PAL
3/5/09

Unified Engineering Problem Set 5
Week 6 Spring 2009

SOLUTIONS

M10 (M6.1)

Given a statically determinate beam made of steel, we consider four cross-sections with different shapes, but of the same area of 15,000 mm^2.

First note that the resultants, $S(x)$ and $M(x)$, are not affected by the cross-section; thus the problem points to look at the geometrical contribution of the cross-sections to the various cases.

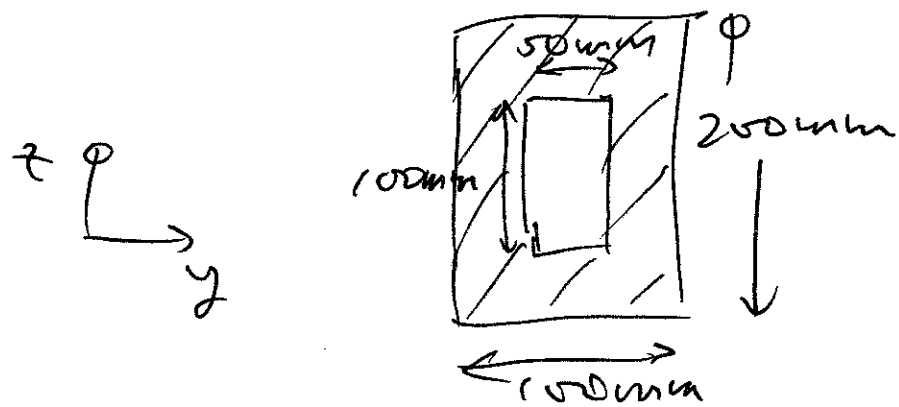
(a) The deflection is determined via:

$$\frac{M}{EI} = \frac{d^2w}{dx^2}$$

Boundary conditions are not affected by cross-sections, so in order to minimize w, one must minimize $\frac{M}{EI}$. $M(x)$ and E do not change with cross-section, so it is a matter of <u>maximizing</u> I.

Consider each cross-section:

(A) <u>The rectangular tube</u>



This is symmetric about the centerpoint, so that is the centroid. The moment of inertia can be found by subtracting the amount of that of the inner material removed. This is because these are concentric geometries of the same shape

$$I_{tube} = I_{outer} - I_{inner}$$

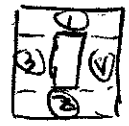
For a rectangle: $I = \frac{bh^3}{12}$

Thus:

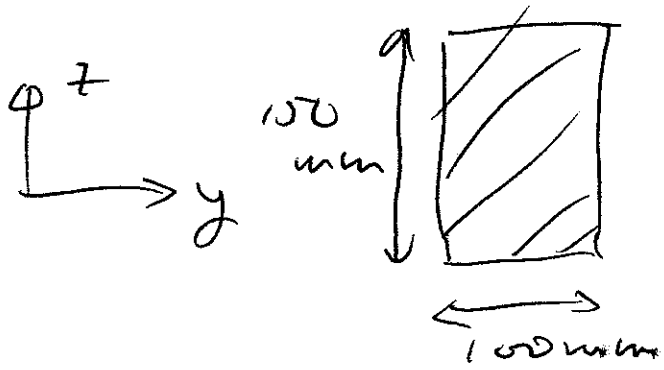
$$I_{(A)} = \frac{(100\,mm)(200\,mm)^3}{12} - \frac{(50\,mm)(100\,mm)^3}{12}$$

$$= \frac{1}{12}\left(8\times10^8\,mm^4 - 5\times10^7\,mm^4\right)$$

$$\Rightarrow \boxed{I_{(A)} = 6.25\times10^7\,mm^4}$$

(Note: can also obtain this result by working all the details of the parallel axis theorem with the four subsections of this cross-section )

(B) Solid Rectangle

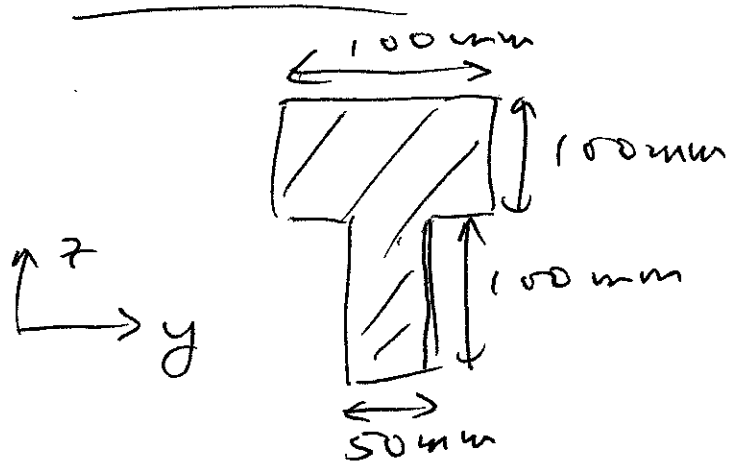


This comes directly from the expression for a rectangular cross-section:

