(a) To determine the distribution of the internal torque resultant, first determine an expression for the distributed applied torque:

\[ t(x) = mx + b \]
Two values are given:

- \( t(x, t = 0) = 100 \, \text{ft} \cdot 15/\text{ft} \)
  \[ b = 100 \, \text{ft} \cdot 15/\text{ft} \]

- \( t(x, t = 10 \, \text{ft}) = 0 \)
  \[ 0 = m \cdot (10 \, \text{ft}) + 100 \, \text{ft} \cdot 15/\text{ft} \]
  \[ m = -10 \, \text{ft} \cdot 15/\text{ft}^2 \]

8: \( t(x) = 100 \, \text{ft} \cdot 15/\text{ft} - (10 \, \text{ft} \cdot 15/\text{ft}^2) x, \)

→ Draw the Free Body Diagram:

\[ X_3 \, 0 \]
\[ t(x) = 100 \, \text{ft} \cdot 15/\text{ft} - (10 \, \text{ft} \cdot 15/\text{ft}^2) x, \]
\[ 35 \, \text{ft} \cdot 15 \, \text{ft} \rightarrow 5 \, \text{ft} \rightarrow 200 \, \text{ft} \cdot 15 \, \text{ft} \rightarrow 10 \, \text{ft} \rightarrow \text{resultant at wall} \]

(Note: \( t(x, t) \) is defined in negative direction.
Can also define in positive direction and express as: \( t(x, t) = (10 \, \text{ft} \cdot 15/\text{ft}^2) x, -100 \, \text{ft} \cdot 15/\text{ft} \))
→ Take equilibrium of the moments/torques about $x_1$. Use right hand rule ($R+R-)$ for $\int x_1$

$\Rightarrow \sum T_{x_1} = 0 \quad \Rightarrow \sum x_1$

gives:

$300 \text{ ft-lb} - 200 \text{ ft-lb} + \int_0^{10} \left(10 \frac{\text{ ft-lb}}{\text{ ft}^2} x_1 - 100 \frac{\text{ ft-lb}}{\text{ ft}} x_1\right) dx_1$

$- T_A = 0$

Evaluating the integral:

$\int_0^{10} \left(10 \frac{\text{ ft-lb}}{\text{ ft}^2} x_1^2 - 100 \frac{\text{ ft-lb}}{\text{ ft}} x_1\right) dx_1$

$= \left[500 \text{ ft-lb} - 1000 \text{ ft-lb}\right]_{x=0}^{x=10}$

$= 5000 \text{ ft-lb} - 10000 \text{ ft-lb} = -5000 \text{ ft-lb}$

Using this in the $\sum T_{x_1} = 0$ equation:

$T_A = -400 \text{ ft-lb}$

→ To determine the torque distribution $T(x)$, make cuts in the shaft/rod in the regions of different loading.

(NOTE: Can start from either end and progress inward.)
First region: \( 0 < x_1 < 5 \) ft

\[ T(x_1) = \begin{cases} \text{ft-lb} \\ \end{cases} \]

(Note: \( t(x) \) drawn in its original negative sense)

Take equilibrium:

\[ \sum T_{x_1} = 0 \quad \Rightarrow \quad 300 \text{ ft-lb} - \int_0^{x_1} t(x_1) + T(x_1) = 0 \]

Evaluate integral:

\[ \int_0^{x_1} \left( 100 \cdot \frac{\text{ft-lb}}{\text{ft}} - 10 \cdot \frac{\text{ft-lb}}{\text{ft}^2} x_1 \right) \, dx_1 \]

\[ = 100 \cdot \frac{\text{ft-lb}}{\text{ft}} x_1 - 5 \cdot \frac{\text{ft-lb}}{\text{ft}^2} x_1^2 \]

Using in equation:

\[ T(x_1) = 100 \cdot \frac{\text{ft-lb}}{\text{ft}} x_1 - 5 \cdot \frac{\text{ft-lb}}{\text{ft}^2} x_1^2 - 300 \text{ ft-lb} \quad 0 < x_1 < 5 \text{ ft} \]

OR

use \( \frac{dT}{dx_1} = -t(x_1) \)

