} - x\right)dx$$

$$= p_0\left[\frac{2x^3}{3L} - \frac{x^2}{2}\right]_0^L$$

$$= p_0\left[\frac{2L^3}{3L} - \frac{L^2}{2}\right] = \frac{p_0 L^2}{6}$$

$$\Rightarrow M = \frac{p_0 L^2}{6}$$

Step 2 - Work to get Shear and Moment Resultants

→ use $\frac{dS}{dx} = q(x) \Rightarrow S(x) = \int q(x)\,dx$

working:

$$S(x) = \int p_0\left(2\frac{x}{L} - 1\right)x$$

$$= p_0\left(\frac{x^2}{L} - x\right) + C_1$$

· Use a boundary condition:

@ $x = 0$, $S = V$ (reaction force and no other point load)

$V = 0 \Rightarrow S(0) = 0$

gives: $C_1 = 0$

Thus: $S(x) = p_0\left(\frac{x^2}{L} - x\right)$

→ use $\frac{dM}{dx} = S(x) \Rightarrow M(x) = \int S(x)\,dx$

working:

$$M(x) = \int p_0\left(\frac{x^2}{L} - x\right)dx$$

$$= p_0\left(\frac{x^3}{3L} - \frac{x^2}{2}\right) + C_2$$

· Use a boundary condition:

@ $x = 0$, $M(x) = M$

$$\Rightarrow \frac{p_0 L^2}{6} = C_2$$

Thus: $M(x) = p_0\left(\frac{x^3}{3L} - \frac{x^2}{2} + \frac{L^2}{6}\right)$

<u>Check</u>: at the tip ($x = L$), moment goes to zero:

$$M(L) = 0 \stackrel{?}{=} p_0\left(\frac{L^3}{3L} - \frac{L^2}{2} + \frac{L^2}{6}\right)$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{6}$$ ✓ <u>yes</u>

<u>Step 3</u> - Proceed to moment - displacement (i.e. curvature) relationship

$$M = EI\frac{d^2w}{dx^2}$$

$$\Rightarrow \frac{d^2w}{dx^2} = \frac{p_o}{EI}\left(\frac{x^3}{3L} - \frac{x^2}{2} + \frac{L^2}{6}\right)$$

$\longrightarrow$ take an integral to get slope $\frac{dw}{dx}$:

$$\frac{dw}{dx} = \int \frac{p_o}{EI}\left(\frac{x^3}{3L} - \frac{x^2}{2} + \frac{L^2}{6}\right)dx$$

$$= \frac{p_o}{EI}\left(\frac{x^4}{12L} - \frac{x^3}{6} + \frac{L^2x}{6}\right) + C_3$$

$\longrightarrow$ take an integral to get displacement w:

$$w = \int\left\{\frac{p_o}{EI}\left(\frac{x^4}{12L} - \frac{x^3}{6} + \frac{L^2x}{6}\right) + C_3\right\}dx$$

$$= \frac{p_o}{EI}\left(\frac{x^5}{60L} - \frac{x^4}{24} + \frac{L^2x^2}{12}\right) + C_3x + C_4$$

$\longrightarrow$ use two boundary conditions on displacement and/or slope to determine the two constants, C_3 and C_4:

For a clamped boundary, displacement and slope are zero. So:

@ $x=0$, $\frac{dw}{dx} = 0 \quad \Rightarrow \quad C_3 = 0$

@ $x=0$, $w = 0 \quad \Rightarrow \quad C_4 = 0$

- Use these results and also normalize the dependence on x by the length of the beam, L: i.e. $\left(\frac{x}{L}\right)$

so: $$w(x) = \frac{p_o L^4}{EI}\left\{\frac{1}{60}\left(\frac{x}{L}\right)^5 - \frac{1}{24}\left(\frac{x}{L}\right)^4 + \frac{1}{12}\left(\frac{x}{L}\right)^2\right\}$$

<u>Step 4</u>: check for the maximum (in magnitude)

Need to check at ends of configuration and within configuration.