$$I_{(B)} = \frac{bh^3}{12} = \frac{(100\,mm)(150\,mm)^3}{12}$$

$$\Rightarrow \boxed{I_{(B)} = 2.82 \times 10^7\ mm^4}$$

(C) "T"-beam

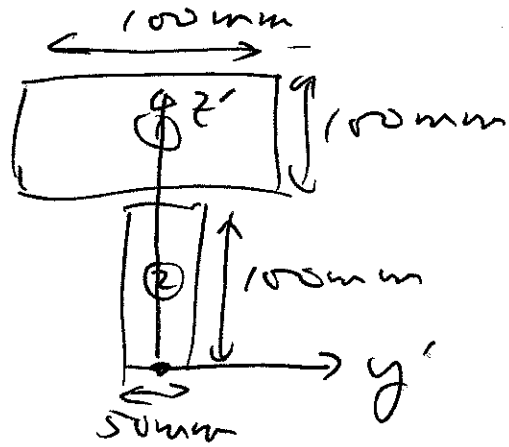


This cross-section does not have a symmetry about y-axis or ones parallel to it. It is therefore necessary to determine the location of the centroid and work from there.

Break the cross-section up into two rectangular sub-sections and place an initial axis system at some location

(bottom of cross-section is chosen here)
to get location of centroid via:

$$z_c = \frac{\iint z\,dA}{\iint dA}$$



The z_c-integrals can be done piecewise. Furthermore, since each subsection of this is a rectangle, the centroid of that subsection is known (its center) and thus:

$$\iint z\,dA = z_c A = z_c bh$$

(Prove via: $\iint z\,dy\,dz = b\int_{z_c-h/2}^{z_c+h/2} z\,dz$

$$= b\left.\frac{z^2}{2}\right]_{z_c-h/2}^{z_c+h/2}$$

$$\vdots$$

$$= z_c bh\ )$$

So:

$$z_c = \frac{(150\,mm)(100\,mm)(100\,mm) + (50\,mm)(50\,mm)(100\,mm)}{(100\,mm)(100\,mm) + (50\,mm)(100\,mm)}$$

Or in:

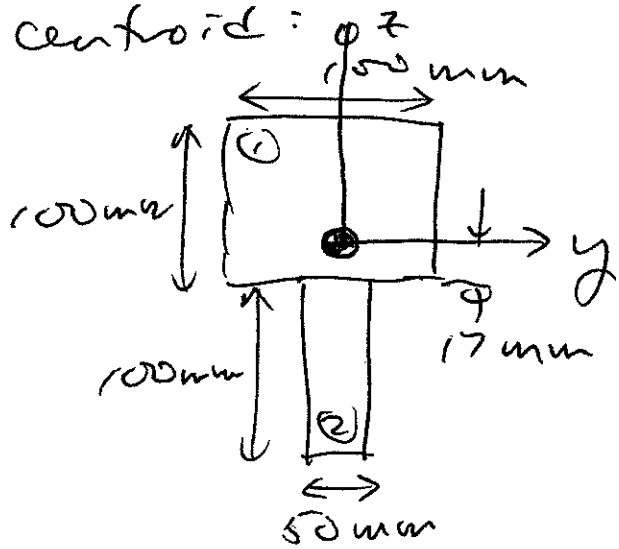
$$z_c = \frac{z'_{c_1} b_1 h_1 + z'_{c_2} b_2 h_2}{b_1 h_1 + b_2 h_2}$$

giving:

$$z_c = \frac{(1.5 \times 10^6 + 2.5 \times 10^5)\,mm^3}{(1 \times 10^4 + 5 \times 10^3)\,mm^2}$$

$$\Rightarrow z_c = 117\,mm \quad \text{from bottom}$$

Draw this and put the table together for the subsections with the axis system at the centroid:



Section	A [mm^2]	z_c [mm]	Az_c [mm^3]	Az_c^2 [mm^4]	I_o [mm^4]
①	1×10^4	$50-17 = 33$	3.3×10^5	1.09×10^7	8.33×10^6
②	5×10^3	$-17-50 = -67$	-3.35×10^5	2.24×10^7	4.17×10^6

(*)

(Note: Az_c of each of the subsections counter as they must)

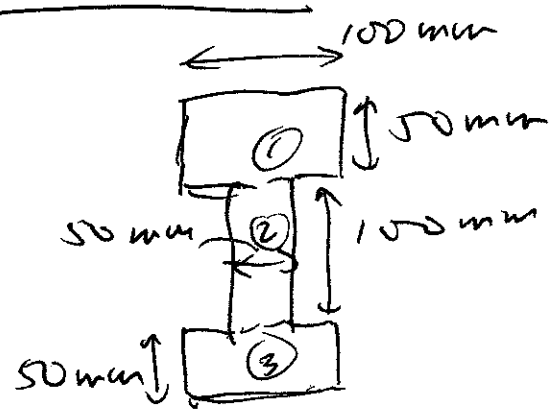
Summing these items:

$$I_{©} = \sum_{\text{sections}} (I_o + Az_c^2)$$

$$= \left\{(8.33\times10^6 + 1.09\times10^7) + (4.17\times10^6 + 2.24\times10^7)\right\} \text{mm}^4$$

$$\Rightarrow \boxed{I_{©} = 4.58\times10^7 \text{ mm}^4}$$

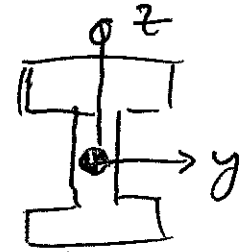
(D) "I"-beam



This is broken into three subsections -- the 2 flanges and the web.

