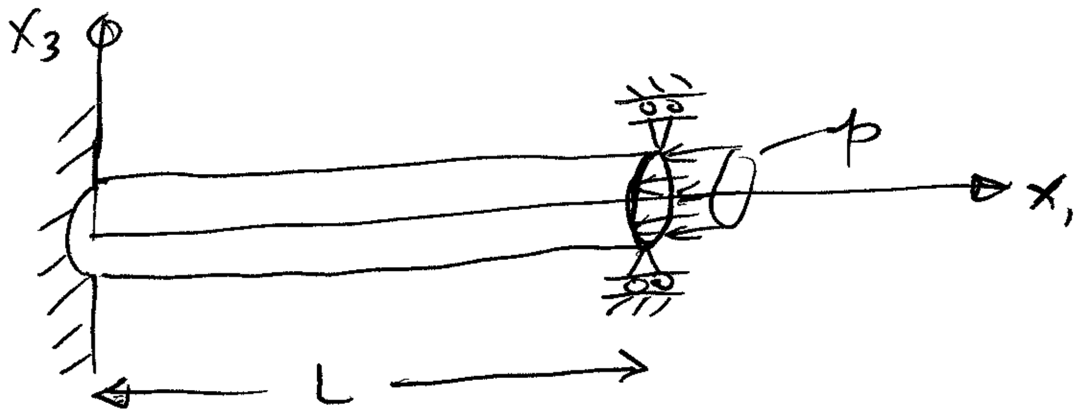
PAL
4/9/09

Unified Engineering Problem Set 9
Week 11 Spring 2009

SOLUTIONS

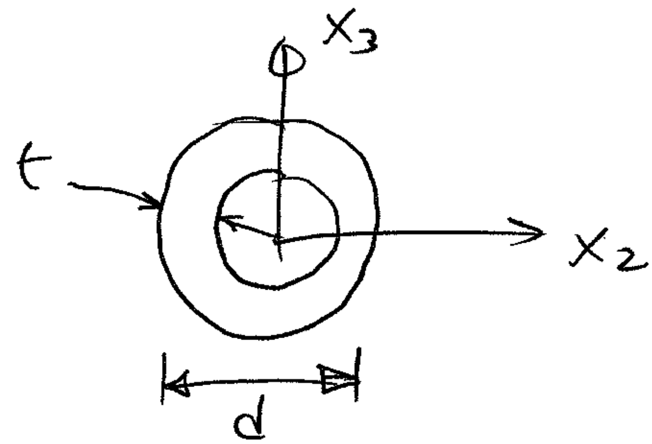
M15 (M11.1)

General configuration:



Note: $x_1 - x_3$ axes placed as in buckling work done in lecture

Consider the cross-section:



Use a general structural property characterization:

Cross-sectional Area = A
Moment of inertia = I
Modulus = E
End Load = P $(= p A)$ ← pressure

With this characterization, the buckling behavior can be characterized without the specifics of the cross-section, but in terms of cross-section representations of I and A, along with the overall load P. The specifics of the cross-section can then be used.

→ Start with the basic governing equation:

$$\frac{d^2 u_3}{dx_1^2} + \frac{P}{EI} u_3 = 0 \qquad (1)$$

with the general homogeneous solution:

$$u_3 = A \sin\left(\sqrt{\frac{P}{EI}}\, x_1\right) + B \cos\left(\sqrt{\frac{P}{EI}}\, x_1\right) + C + D x_1 \qquad (2)$$

→ Look at the boundary conditions for this configuration

- At the clamped end ($x_1 = 0$):

$$u_3 = 0$$

$$\frac{du_3}{dx_1} = 0$$

- At the roller-supported end with applied load ($x_1 = l$):

$$u_3 = 0$$

$$M = 0 \Rightarrow \frac{d^2 u_3}{dx_1^2} = 0$$

→ To facilitate writing the expression for the solution, represent:

$$\lambda = \sqrt{\frac{P}{EI}}$$

so:

$$\frac{d^2 u_3}{dx_1^2} + \lambda^2 u_3 = 0 \qquad (1')$$

and:
$$u_3 = A\sin\lambda x_1 + B\cos\lambda x_1 + C + Dx_1 \quad (2')$$

take the first two derivatives:

$$\frac{du_3}{dx_1} = \lambda A\cos\lambda x_1 - \lambda B\sin\lambda x_1 + D$$

$$\frac{d^2u_3}{dx_1^2} = -\lambda^2 A\sin\lambda x_1 - \lambda^2 B\cos\lambda x_1$$