**Note:** This equation is defined with \( t(x_1) \) defined in a positive sense.
So be sure that it is properly defined
in applying this

$$\frac{dT}{dx_1} = -T(x_1) = \begin{pmatrix} 10 \frac{ft}{sec} \ x_1, & -100 \frac{ft}{sec} \end{pmatrix}$$

indicating "positive"

$$\Rightarrow T(x_1) = \int \begin{pmatrix} 10 \frac{ft}{sec} \ x_1, & -100 \frac{ft}{sec} \end{pmatrix} \ dx_1,$$

$$= -5 \frac{ft}{sec} \ x_1^2 + 100 \frac{ft}{sec} \ x_1 + C,$$

Boundary Condition is that at \( x_1 = 0 \) \( T \) opposes the (positive) applied torque

$$\Rightarrow T(0) = -300 \ \frac{ft}{sec}$$

from equilibrium sense

$$\Rightarrow C = -300 \ \frac{ft}{sec}$$

Giving: \( T(x_1) = \begin{pmatrix} 100 \ \frac{ft}{sec} \ x_1, & -5 \frac{ft}{sec} \ x_1^2 
-300 \ \frac{ft}{sec} \end{pmatrix} \)

As before (as it vanishes)

Second region

taking a cut:

5 ft < \( x_1 \) < 10 ft
\[ T(x_i) \] \[ \text{NOTE: same issue with sense of direction} \]

\[ 300 \text{ ft. lb} \]

\[ T(x_i) = 300 \text{ ft. lb} - 200 \text{ ft. lb} - \int_0^{x_i} t(x_i) \]

\[ + T(x_i) = 0 \]

Take equilibrium:

\[ \Sigma T_{x_i} = 0 \]

\[ \Rightarrow \quad 300 \text{ ft. lb} - 200 \text{ ft. lb} - \int_0^{x_i} t(x_i) \]

\[ + T(x_i) = 0 \]

\[ \text{the integral comes out far before so:} \]

\[ T(x_i) = 100 \frac{\text{ft. lb}}{\text{ft}} x_i - 5 \frac{\text{ft. lb}}{\text{ft}^2} x_i^2 - 100 \text{ ft. lb} \]

\[ 5 \text{ ft} < x_i < 10 \text{ ft} \]

\[ \text{again the other procedure of} \]

\[ \frac{dT}{dx_i} = -t(x_i) \] \[ \text{can be used} \]

\[ \text{Using these results, the distribution of} \]
\[ \text{the internal torque resultant can be} \]
\[ \text{sketched as a function of } x_i. \]
\[ \text{At a few} \]
\[ \text{key values:} \]

1. \[ @ x_i = 0 \text{ ft} \]
\[ T = -300 \text{ ft. lb} \]
\[ \text{(equal and opposite to applied torque)} \]

2. \[ @ x_i = 5 \text{ ft} \]
\[ T = +75 \text{ ft. lb} \]
(a) \( x_1 = 5 \text{ ft} \) (+) \( T = 275 \text{ ft} \cdot \text{lb} \) (jump equal and opposite to applied torque at that point)

(b) \( x_1 = 10 \text{ ft} \) (-) \( T = 400 \text{ ft} \cdot \text{lb} \) (equal to resultant torque in positive sense)

(Note: was originally defined + in negative sense)

\[ T(x_1) \]

\[ T(x_1) \]

\[ +400 \]

\[ +275 \]

\[ +75 \]

\[ -300 \]

\[ +400 \]

\[ +275 \]

\[ +75 \]

\[ -300 \]

\[ x_1 \text{ [ft]} \]

(b) To determine the twist angle of the rod, the key equation is:

\[ \frac{d\phi}{dx_1} = \frac{T}{GJ} \]

→ One needs to choose the end from which to work. The known boundary condition is at the wall (\( \phi = 0 \) @ \( x_1 = 0 \) feet).
One can start from there are work with decreasing $x_1$ or one can start at $x_1 = 0$, carry constants in the work and then back calculate constants when getting to the Boundary Condition. Use the latter here.

for $0 < x < 5 \text{ft}: \quad T(x) = 100 \frac{\text{ft} \cdot 16}{\text{ft}^2} x_1 - 5 \frac{\text{ft} \cdot 16}{\text{ft}^2} x_1^2 - 300 \text{ft} \cdot 16$