The I-beam is symmetric in y about its center point, so this is the centroid to work about and to use the parallel axis theorem as used before:

$$I = I_0 + A z_c^2$$



Assemble the table and work from there:

Section	$A\,[mm]^2$	$z_c\,[mm]$	$A z_c^2\,[mm^4]$	$I_0\,[mm^4]$
①	$100mm \times 50mm = 5\times10^3$	$50+25 = 75$	2.82×10^7	$(100mm)\times(50mm)^3/12$ 1.04×10^6
②	$50mm \times 100mm = 5\times10^3$	0	0	$(50mm)\times(100mm)^3/12$ 4.17×10^6
③	$100mm \times 50mm = 5\times10^3$	$-50-25 = -75$	2.82×10^7	$(100mm)\times(50mm^3)/12$ 1.04×10^6

Now proceed to:

$$I_{①} = \sum_{\substack{\#\text{ of}\\ \text{sections}}} (I_0 + A z_c^2)$$

$$= \left[(1.04\times10^6 + 2.82\times10^7) + (4.17\times10^6 + 0) + (1.04\times10^6 + 2.82\times10^7)\right] mm^4$$

$$\Rightarrow \boxed{I_{①} = 6.27\times10^7\ mm^4}$$

Summarizing for all 4 sections:

$I_{Ⓐ} = 6.25 \times 10^7 \text{ mm}^4$ rectangular tube

$I_{Ⓑ} = 2.82 \times 10^7 \text{ mm}^4$ solid rectangle

$I_{Ⓒ} = 4.58 \times 10^7 \text{ mm}^4$ T-beam

$I_{Ⓓ} = 6.27 \times 10^7 \text{ mm}^4$ I-beam

<u>NOTE</u>: within calculation, the I-beam and rectangular tube have the same I about the y-axis. This is because their geometry is fundamentally the same about the y-axis with the "ears" of the rectangular tube split into two equal sections summing to 50 mm total.

So, the smallest deflection occurs when the value of I is maximized. This occurs for

rectangular tube (section Ⓐ)

<u>and</u>

I-beam (section Ⓓ)

(b) The axial stress is determined via:

$$\sigma_{xx} = \frac{-Mz}{I}$$

Thus, the maximum magnitude (i.e. absolute value) will occur for <u>maximizing</u> $\left|\frac{z}{I}\right|$ where z is the distance from the centroid. I is a constant for any cross-section and the maximum value of $M(x)$ is unaffected by this.

So consider this for each case:

(A): $I_{(A)} = 6.25 \times 10^7 \text{ mm}^4$ $\quad |z_{max}| = 100 \text{ mm}$

$$\Rightarrow \left|\frac{z}{I}\right|_{(A)} = 1.60 \times 10^{-6} \text{ mm}^{-3}$$

(B): $I_{(B)} = 2.82 \times 10^7 \text{ mm}^4$ $\quad |z_{max}| = 75 \text{ mm}$

$$\Rightarrow \left|\frac{z}{I}\right|_{(B)} = 2.66 \times 10^{-6} \text{ mm}^{-3}$$

(C): $I_{(C)} = 4.58 \times 10^7 \text{ mm}^4$ $\quad |z_{max}| = 117 \text{ mm}$

$$\Rightarrow \left|\frac{z}{I}\right|_{(C)} = 2.55 \times 10^{-6} \text{ mm}^{-3}$$

Ⓓ : $I_{Ⓓ} = 6.27 \times 10^7 \, mm^4$ $\quad |z_{max}| = 100 \, mm$

$$\Rightarrow \left| \frac{z}{I} \right|_{Ⓓ} = 1.59 \times 10^{-6} \, mm^{-3}$$

Note that once again the rectangular tube and the I-beam yield the same result and...

the cross-sections with the smallest value of the maximum magnitude of σ_{xx} are the

$$\boxed{\begin{array}{c} \text{rectangular tube (section Ⓐ)} \\ \underline{\underline{\text{and}}} \\ \text{I-beam (section Ⓓ)} \end{array}}$$

(c) The shear stress is determined via:

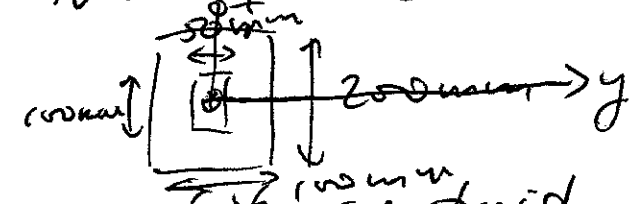
$$\sigma_{xz} = -\frac{SQ}{Ib}$$

$S(x)$ is a constant (i.e. the same) for all cross-sections. To find the maximum magnitude of σ_{xz} for any case, it is necessary to <u>maximize</u> $\left| \frac{Q}{Ib} \right|$ where

I is a constant, but Q and b can change with z.