- check ends by calculating values there:

@ $x = 0$, $w = 0$ (as noted in boundary condition)

$$@\ x = L,\ w = \frac{p_o L^4}{EI}\left\{\frac{1}{60} - \frac{1}{24} + \frac{1}{12}\right\}$$

$$= \frac{p_o L^4}{EI}\left\{\frac{2 - 5 + 10}{120}\right\}$$

$$\Rightarrow w(L) = \frac{7}{120}\frac{p_o L^4}{EI}$$

- check within structure by taking first derivative and set to zero within

$0 < x < L$:

$$\frac{dw}{dx} = \frac{p_o L^3}{EI}\left\{\frac{1}{12}\left(\frac{x}{L}\right)^4 - \frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{1}{6}\left(\frac{x}{L}\right)\right\}$$

This does not equal zero within $0 < x < L$, only at $x = 0$.

Thus:

$$\boxed{\begin{gathered} w_{max} = \frac{7}{120}\frac{p_o L^4}{EI} \\ \text{at } x = L \end{gathered}}$$

(NOTE: Cantilevered beams generally have maximum deflection at tip except for very specific load configurations).

Also, check units:

$$[L] \stackrel{?}{=} \frac{\left[\frac{F}{L}\right][L^4]}{\left[\frac{F}{L^2}\right][L^4]} = [L] \checkmark$$

OK

(6) To find the maximum magnitude of the axial stress, start with:

$$\sigma_{xx} = -\frac{Mz}{I}$$

• z and I do not vary in x, so look for the maximum magnitude of:

$$\frac{\sigma_{xx}}{(z/I)} = -M(x)$$

Earlier in (a) found:

$$M(x) = p_0\left(\frac{x^3}{3L} - \frac{x^2}{2} + \frac{L^2}{6}\right)$$

Normalizing: $= p_0 L^2 \left\{\frac{1}{3}\left(\frac{x}{L}\right)^3 - \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{1}{6}\right\}$

To find maximum magnitude, first evaluate values at end points:

$$M(0) = \frac{p_0 L^2}{6}$$

$$M(L) = p_0 L^2 \left\{\frac{1}{3} - \frac{1}{2} + \frac{1}{6}\right\} = 0$$

(as earlier noted)

Second, check within $0 < x < L$ by taking derivative and setting to zero:

$$\frac{dM(x)}{dx} = p_0 L \left\{\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)\right\}$$

Its value is zero only at the ends ($x = 0, L$)

Thus, maximum magnitude of axial stress, σ_{xx}, occurs at $x = 0$:

Thus:

$$\boxed{\left|\sigma_{xx_{max}}\right| = \frac{p_0 L^2}{6}\left(\frac{z}{I}\right) \quad \text{at } x = 0}$$

again, check units:

$$\left[\frac{F}{L^2}\right] \stackrel{?}{=} \left[\frac{F}{L}\right]\left[L^2\right] \cdot \frac{[L]}{[L^4]}$$

$$= \left[\frac{F}{L^2}\right] \quad ✓ \text{ yes}$$

(c) To find the maximum magnitude of the shear stress, start with:

$$\sigma_{xz} = -\frac{SQ}{Ib}$$

- as for the σ_{xx} case, Q, I, and b do not vary in x, so look for the maximum magnitude of:

$$\frac{\sigma_{xz}}{(Q/Ib)} = -S(x)$$

Earlier in (a) found:

$$S(x) = p_0\left(\frac{x^2}{L} - x\right)$$

Normalizing: $= p_0 L\left\{\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)\right\}$

To find maximum magnitude, first evaluate values at end points:

$$S(0) = 0$$
$$S(L) = p_0 L\,(1-1) = 0$$

(proper Boundary Conditions)

Second, check within $0<x<L$ by taking derivative and setting to zero:

$$\frac{dS(x)}{dx} = p_0\left(2\frac{x}{L} - 1\right)$$

(NOTE: This is expression for $q(x)$ as expected)

setting to zero:

$$0 = p_0\left(2\frac{x}{L} - 1\right)$$
$$\Rightarrow \frac{x}{L} = \frac{1}{2} \Rightarrow x = \frac{L}{2}$$

Evaluate at this point:

$$S(L/2) = p_o L \left\{ \frac{1}{4} - \frac{1}{2} \right\} = -\frac{p_o L}{4}$$

Thus, maximum magnitude of shear stress, σ_{xz}, occurs at $x = L/2$:

$$\boxed{|\sigma_{xz_{max}}| = \frac{p_o L}{4}\left(\frac{Q}{Ib}\right) \quad \text{at } x = L/2}$$