(Note: must be defined in positive sense as it is)

$\Rightarrow \quad \phi (x_1) = \frac{1}{GJ} \int (100 \frac{\text{ft} \cdot 16}{\text{ft}^2} x_1 - 5 \frac{\text{ft} \cdot 16}{\text{ft}^2} x_1^2 - 300 \text{ft} \cdot 16) \, dx$

$\therefore \quad \phi (x_1) = \frac{1}{GJ} (50 \frac{\text{ft} \cdot 16}{\text{ft}^2} x_1^2 - \frac{5}{3} \frac{\text{ft} \cdot 16}{\text{ft}^2} x_1^3 - 300 \text{ft} \cdot 16 x_1) + C_2$

and for $5 \text{ft} < x < 10 \text{ft}: \quad T(x) = 100 \frac{\text{ft} \cdot 16}{\text{ft}^2} x_1 - 5 \frac{\text{ft} \cdot 16}{\text{ft}^2} x_1^2 - 100 \text{ft} \cdot 16$

$\Rightarrow \quad \phi_2 (x_1) = \frac{1}{GJ} (50 \frac{\text{ft} \cdot 16}{\text{ft}^2} x_1^2 - \frac{5}{3} \frac{\text{ft} \cdot 16}{\text{ft}^2} x_1^3 - 200 \text{ft} \cdot 16 x_1) + C_3$

$\therefore \text{Now evaluate } C_3 \text{ by using the Boundary Condition:}$

$\phi (10 \text{ ft}) = 0$

$\Rightarrow \quad 0 = \frac{1}{GJ} (5000 \frac{\text{ft} \cdot 16^2}{\text{ft}^4} - 1667 \frac{\text{ft} \cdot 16}{\text{ft}^2} - 1000 \frac{\text{ft} \cdot 16}{\text{ft}^2}) + C_3$

$\Rightarrow \quad C_3 = \frac{1}{GJ} (2333 \frac{\text{ft} \cdot 16}{\text{ft}^2})$
Now match the displacement at the junction of the two regions \((x_i = 5\text{ ft})\) to determine \(C_2 = \Phi_2(x_i)\):

\[
\Phi_2(5\text{ ft}) = \frac{1}{6J} \left[ 50 \frac{ft}{x_i^2} \left( \frac{ft}{x_i} \right)^2 - \frac{5}{3} \frac{ft}{x_i^2} \left( \frac{ft}{x_i} \right)^3 - 300 \frac{ft}{x_i^2} \left( \frac{ft}{x_i} \right)^3 \right. \\
+ C_2 \right] = \Phi_2(5\text{ ft}) = \frac{1}{6J} \left[ 50 \frac{ft}{x_i^2} \left( \frac{ft}{x_i} \right)^2 \\
- \frac{5}{3} \frac{ft}{x_i^2} \left( \frac{ft}{x_i} \right)^3 - 100 \frac{ft}{x_i^2} \left( \frac{ft}{x_i} \right)^3 - 2333 \frac{ft^2}{x_i^3} \right]
\]

Working through this:

\[
C_2 = \frac{1}{6J} \left[ +1500 \frac{ft^2}{x_i^3} - 500 \frac{ft^2}{x_i^3} - 2333 \frac{ft^2}{x_i^3} \right] \\
\Rightarrow C_2 = -\frac{1}{6J} \ 1333 \frac{ft^2}{x_i^3}
\]

This gives

\[
\Phi_2(x_i) = \frac{1}{6J} \left[ 50 \frac{ft}{x_i^2} x_i^2 - \frac{5}{3} \frac{ft}{x_i^2} x_i^3 - 300 \frac{ft}{x_i^2} x_i^3 - 1333 \frac{ft^2}{x_i^3} \right] \\
0 < x_i < 5\text{ ft}
\]

→ to get the twist at the tip, use \(x_i = 0\)

\[
\Rightarrow \Phi_2(x_i = 0) = -\frac{1333}{6J} \ \text{ft}^2 / \text{lb}
\]

→ finally, need to calculate \(6J\) to get the torsional stiffness of the rod.
In general: \[ J = \iint (x_2^2 + x_3^2) \, dA \]

For a circular cross-section:
\[ J = \frac{\pi R^4}{2} \]

Here \( R = 3 \text{ in} = 0.25 \text{ ft} \)
\[ \Rightarrow J = 127.2 \text{ in}^4 = 6.136 \times 10^{-3} \text{ ft}^4 \]

The material properties given are:
\[ E = 30 \text{ Msi} \quad \nu = 0.3 \]

while the shear modulus \( G \) is needed.