Consider for each of the cross-sections....

(A) <u>Rectangular tube</u>



By inspection, Q_{max} is at the centroid (due to symmetry) and this is the point of minimum width (recommendation of width of 2 edges = 50mm). So the maximum value of $\frac{Q}{b}$ is at $z=0$. Find this:

Use: $Q = \int_z^{z_{top}} b\, z\, dz$

$$Q(z=0) = \int_0^{50mm} (50mm)\, z\, dz + \int_{50mm}^{100mm} (100mm)\, z\, dz$$

$$= \left\{ (50mm)\left.\frac{z^2}{2}\right]_0^{50mm} + (100mm)\left.\frac{z^2}{2}\right]_{50mm}^{100mm} \right\}$$

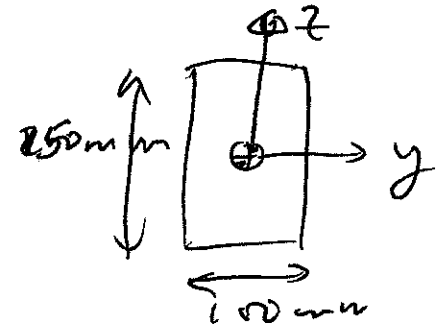
$$= \left\{ 6.25\times10^4 mm^3 + 5\times10^5 mm^3 - 1.25\times10^5 mm^3 \right\}$$

$$\Rightarrow Q(z=0) = 4.375\times10^5 mm^3$$

So for the rectangular tube, the maximum $\left|\frac{Q}{Ib}\right|$ is at $z=0$ with

$$\left|\frac{Q}{Ib}\right|_{max_{(A)}} = 1.40\times10^{-4} mm^{-2}$$

(B) Solid Rectangle



Q - This is maximized at the center point ($z = 0$) which is the location of the centroid. The width, b, is a constant, so there is no need to consider that item. So:

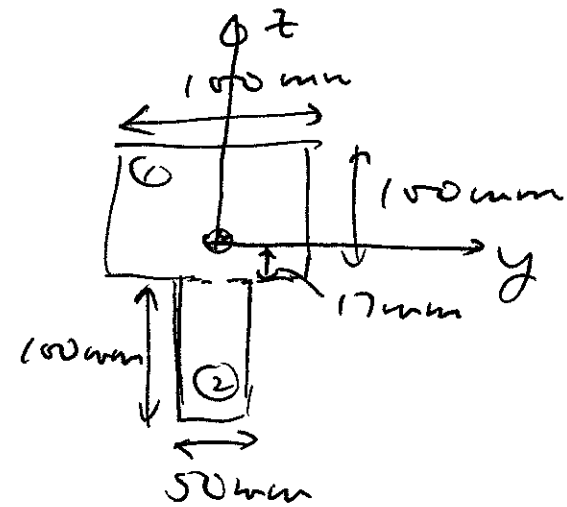
$$Q_{max} = \int_0^{75mm} (100\,mm)\, z\, dz$$

$$= (100\,mm) \frac{z^2}{2} \Big]_0^{75mm}$$

$$= 2.82 \times 10^5\, mm^3$$

So: $$\left| \frac{Q}{Ib} \right|_{max_{(B)}} = 9.97 \times 10^{-5}\, mm^{-3}$$

(C) T-beam



The width changes between the two sub-sections of the cross-section, so this needs to be considered for each cross-section

for subsection ①, the maximum value of Q occurs at the centroid. So:

$$Q_{max_{①}} = \int_0^{83\,mm} (100\,mm)\, z\, dz$$

$$= (100\,mm) \left.\frac{z^2}{2}\right]_0^{83\,mm}$$

$$= 3.44 \times 10^5\, mm^3$$

$$\Rightarrow \left|\frac{Q}{s}\right|_{max_{①}} = \frac{3.44 \times 10^5\, mm^3}{100\,mm} = 3.44 \times 10^3\, mm^2$$

for section ②, Q will be maximum nearest the centroid, thus where this subsection starts at $z = -17\,mm$:

$$Q_{max_{②}} = \left| \int_{-117\,mm}^{-17\,mm} (50\,mm)\, z\, dz \right|$$

$$= \left| (50\,mm) \left.\frac{z^2}{2}\right]_{-117\,mm}^{-17\,mm} \right|$$

$$= \left| 25\,mm \left(2.89 \times 10^2 - 1.37 \times 10^4\right) mm^2 \right|$$