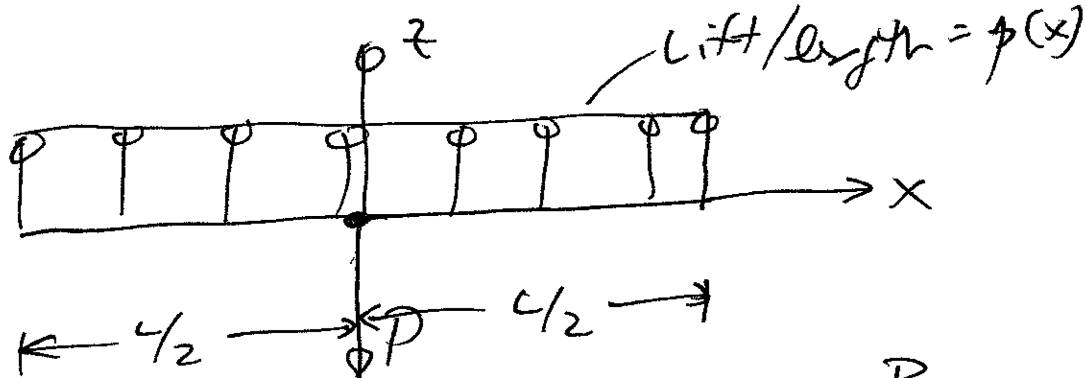
again, check units:

$$\left[\frac{F}{L^2}\right] = \left[\frac{F}{L}\right][L] \frac{[L^3]}{[L^4][L]}$$

$$= \left[\frac{F}{L^2}\right] \quad \checkmark \quad \underline{\text{yes}}$$

M9 (M5.2)

Continue considering the wing configuration of M7 (M4.2). The case of constant lift load is shown:



$$p(x) = q(x) = \frac{P}{L}$$

x = distance from root

→ Use the results for the internal load resultants determined in M7 (M4.2) for each of the three load cases.

Summarize therefrom those results for one wing: $0 < x < L/2$ ....

(Note: use subscript on expressions to indicate case)

.... [All for $0 < x < L/2$]

Case 1:

$$q_1(x) = p_0 = P/L$$
$$F_1(x) = 0$$
$$S_1(x) = P\left(\frac{x}{L} - \frac{1}{2}\right)$$
$$M_1(x) = \frac{PL}{2}\left\{\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right) + \frac{1}{4}\right\}$$

Case 2:

$$q_2(x) = \frac{4P}{3L}\left(1 - \frac{x}{L}\right)$$
$$F_2(x) = 0$$
$$S_2(x) = \frac{4P}{3}\left\{-\frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{x}{L} - \frac{3}{8}\right\}$$
$$M_2(x) = \frac{4}{3}PL\left\{-\frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{3}{8}\left(\frac{x}{L}\right) + \frac{1}{12}\right\}$$

Case 3:

$$q_3(x) = \frac{3P}{2L}\left\{1 - 4\left(\frac{x}{L}\right)^2\right\}$$
$$F_3(x) = 0$$
$$S_3(x) = \frac{3P}{2}\left\{-\frac{4}{3}\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right) - \frac{1}{3}\right\}$$
$$M_3(x) = \frac{3PL}{2}\left\{-\frac{1}{3}\left(\frac{x}{L}\right)^4 + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{3}\left(\frac{x}{L}\right) + \frac{1}{16}\right\}$$

Now proceed .....

(a) The axial stress is related to the moment via:

$$\sigma_{xx} = \frac{-M(x)\,z}{I}$$

This stress varies at any point x, along the beam with distance from the axis z. The specifics of the cross-section are not given and thus z and I cannot be determined. However, it is given that the cross-sectional shape and properties (i.e. I) are constant along the beam. Thus z/I does not affect the distribution of σ_{xx} along x, with the maximum stress occurring where z is a maximum value. Thus, the distribution of σ_{xx} with x is the same as M(x) with the value modified by $-z/I$:

$$\frac{\sigma_{xx}(x)}{(-z/I)} = M(x)$$

Note that the moment is always positive, so the stress will be negative (compressive) for +z and positive (tensile) for −z. This is consistent for a beam that bends up.