For isotropic materials:
\[ G = \frac{E}{2(1+\nu)} \]
\[ \Rightarrow G = \frac{30 \times 10^6 \text{ lb/in}^2}{2(1+0.3)} = 11.5 \times 10^6 \text{ lb/in}^2 \]

All the work needs to be done in consistent units (inches or feet). Displacement is small, so convert \( \phi \) to inches.
\[ \phi(0) = -\frac{1333}{60} \text{ ft}^2 \text{ in} \cdot \frac{(12 \text{ in})^2}{\text{ft}^2} \]
\[ = -\frac{181950}{60} \text{ in}^2 \cdot \text{in} \]
and:
\[ GJ = (11.5 \times 10^6 \text{ in}^2)(127.2 \text{ in}^2) \]
\[ = 1.463 \times 10^9 \text{ lb in}^2 \]

Using this in the expression for tip deflection:
\[ \phi (x_i = 0) = \frac{-191.950 \text{ in}^2 \cdot \text{Ib}}{1.463 \times 10^9 \text{ lb in}^2} \]
\[ \Rightarrow \phi_{tip} = -7.52 \times 10^{-3} \text{ radians} \]

with \( 2\pi \text{ radians} = 360^\circ \)
\[ \Rightarrow 57.3\%\text{ radian} \]
\[ \Rightarrow \phi_{tip} = -7.52 \times 10^{-3} \]

(c) To find the shear stress, use:
\[ T_{res} = \frac{T}{J} \]

To determine the maximum magnitude and its location, find the maximum magnitude of \( T(x) \) and \( r \). The value of \( r \) is maximized along the outer surface of the rod \((r = 3 \text{ in})\).
Sketch of \( T(x, r) \) showed its maximum magnitude at the wall \((x = 10\,\text{ft})\) with

\[
T(10\,\text{ft}) = +400\,\text{ft} \cdot 16
\]

have the value of \( T \) of 127.2 in.

Again convert ft to in and proceed:

maximum magnitude \( T_{res} = \frac{(400\,\text{ft} \cdot 16)(12\,\text{in}/16)(3\,\text{in})}{127.2\,\text{in}^4} \)

\[
T_{res} = 113\,\text{ps\textit{i}} \text{ \ at } r = 3\,\text{in} \text{ \ at } x = 10\,\text{ft}
\]

(d) Rod is now a hollow tube with wall thickness of \( \frac{1}{2} \) inch:

\[
R_o = 3\,\text{in} \\
R_i = 2R_o - 2t = 2.5\,\text{in}
\]
The only thing that changes from the solid rod case is the cross-sectional polar moment of inertia. So...

for (a) **Torsion Distribution does not change**

This is statically determinate and does not depend upon the cross-sectional properties.

for (b) All remains the same in considering the twist except $J$ changes. Calculate the new $J$ via superposition (remove inner section):

\[
J = \frac{\pi R_o^4}{2} - \frac{\pi R_i^4}{2} = \frac{\pi}{2} (R_o^4 - R_i^4)
\]

\[
= \frac{\pi}{2} (3^4 - 2.5^4) \text{ [in}^4]\]

\[
\Rightarrow J = 65.9 \text{ in}^4
\]

So: $\phi$ will change by the change in the inverse of $J$ since:

\[
\phi = \frac{1}{6J} \int T(x) dx
\]

\[
\frac{J_{\text{new}}}{J_{\text{old}}} = \frac{65.9 \text{ in}^4}{127.2 \text{ in}^4} = 0.518
\]
\[ \frac{J_{\text{new}}}{J_{\text{old}}} = \frac{J_{\text{old}}}{J_{\text{new}}} = 1.93 \]

So: \( \phi_{\text{tip}} \) increases by 93%.