$$= 3.35 \times 10^5\, mm^3$$

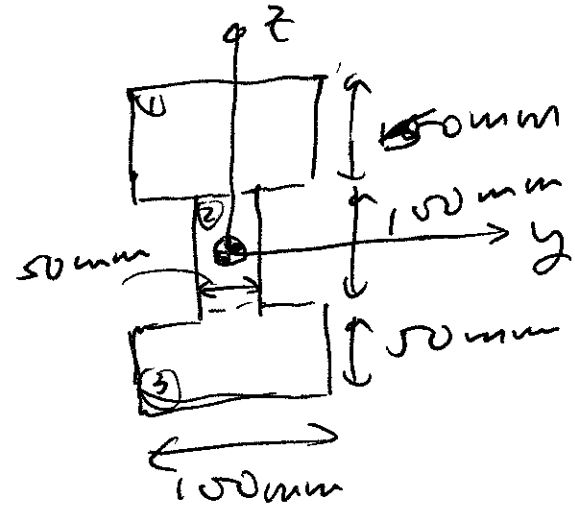
(NOTE: Can also obtain by subtracting $\int_0^{17\,mm} (100\,mm)\, z\, dz$ from $Q_{max_{①}}$)

$$\Rightarrow \left|\frac{Q}{b}\right|_{max_{②}} = \frac{3.35 \times 10^5 \text{ mm}^3}{50 \text{ mm}} = 6.70 \times 10^3 \text{ mm}^2$$

So for the T-beam, the maximum $\left|\frac{Q}{Ib}\right|$ is at $z = -17$ mm in the second sub-section with the value:

$$\left|\frac{Q}{Ib}\right|_{max_{②}} = 1.46 \times 10^{-4} \text{ mm}^{-2}$$

ⓓ I-beam



It is necessary to consider the Q's for the separate sections since b changes. However, one knows that Q will reach its maximum value at the centroid which is the center here. This is where the width is a minimum, so $\left|\frac{Q}{b}\right|$ being a maximum must be at this point ($z = 0$).

So calculate Q at this point:

$$Q_{max_{(D)}} = \int_{50mm}^{100mm} (100mm) z\,dz + \int_{0}^{50mm} (50mm) z\,dz$$

$$= (100mm) \left.\frac{z^2}{2}\right]_{50mm}^{100mm} + (50mm) \left.\frac{z^2}{2}\right]_{0}^{50mm}$$

$$= 3.75 \times 10^5 mm^3 + 6.25 \times 10^4 mm^3$$

$$Q_{max_{(D)}} = 4.38 \times 10^5 mm^3$$

giving $\left|\frac{Q}{b}\right|_{max_{(D)}} = 8.75 \times 10^3 mm^2$

So, for the I-beam, the maximum value is at $z=0$ with the value of:

$$\left|\frac{Q}{Ib}\right|_{max_{(D)}} = 1.40 \times 10^{-4} mm^{-2}$$

Summarizing for the 4 cases:

(A) $1.40 \times 10^{-4} mm^{-2}$ at $z=0$ rectangular tube

(B) $9.97 \times 10^{-5} mm^{-2}$ at $z=0$ solid rectangle

(C) $1.46 \times 10^{-4} mm^{-2}$ at $z=-17mm$ T-beam

(D) $1.40 \times 10^{-4} mm^{-2}$ at $z=0$ I-beam

So the cross-section with the smallest value of the maximum magnitude of σ_{xz} is the

$$\boxed{\text{solid rectangle - section (B)}}$$

and this occurs at the center ($\eta = 0$)

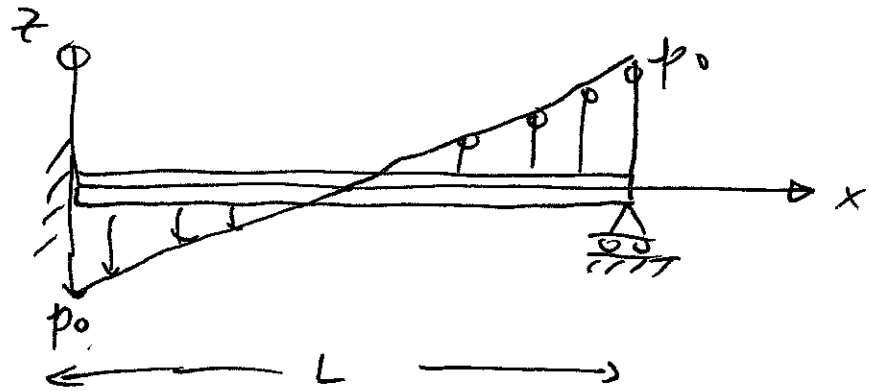
(NOTE: Once again the rectangular tube and I-beam give the same result and due to the same reasons).

(d) Some items to note:

- An I-beam is generally the most efficient configuration with regard to bending stiffness and to get the minimum deflection. This shows getting the most material as far from the centroid as possible results in greatest effectiveness. This also helps in minimizing stresses as the stresses can create a larger moment as they have a larger momentum.