The maximum moment occurs at the root in all cases. Thus:

$$\boxed{\text{Maximum } |\sigma_{xx}| \text{ at } x=0,\ |z| \text{ maximum}}$$

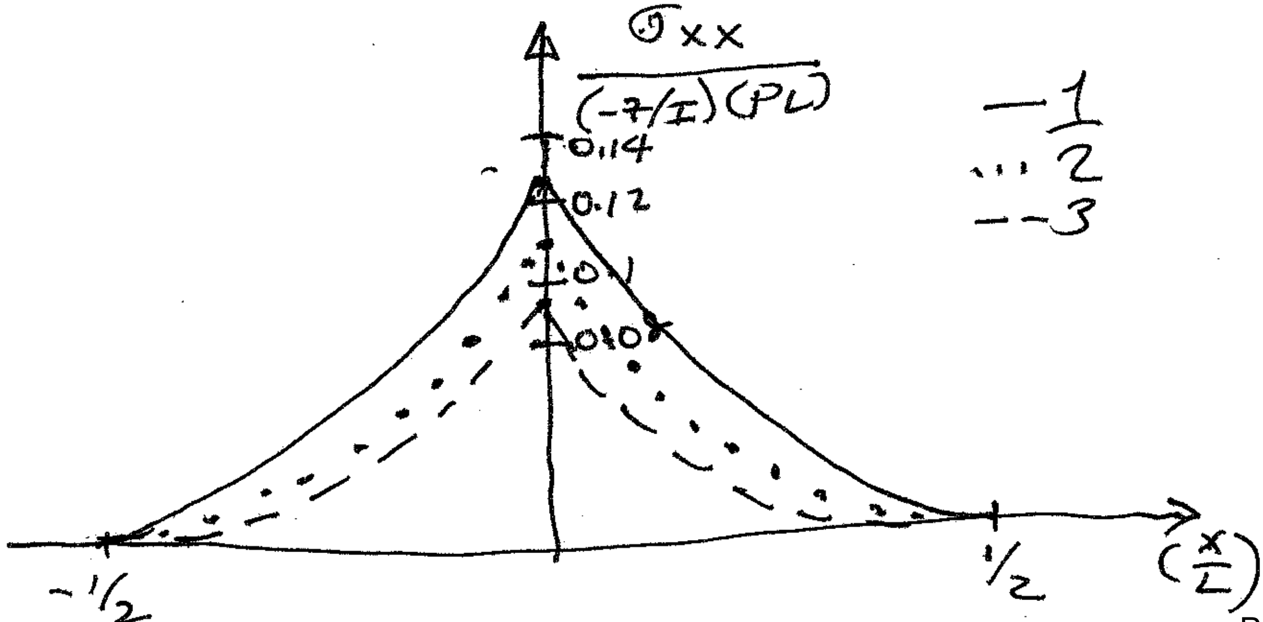
Consider the maximum values of the moment that occurs at the root:

$$M_{1\,max} = \frac{PL}{8} = 0.125\,PL$$

$$M_{2\,max} = \frac{PL}{9} = 0.111\,PL$$

$$M_{3\,max} = \frac{3PL}{32} = 0.094\,PL$$

Plot is same as for $M(x)$ as in previous week with $M(x)$ being symmetric about the root. It is $\sigma_{xx}(x)$ here:



As always, do <u>unit check</u>:

$$\frac{\sigma_{xx}}{(-z/I)(PL)} \text{ is normalized?}$$

$$\frac{[F/L^2]}{\frac{[L]}{[L^4]}\,[F][L]} = \frac{[F/L^2]}{[F]/[L^2]} \quad \checkmark \ \underline{\text{yes}}$$

(b) The shear stress is related to the shear resultant via:

$$\sigma_{xz} = -\frac{S(x)\,Q}{Ib}$$

Again, the specifics of the cross-section are not known, but they do not vary along the beam (in x). Thus, it can be said that σ_{xz} varies in x in the same way as $S(x)$ and that the maximum in z occurs where Q/b is a maximum. Thus, modify the distribution of σ_{xz} with x by $-Q/Ib$:

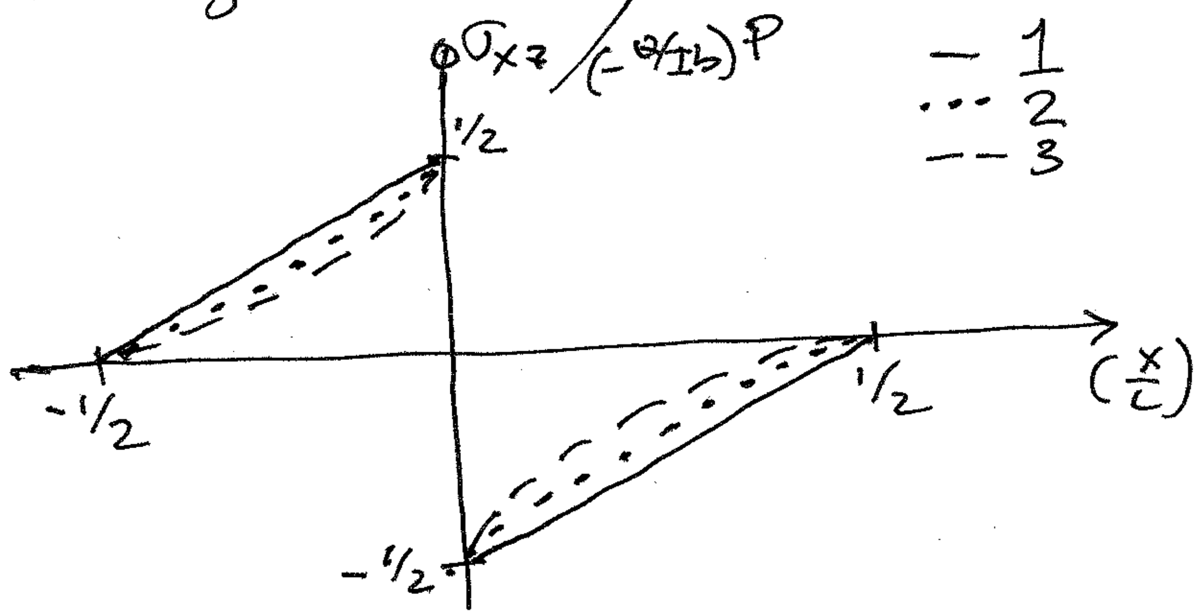
$$\frac{\sigma_{xz}(x)}{(-Q/Ib)} = S(x)$$

The maximum absolute value of the shear resultant $|S(x)|$, occurs at the root in all cases and is the same value of $P/2$ in all cases. Thus

$$\boxed{\text{Maximum } |\sigma_{xz}| \text{ at } x=0, \ \left|\frac{Q}{b}\right| \text{ maximum}}$$

$$S_{1max} = S_{2max} = S_{3max} = P/2$$

Plot is the same as for $S(x)$ as in previous week with $S(x)$ being asymmetric about the root -- negative for $0<x$; positive for $x<0$



Again, check units:

$$\frac{\sigma_{xz}}{(-Q/Ib)(P)} \text{ is normalized?}$$

$$\frac{[F/L^2]}{\frac{[L^3]}{[L^4][L]}\,[F]} = \frac{[F/L^2]}{[F]/[L^2]} \quad \checkmark \; \underline{\text{yes}}$$

(c) The deflection of a beam, $\underline{w}$ is related to the moment via:

$$EI\,\frac{d^2w}{dx^2} = M(x)$$

$M(x)$ is symmetric in x, so $w(x)$ must be as well. Thus, integrate the various moment equations for $x \geq 0$ only and similar results accounting for sign change will occur for $x < 0$.

Also note that EI does not change in x or for the various cases, so this can be ~~factored~~ in all cases.

In integrating twice, there will be a need for 2 Boundary Conditions. Refer the deflection to the attachment to the fuselage at the root. So:

$$@\ x = 0,\ w = 0$$

The other Boundary Condition comes from symmetry. Since the wing is continuous it must have the same slope on each side of the fuselage (at $x=0$). Due to symmetry, the slope must thus be zero, so:

$$@\ x = 0,\ \frac{dw}{dx} = 0$$

→ Work for each case:

Case 1: $$\left(\frac{d^2w}{dx^2}\right)_1 = \frac{PL}{2EI}\left\{\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right) + \frac{1}{4}\right\}$$

$$\Rightarrow \left(\frac{dw}{dx}\right)_1 = \frac{PL^2}{2EI}\left\{\frac{1}{3}\left(\frac{x}{L}\right)^3 - \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{1}{4}\left(\frac{x}{L}\right)\right\} + C_1$$

$$@\ x = 0,\ \frac{dw}{dx} = 0 \Rightarrow C_1 = 0$$

progressing:

$$w_1 = \frac{PL^3}{2EI}\left\{\frac{1}{12}\left(\frac{x}{L}\right)^4 - \frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{1}{8}\left(\frac{x}{L}\right)^2\right\} + C_2$$