\[ \rightarrow \text{for (c) Again, all remains the same in considering the shear stress except the polar moment of inertia as } T_{\text{new}} = T_{\text{old}} \text{ as the maximum value of } \rho \text{ is the same as is the maximum value of } T \text{ and its location. So the change is via the inverse of } J \text{ again.} \]

\[ \rightarrow \text{maximum magnitude} \]

\[ T_{\text{new}} \text{ increases by 93%} \]

\[ \text{stays at} \]

\[ \frac{r}{r = 3 \text{ in}} \]

\[ x_1 = 60 \text{ ft} \]

(e) Rod continues to be a hollow tube but wall thickness varies.
The explicit equation for $R_i$ is:

$$R_i = R_0 - t$$

Use the result of (d) as a basis for only the cross-sectional polar moment of inertia changes.

$\rightarrow$ for (a) again, the torque distribution does not change.

$\rightarrow$ for (b) the twist...

and

$\rightarrow$ for (c) the maximum shears...

...the change is via the inverse of $J$. For the tube:

$$J_{\text{new}} = \frac{\pi R_0^4}{2} - \frac{\pi R_i^4}{2}$$

$$= \frac{\pi}{2} (R_0^4 - (R_0 - t)^4)$$

So:

$$\frac{J_{\text{new}}}{J_{\text{old}}} = \frac{\tau_{\text{res, new}}}{\tau_{\text{res, old}}} = \frac{\phi_{\text{old}}}{\phi_{\text{new}}} = \frac{\frac{\pi}{2} (R_0^4)}{\frac{\pi}{2} (R_0^4 - (R_0 - t)^4)}$$

$$\Rightarrow 1 - \left(1 - \frac{t}{R_0}\right)^4 = \frac{\phi_{\text{new}}}{\phi_{\text{old}}} = \frac{\tau_{\text{res, new}}}{\tau_{\text{res, old}}}$$.****
One can do a quick numerical evaluation by evaluating this ratio $t/R_0$ to determine the ratio of the result for the tube to the solid case. This will approach 1 as $t$ approaches $R_0$ (the solid cross-section) and approach 0 as $t \to 0$.

\[ \text{Ratio} \]

\[ 3 \]

\[ 2 \]

\[ 1 \]

\[ 0.5 \]

\[ t/R_0 \]

→ The applicability

Suppose six in is exact and for the circular cross-section there were no approximations beyond the initial assumptions. The only issue occurs if the overall displacements became large such that small-angle assumptions cannot be made.
It was determined that the true solid case: 
\[ \phi_{\text{tip}} = -7.52^\circ \times 10^{-3} \]
(maximum value)

Thus, a ratio of \( \frac{J_{\text{old}}}{J_{\text{new}}} \) of 1000 (i.e. \( 10^3 \)) yields a \( \phi_{\text{tip,new}} \) of 7.52\(^\circ\). This is still relatively small. One can solve for the value of \( t \) that yields this and continue to assess applicability by considering the tip angle.
The preliminary design of a circular rod to measure torque has been evaluated as per your request. The objective is to maximize the sensitivity of the measuring capability of the rod provided by a pointer at the far end of the rod calibrated to a linear angular scale. The rod is clamped at one end and the torque is applied through a gear, at some point along the shaft, connected by a chain. The rod has a circular cross-section. These considerations yield the following physical set-up for the shaft:

\[ \phi_{\text{tip}} = \frac{TPL}{100GJ} \]

where \( P \) is the percentage of the length along which the gear is placed. The equation to determine the twist at the tip of the rod is:

and was derived through the approach documented in the accompanying Appendix. The parameters are the applied torque, \( T \), the total rod length, \( L \), the material shear modulus, \( G \), the point at which the gear is placed, \( P \) (expressed as a percentage with value from 0 to 100), and the polar moment of inertia of the cross-section, \( J \). The torque is applied and you have indicated the length is constrained. Thus, in order to maximize the sensitivity of the arrangement, one must maximize the twist measured for a given torque and this implies minimizing the shear modulus of the material, \( G \), and the polar moment of inertia of the cross-section, \( J \), and maximizing the length, \( P \), at which the gear is located to apply the torque.