- In this case, the rectangular tube and I-beam have the same geometry relative to the y-axis. Thus, these two cross-section yield the same results for this particular case
- shear stresses are lower in magnitude for "bulkier" cross-sections such as the solid rectangle. Thus, if this becomes a key consideration, such factors must become involved in the overall design.

M 11 (M6.2)

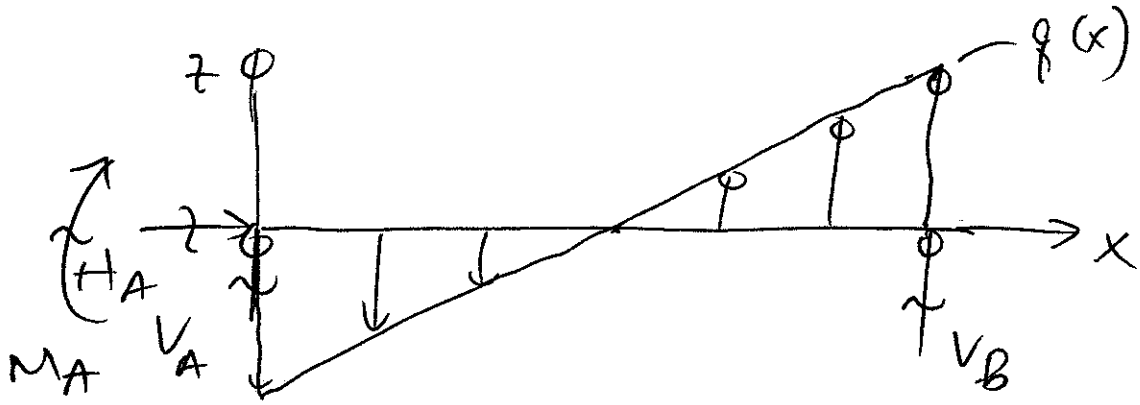


This is the same as in last week's (week 5) (Problem set 4) problem set <u>except</u> that the far tip has a roller support as opposed to being free. Thus, the problem is now <u>statically</u> <u>indeterminate</u> and equations will need to be solved simultaneously. Much of what was determined last week, but the boundary conditions definitely change.

(a) Begin with the loading being the same. So:

$$q(x) = p_0\left(\frac{2x}{L} - 1\right)$$

Now draw the Free Body Diagram, but this is a bit different:



→ Apply Equilibrium Equations to get expressions for the reactions:

$$\sum F_x = 0 \xrightarrow{+} \quad \Rightarrow H_A = 0$$

$$\sum F_z = 0 \uparrow + \Rightarrow V_A + V_B + \int_0^L q(x)\,dx = 0$$

$$\Rightarrow V_A + V_B = 0 \qquad (1)$$

(NOTE: Number the equations that will need to be solved)

$$\sum M_o = 0 \quad \circlearrowleft + \Rightarrow -M_A + \int_0^L q(x)\,x\,dx + V_B L = 0$$

$$\Rightarrow M_A - V_B L = \frac{p_o L^2}{6} \qquad (2)$$

→ Now work to get the Shear and Moment Resultants in a similar way, except the Boundary Conditions are different:

$$\frac{dS}{dx} = q(x) \Rightarrow S(x) = \int q(x)\,dx$$

as per last week's solution:

$$S(x) = p_0\left(\frac{x^2}{L} - x\right) + C_1$$

and $S(0) = V_A$ and this value is an unknown at this point in this case. So:

$$V_A = C_1$$

$$\Rightarrow S(x) = p_0\left(\frac{x^2}{L} - x\right) + V_A \qquad (3)$$

Now progress to:

$$\frac{dM}{dx} = S(x) \Rightarrow M(x) = \int S(x)\,dx$$

working:

$$M(x) = \int \left\{ p_0\left(\frac{x^2}{L} - x\right) + V_A \right\} dx$$

$$= \frac{p_0 x^3}{3L} - \frac{p_0 x^2}{2} + V_A x + C_2$$

Use the Boundary Condition that there is no moment at the roller:

$$M(L) = 0$$

$$\Rightarrow \frac{p_0 L^2}{3} - \frac{p_0 L^2}{2} + V_A L + C_2 = 0$$

$$\Rightarrow C_2 = \frac{p_0 L^2}{6} - V_A L$$

Finally:

$$M(x) = \frac{p_0 x^3}{3L} - \frac{p_0 x^2}{2} + V_A x + \frac{p_0 L^2}{6} - V_A L \quad (4)$$

→ Now move on to get an expression for the deflection via the <u>Moment-Curvature Relation</u>

$$M = EI \frac{d^2 w}{dx^2}$$

$$\Rightarrow \frac{d^2 w}{dx^2} = \frac{1}{EI}\left(\frac{p_0 x^3}{3L} - \frac{p_0 x^2}{2} + V_A x + \frac{p_0 L^2}{6} - V_A L\right)$$