$$@\ x = 0,\ w = 0 \Rightarrow C_2 = 0$$

Case 2: $\left(\frac{d^2w}{dx^2}\right)_2 = \frac{4PL}{3EI}\left\{-\frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{3}{8}\left(\frac{x}{L}\right) + \frac{1}{12}\right\}$

$$\Rightarrow \left(\frac{dw}{dx}\right)_2 = \frac{4PL^2}{3EI}\left\{-\frac{1}{24}\left(\frac{x}{L}\right)^4 + \frac{1}{6}\left(\frac{x}{L}\right)^3 - \frac{3}{16}\left(\frac{x}{L}\right)^2 + \frac{1}{12}\left(\frac{x}{L}\right)\right\} + C_3$$

$$@\ x = 0\quad \frac{dw}{dx} = 0 \Rightarrow C_3 = 0$$

progressing:

$$w_2 = \frac{4PL^3}{3EI}\left\{-\frac{1}{120}\left(\frac{x}{L}\right)^5 + \frac{1}{24}\left(\frac{x}{L}\right)^4 - \frac{1}{16}\left(\frac{x}{L}\right)^3 + \frac{1}{24}\left(\frac{x}{L}\right)^2\right\} + C_4$$

$$@\ x = 0\quad w = 0 \Rightarrow C_4 = 0$$

Case 3: $\left(\frac{d^2w}{dx^2}\right)_3 = \frac{3PL}{2EI}\left\{-\frac{1}{3}\left(\frac{x}{L}\right)^4 + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{1}{3}\left(\frac{x}{L}\right) + \frac{1}{16}\right\}$

$$\Rightarrow \left(\frac{dw}{dx}\right)_3 = \frac{3PL^2}{2EI}\left\{-\frac{1}{15}\left(\frac{x}{L}\right)^5 + \frac{1}{6}\left(\frac{x}{L}\right)^3 - \frac{1}{6}\left(\frac{x}{L}\right)^2 + \frac{1}{16}\left(\frac{x}{L}\right)\right\} + C_5$$

$$@\ x = 0\quad \frac{dw}{dx} = 0 \Rightarrow C_5 = 0$$

progressing:

$$w_3 = \frac{3PL^3}{2EI}\left\{-\frac{1}{90}\left(\frac{x}{L}\right)^6 + \frac{1}{24}\left(\frac{x}{L}\right)^4 - \frac{1}{18}\left(\frac{x}{L}\right)^3 + \frac{1}{32}\left(\frac{x}{L}\right)^2\right\} + C_6$$

$$@\ x = 0,\quad w = 0 \Rightarrow C_6 = 0$$

$\boxed{\text{The maximum value must occur at the tip in all cases } (x = L/2)}$

In all cases, these are in terms of $\frac{PL^3}{EI}$. Thus, check units:

$$[L] \stackrel{?}{=} \frac{[F][L^3]}{[F/L^2][L^4]} = \frac{[L^3]}{[L^2]} = [L] \checkmark$$

$\underline{\underline{\text{yes}}}$

So express in terms of these. Determine maximum w at $x = L/2$:

$$w_{max_1} = \frac{PL^3}{EI} \cdot \frac{1}{2} \left\{ \frac{1}{12}\left(\frac{1}{16}\right) - \frac{1}{6}\left(\frac{1}{8}\right) + \frac{1}{8}\left(\frac{1}{4}\right) \right\}$$

$$= \frac{PL^3}{EI} \cdot \frac{1}{2} \left\{ \frac{1}{192} - \frac{1}{48} + \frac{1}{32} \right\}$$

$$= \frac{PL^3}{EI} \cdot \frac{1}{2} \left\{ \frac{1 - 4 + 6}{192} \right\}$$

$$\Rightarrow w_{max_1} = \frac{PL^3}{EI} \left(\frac{3}{384} \right) = 0.00781 \frac{PL^3}{EI}$$

$$w_{max_2} = \frac{PL^3}{EI} \cdot \frac{4}{3} \left\{ -\frac{1}{120}\left(\frac{1}{32}\right) + \frac{1}{24}\left(\frac{1}{16}\right) - \frac{1}{16}\left(\frac{1}{8}\right) + \frac{1}{24}\left(\frac{1}{4}\right) \right\}$$