We would need more information and design constraints from your company to be better able to recommend a material to be used. We expect that you will use a typical isotropic material, such as a metal, for your application. The shear modulus is directly related to the longitudinal modulus, \( E \), and inversely related to the Poisson’s ratio. Poisson’s ratio is equal to 0.3 for most materials, so we recommend you minimize the value of the longitudinal modulus, \( E \), within your other constraints. In working to minimize the polar moment of inertia of the cross-section, \( J \), we note that a hollow tube, as opposed to one with a solid cross section, produces a lower value of this parameter. For a hollow tube, the value of the parameter is directly related to the fourth power of the outer radius minus the fourth power of the inner radius. Thus, within other considerations, the cross-section should have a wall as thin as possible, and with the smallest outer radius possible.

In summary, the recommendations for the rod are as follows:

- Given more design constraints by your company, we will be better able to downselect material for the rod.
- The rod should be as long as possible.
- The placement of the attachment gear should be as outboard as possible.
- The cross-section of the rod should be hollow, designed to have walls as thin as possible, and with as small a radius as possible, within other design constraints.
APPENDIX

Drawing of physical situation:
- Circular rod (length $L$)
- Clamped at one end
- Torque applied at some point along length

Define point along length is percentage $P$ of the length $L$. Decimal value is $\frac{P \cdot L}{100}$

$X_3 \quad \bullet \quad L \quad \rightarrow \quad X_1$

$\frac{P \cdot L}{100}$

$T$

→ want to measure angle $\phi$ at rod tip

Twist of rod is determined via:

$$\frac{d\phi}{dx_1} = \frac{T(x_1)}{GJ} \quad (1)$$

⇒ Determine $T(x_1)$

Draw Free Body Diagram:

$\begin{array}{c}
T_A \quad \leftarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \quad X_1 \\
\frac{P \cdot L}{100} \quad \rightarrow \quad T
\end{array}$
\[ \Sigma T_{x_i} = 0 \rightarrow -T_A + T = 0 \]
\[ \Rightarrow T_A = T \]

Next make cuts along rod. Need two sections, one to each side of concentrated applied torque.

**Section 1:**

\[ 0 < x_i < \frac{PL}{100} \]
\[ T_A = T \]
\[ \Sigma T_{x_i} = 0 \rightarrow -T_A + T(x_i) = 0 \]
\[ \Rightarrow T(x_i) = T \quad 0 < x_i < \frac{PL}{100} \]

**Section 2:**

\[ \frac{PL}{100} < x_i < L \]
\[ T_A = T \]
\[ \Sigma T_{x_i} = 0 \rightarrow -T_A + \frac{PL}{100} + T(x_i) = 0 \]
\[ \Rightarrow T(x_i) = 0 \quad \frac{PL}{100} < x_i < L \]

**Using (1):**

\[ 0 < x_i < \frac{PL}{100} \text{ with } \frac{d\phi}{dx_i} = \frac{T}{GJ} \]
For Section 1:
\[
\phi = \frac{T}{GJ} \int d\phi = \frac{T x_1}{GJ} + C_1
\]

At the wall \((x_1 = 0)\), the twist is zero \((\phi = 0)\)

\[
\Rightarrow C_1 = 0
\]

So:
\[
\phi = \frac{T x_1}{GJ} \quad 0 < x_1 < \frac{PL}{100}
\]

Evaluate at \(x_1 = \frac{PL}{100}\) for matching:
\[
\phi = \frac{T P L}{100 G J}
\]

Now for Section 2:
\[
\frac{d\phi}{dx_1} = 0 \Rightarrow \phi = C_2
\]

Match at \(x_1 = \frac{PL}{100}\) \(\Rightarrow C_2 = \frac{T P L}{100 G J}
\]

So:
\[
\phi_{tip} = \frac{T P L}{100 G J}
\]

Now consider the shear modulus, \(G\), and the polar moment of inertia, \(J\):

Allowing for a hollow tube:
\[
J = \frac{\pi R_0^4}{2} - \frac{\pi R_i^4}{2} = \frac{\pi}{2} (R_0^4 - R_i^4)
\]
for an isotropic material:

\[ G = \frac{E}{2(1+\nu)} \]

We want to maximize \( \phi_{\text{tip}} \) (reading) in order to maximize reading sensitivity for a given applied torque. Thus, maximize:

\[ \frac{100 \phi_{\text{tip}}}{T} = \frac{PL}{GJ} \]

constant with:

\[ G = \frac{E}{2(1+\nu)} \]

\[ J = \frac{\pi}{2} (R_o^4 - R_i^4) \]
M 14 (M/10.3)