V_A and p_0 do not vary with x, so their can be integrated:

$$\frac{dw}{dx} = \frac{1}{EI}\left(\frac{p_0 x^4}{12L} - \frac{p_0 x^3}{6} + V_A \frac{x^2}{2} + \frac{p_0 L^2}{6} x - V_A L x\right) + C_3$$

Use the boundary condition that $\frac{dw}{dx} = 0$ @ the root $(x=0)$

$$\Rightarrow C_3 = 0$$

Proceeding:

$$w(x) = \frac{1}{EI}\left(\frac{p_0 x^5}{60L} - \frac{p_0 x^4}{24} + V_A \frac{x^3}{6} + \frac{p_0 L^2 x^2}{12} - \frac{V_A L x^2}{2}\right) + C_4$$

again use the Boundary Condition that $w = 0$ at $x = 0$

So:

$$w(x) = \frac{1}{EI}\left(\frac{p_0 x^5}{60L} - \frac{p_0 x^4}{24} + V_A \frac{x^3}{6} + \frac{p_0 L^2 x^2}{12} - \frac{V_A L x^2}{2}\right) \quad (5)$$

There is a third boundary condition on the deflection:

$$w(L) = 0$$

Using this in (5):

$$0 = \frac{1}{EI}\left(\frac{p_0 L^4}{60} - \frac{p_0 L^4}{24} + V_A \frac{L^3}{6} + \frac{p_0 L^4}{12} - \frac{V_A L^3}{2}\right)$$

$$\Rightarrow V_A\left(\frac{3}{6} - \frac{1}{6}\right) = p_0 L\left(\frac{2 - 5 + 10}{120}\right)$$

$$V_A / 3 = \frac{7}{120} p_0 L$$

Finally: $\boxed{V_A = \frac{7}{40} p_0 L}$

use this in (1):

$$V_A + V_B = 0$$

$$\Rightarrow \boxed{V_B = -\frac{7}{40} p_0 L}$$

and then this in (2):

$$M_A = \frac{p_0 L^2}{6} + V_B L$$

$$\Rightarrow M_A = p_0 L^2 \left(\frac{1}{6} - \frac{7}{40}\right)$$

$$= p_0 L^2 \left(\frac{20-21}{120}\right)$$

$$\Rightarrow \boxed{M_A = -\frac{1}{120} p_0 L^2}$$

Note: Reactions not needed to get $w(x)$

Return to (5) to find the expression for $w(x)$ using the value for V_A:

$$w(x) = \frac{1}{EI}\left(\frac{p_0 x^5}{60L} - \frac{p_0 x^4}{24} + \frac{7 p_0 L x^3}{240} + \frac{p_0 L^2 x^2}{12} - \frac{7 p_0 L^2 x^2}{80}\right)$$

and combining the last two terms:

$$\boxed{w(x) = \frac{p_0}{EI}\left(\frac{x^5}{60L} - \frac{x^4}{24} + \frac{7Lx^3}{240} - \frac{L^2x^2}{240}\right)}$$

Check that the units are correct:

$$[L] \stackrel{?}{=} \frac{[F/L]}{[F/L^2][L^4]}\left([L^4]\right) = [L] \checkmark \text{ yes}$$

So proceed to find the maximum magnitude of deflection and its location by taking the derivative and setting it to zero:

$$\frac{dw}{dx} = \frac{p_0}{EI}\left(\frac{x^4}{12L} - \frac{x^3}{6} + \frac{7Lx^2}{80} - \frac{L^2x}{120}\right) = 0$$

Either explicitly solve the quartic or try values $(0 < \frac{x}{L} < 1)$ in a nondimensionalized expression. Using the latter:

$$0 = \frac{1}{12}\left(\frac{x}{L}\right)^4 - \frac{1}{6}\left(\frac{x}{L}\right)^3 + \frac{7}{80}\left(\frac{x}{L}\right)^2 - \frac{1}{120}\left(\frac{x}{L}\right)$$

$$\Rightarrow \quad 0 = 10\left(\frac{x}{L}\right)^4 - 20\left(\frac{x}{L}\right)^3 + 105\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)$$

Approximate the middle to begin and work from there (closer to the less restrained end -- the roller)

use $\frac{x}{L}$ as a first value:

$$\frac{x}{L} = 0.5 \Rightarrow 0.25$$

then go to $\frac{x}{L} = 0.6 \Rightarrow 0.156$

on to $\frac{x}{L} = 0.7 \Rightarrow -0.014$

So come back a bit to $\frac{x}{L} = 0.69 \Rightarrow 0.006$

use this as good enough.