$$= \frac{PL^3}{EI} \cdot \frac{1}{24} \left\{ -\frac{1}{120} + \frac{1}{12} - \frac{1}{4} + \frac{1}{3} \right\}$$

$$= \frac{PL^3}{EI} \cdot \frac{1}{24} \left\{ \frac{-1 + 10 - 30 + 40}{120} \right\}$$

$$\Rightarrow w_{max_2} = \frac{PL^3}{EI} \left\{ \frac{19}{2880} \right\} = 0.00660 \frac{PL^3}{EI}$$

$$w_{max_3} = \frac{PL^3}{EI} \cdot \frac{3}{2} \left\{ -\frac{1}{90}\left(\frac{1}{64}\right) + \frac{1}{24}\left(\frac{1}{16}\right) - \frac{1}{18}\left(\frac{1}{8}\right) + \frac{1}{32}\left(\frac{1}{4}\right) \right\}$$

$$= \frac{PL^3}{EI} \cdot \frac{3}{32} \left\{ -\frac{1}{90}\left(\frac{1}{4}\right) + \frac{1}{24} - \frac{1}{9} + \frac{1}{2}\left(\frac{1}{4}\right) \right\}$$

$$= \frac{PL^3}{EI} \cdot \frac{3}{32} \left\{ -\frac{1}{360} + \frac{1}{24} - \frac{1}{9} + \frac{1}{8} \right\}$$

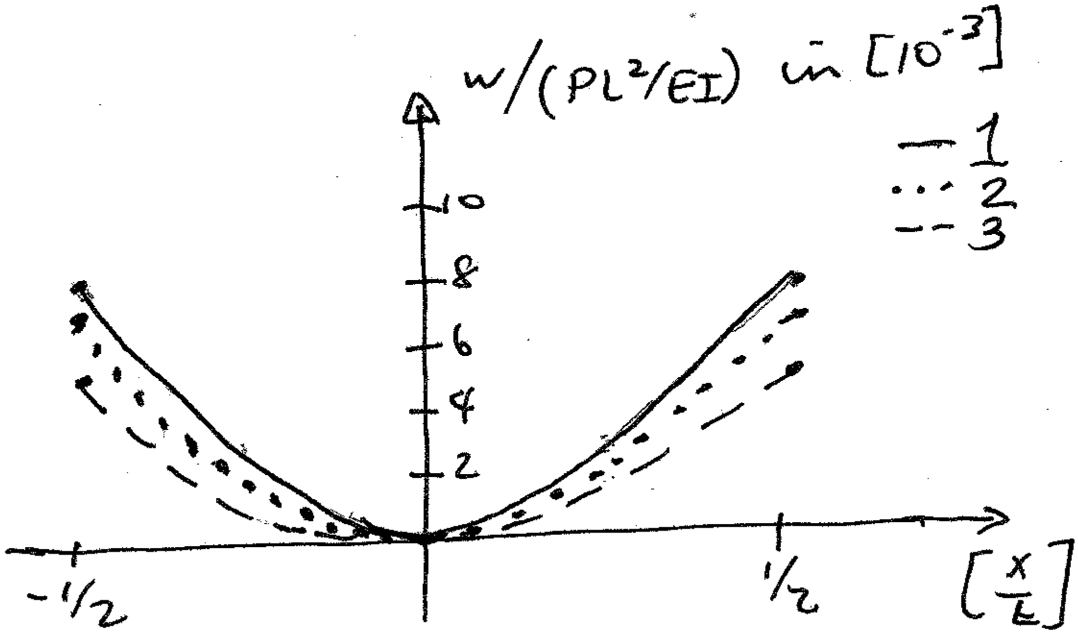
$$= \frac{PL^3}{EI} \cdot \frac{3}{32} \left\{ \frac{-1+15-40+45}{360} \right)$$

$$= \frac{PL^3}{EI} \left(\frac{19}{3840} \right) = 0.00495 \frac{PL^3}{EI}$$

Summarizing:

$$\boxed{\begin{aligned} w_{max_1} &= 0.00781 \; PL^3/EI \\ w_{max_2} &= 0.00660 \; PL^3/EI \\ w_{max_3} &= 0.00495 \; PL^3/EI \end{aligned}}$$

Proceed to a plot of approximate sketches


w/(PL²/EI) in [10⁻³]
1
2
3
10
8
6
4
2
-1/2
1/2
[x/L]