Begin with general configuration:

\[ \Phi \]
\[ I \]
\[ L \]
\[ \phi \]
\[ k \]
\[ L/2 \]
\[ x_3 \to x_1 \]

(a) Determine angular rotation of rigid column by defining details of displacement:

\[ x_3 \]
\[ \Theta \]
\[ \phi \]
\[ L/2 \]
\[ x_1 \]
- Applied loads $P_a$ and $P_b$, stay with tip and point in their original directions
- Any change in orientation of of spring can be ignored.

→ Determine equation governing angular motion $\theta$ (defined as $+$ clockwise is this configuration) by taking moment equilibrium about point of attaching

Requires moment arms for each force.

Geometry

![Diagram showing forces and geometry](image-url)
\[
\begin{align*}
\alpha &= L \cos \theta = \text{moment arm of } P_A \\
\beta &= \frac{L}{2} \cos \theta = \text{moment arm of spring force} \\
\gamma &= L \sin \theta = \text{moment arm of } P_B \\
\delta &= \frac{L}{2} \sin \theta = \text{displacement of spring}
\end{align*}
\]

Let the spring force act in \( u \) \& \( x \)-directions.

\[
\begin{align*}
X_3 & \quad \Phi \\
\rightarrow & + F_s \\
\rightarrow & X_1
\end{align*}
\]

For a positive \( d \), the spring is compressed, so \( F_s \) must be negative.

\[
\Rightarrow F_s = -kd
\]

\( \Rightarrow \) Take moment equilibrium:

\[
\sum M_0 = 0 \Rightarrow P_A \alpha + P_B \gamma + F_s \beta = 0
\]

\[
\Rightarrow P_A L \cos \theta + P_B L \sin \theta - k \left( \frac{L}{2} \sin \theta \right) \left( \frac{L}{2} \cos \theta \right) = 0
\]
working gives:

$$\left( \frac{kL}{4} \cos \theta - P_B \right) \sin \theta = P_A \cos \theta$$

(b) small angle approximation

$$\Rightarrow \cos \theta \approx 1$$

$$\sin \theta \approx \theta$$

Governing equation becomes:

$$\left( \frac{kL}{4} - P_B \right) \theta = P_A$$

This governs the stability of the configuration as it is in the form:

$$k_{\text{eff}} \theta = P_A$$

with $$k_{\text{eff}} = \frac{kL}{4} - P_B$$
Instability occurs when $k_{eff} = 0$

$$\Rightarrow \frac{kl}{4} - P_B = 0 \quad (\ast)$$

$$\Rightarrow P_B = \frac{kl}{4} \quad \text{point of instability}$$

- $P_B < \frac{kl}{4} \Rightarrow \text{stable}$
- $P_B > \frac{kl}{4} \Rightarrow \text{unstable}$

(c) Return to:

$$\left(\frac{kl}{4} - P_B\right) \theta = P_A$$

Normalize using $k$ and $l$:

$$\left(\frac{1}{4} - \frac{P_B}{kl}\right) \theta = \frac{P_A}{kl}$$

$\Rightarrow$ first consider $P_A = 0$

- Stable: $P_B < \frac{1}{4} \Rightarrow$ spring returns rod to initial orientation of $\theta = 0$ regardless of $P_B$
- Instability: $\frac{P_B}{kl} = \frac{1}{4} \Rightarrow \theta \to \infty$
Now consider various values of $\frac{P_A}{KL}$

- First for $P_B = 0$
  
  \[ \Rightarrow \theta = \frac{4P_A}{KL} \]

- Then for stable $P_B$ (same condition):
  
  \[ \frac{P_B}{KL} < \frac{1}{4} \Rightarrow \text{spring returns rod to initial offset orientation of } \theta = \frac{4P_A}{KL} \]

- Instability: $\frac{P_B}{KL} = \frac{1}{4} \Rightarrow \theta \to \infty$

(Note: If had $P_A < 0$, the same occurs except $\theta$ initially is $< 0$ and at instability $\theta \to -\infty$)

Plot of behavior for this case:

(Note: Within limits of small angular deformation)