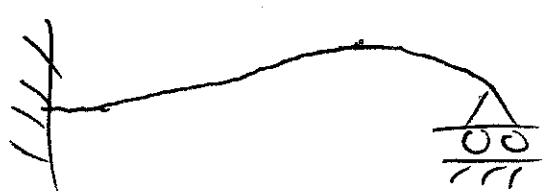
Normalize the $w(x)$ expression:

$$w(x) = \frac{p_0 L^4}{EI}\left\{\frac{1}{60}\left(\frac{x}{L}\right)^5 - \frac{1}{24}\left(\frac{x}{L}\right)^4 + \frac{7}{240}\left(\frac{x}{L}\right)^3 - \frac{1}{240}\left(\frac{x}{L}\right)^2\right\}$$

So:

$$\boxed{w_{max} = \frac{p_0 L^4}{EI}(7.60 \times 10^{-4}) \quad @ \quad \frac{x}{L} = 0.69}$$

As a sketch, the deflection is:



(b) To find the maximum magnitude of the axial stress, start with:

$$\sigma_{xx} = -\frac{Mz}{I}$$

z and I do not vary in x, so look for the maximum of $\frac{\sigma_{xx}}{(z/I)} = -M(x)$

From equation (*):

$$M(x) = \frac{p_0 x^3}{3L} - \frac{p_0 x^2}{2} + V_A x + \frac{p_0 L^2}{6} - V_A L$$

with $V_A = \frac{7}{40} p_0 L$

$$\Rightarrow M(x) = \frac{p_0 x^3}{3L} - \frac{p_0 x^2}{2} + \frac{7 p_0 L x}{40} + \frac{p_0 L^2}{6} - \frac{7 p_0 L^2}{40}$$

manipulating gives

$$M(x) = p_0 L^2 \left\{ \frac{1}{3}\left(\frac{x}{L}\right)^3 - \frac{1}{2}\left(\frac{x}{L}\right)^2 + \frac{7}{40}\left(\frac{x}{L}\right) - \frac{1}{120} \right\}$$

Check units:

$$[F \cdot L] \stackrel{?}{=} [F/L] \cdot [L^2] = [F/L] \quad \checkmark \text{ Yes}$$

First check to see if there is a maximum magnitude within the $0 < x < L$

by taking the derivative and setting it to zero: $\frac{dM(x)}{dx} = 0$

first: $\frac{dM(x)}{dx} = p_0 L \left\{ \left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right) + \frac{7}{40} \right\}$

$= 0$ solve via quadratic equation:

$$\frac{x}{L} = \frac{1 \pm \sqrt{1 - 0.7}}{2} = \frac{1 \pm \sqrt{0.3}}{2}$$

$$= \frac{1 \pm 0.548}{2}$$

$$= 0.774,\ 0.226$$

So check values at these two locations and at the endpoints

$\frac{x}{L} = 0,\quad M(x) = -\frac{p_0 L^2}{120} = (-8.33 \times 10^{-3}) p_0 L^2$

$\frac{x}{L} = 1,\quad M(x) = 0$

$\frac{x}{L} = 0.226,\quad M(x) = (4.53 \times 10^{-3}) p_0 L^2$

$\frac{x}{L} = 0.774,\quad M(x) = (-1.76 \times 10^{-2}) p_0 L^2$

$\Rightarrow$ maximum magnitude of the moment and thus σ_{xx} occurs

$$\boxed{\text{at } \frac{x}{L} = 0.77d \text{ with value of } \quad |\sigma_{xx_{max}}| = (1.76 \times 10^{-2}) \frac{p_0 L^2 z_{max}}{I}}$$

(c) To find the maximum magnitude of the shear stress, start with:

$$\sigma_{xz} = -\frac{SQ}{Ib}$$

As for the case of σ_{xx}, Q, I and b do not vary in x. Thus, look for the maximum magnitude of:

$$\left| \frac{\sigma_{xz}}{(Q/Ib)} \right| = |-S(x)|$$

From equation (3):

$$S(x) = p_0 \left(\frac{x^2}{L} - x \right) + V_A$$

and with $V_A = \frac{7}{40} p_0 L$

$$\Rightarrow \quad S(x) = p_0 L \left\{ \left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right) + \frac{7}{40} \right\}$$

Again, take the derivative and set to zero to check for locations within the beam and also look at the end points.

$$\frac{dS(x)}{dx} = p_0 \left\{ 2\left(\frac{x}{L}\right) - 1 \right\}$$

setting to 0:

$$2\left(\frac{x}{L}\right) - 1 = 0$$

$$\Rightarrow \left(\frac{x}{L}\right) = \frac{1}{2}$$

Now check values

$$\frac{x}{L} = 0, \quad S(x) = \frac{7}{40} p_0 L = 0.175\, p_0 L$$

$$\frac{x}{L} = 1, \quad S(x) = \frac{7}{40} p_0 L = 0.175\, p_0 L$$

$$\frac{x}{L} = \frac{1}{2}, \quad S(x) = -0.075\, p_0 L$$

$\Rightarrow$ maximum magnitude of the shear and thus σ_{xz} occurs

$$\boxed{\text{at each end } (x=0, L) \text{ with value of } \left|\sigma_{xz_{max}}\right| = (0.175)\frac{p_o L}{I}\left|\frac{Q}{b}\right|_{max}}$$