→ Now apply each of the 4 Boundary conditions to get 4 equations:

$$@\ x_1 = 0,\ u_3 = 0 \Rightarrow B + C = 0 \quad (3)$$

$$@\ x_1 = 0,\ \frac{du_3}{dx_1} = 0 \Rightarrow \lambda A + D = 0 \quad (4)$$

$$@\ x_1 = L,\ u_3 = 0 \Rightarrow A\sin\lambda L + B\cos\lambda L + C + DL = 0 \quad (5)$$

$$@\ x_1 = L,\ \frac{d^2u_3}{dx_1^2} = 0 \Rightarrow -\lambda^2 A\sin\lambda L + \lambda^2 B\cos\lambda L = 0$$

yielding: $A\sin\lambda L + B\cos\lambda L = 0 \quad (6)$

→ Manipulate these resulting equations....

use (6) in (5)

$$\Rightarrow C + DL = 0$$

giving: $\boxed{C = -DL}$

use this result in (3) to get:

$$\boxed{B = DL}$$

Work directly with (4) to get:

$$\boxed{A = -D/\lambda}$$

→ All constants are now in terms of one constant D. Use these expressions the overall solution (2′):

$$u_3 = -\frac{D}{\lambda}\sin\lambda x_1 + DL\cos\lambda x_1 - DL + Dx_1$$

$$\Rightarrow u_3 = D\left(-\frac{1}{\lambda}\sin\lambda x_1 + L\cos\lambda x_1 - L + x_1\right)$$

This give two possible solutions:

$$D = 0 \quad \text{(trivial)}$$

or

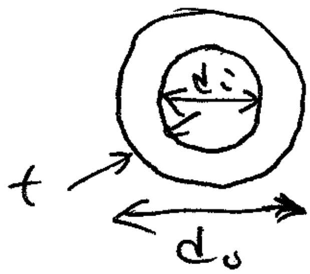
$$-\tfrac{1}{\lambda}\sin\lambda x_1 + L\cos\lambda x_1 - L + x_1 = 0$$

bringing back $\lambda = \sqrt{\frac{P}{EI}}$ yields:

$$-\sqrt{\frac{EI}{P}}\sin\sqrt{\frac{P}{EI}}\,x_1 + L\cos\left(\sqrt{\frac{P}{EI}}\,x_1\right) - L + x_1 = 0$$

→ Go to the cross-section details to incorporate p, d, and E into λ

→ With a tube, area and moment of inertia can be determined by subtracting the inner portion from the outer portion.



with $d_i = d_o - 2t$

giving: $r_o = \frac{d_o}{2}$ $\quad r_i = \frac{d_i}{2} = \frac{d_o}{2} - t$

$= r_o - t$

circular area $= \pi r^2$

$$\Rightarrow A = \pi\left[\left(\frac{d_o}{2}\right)^2 - \left(\frac{d_o}{2} - t\right)^2\right]$$

$$= \pi\left[\frac{d_o^2}{4} - \frac{d_o^2}{4} + d_o t - t^2\right]$$

$$= \pi t (d_o - t)$$

circular moment of inertia $= \frac{\pi r^4}{4}$

$$\Rightarrow I = \pi\left[\frac{(d_o/2)^4}{4} - \frac{1}{4}\left(\frac{d_o}{2} - t\right)^4\right]$$

$$= \frac{\pi}{4}\left[\frac{d_o^4}{16} - \left(\frac{d_o}{2} - t\right)^4\right]$$

with $\lambda = \sqrt{\frac{P}{EI}}$

$$\Rightarrow \lambda = \left[\frac{p \pi t (d_o - t)}{E \frac{\pi}{4} \left\{ \frac{d_o^4}{16} - \left(\frac{d_o}{2} - t \right)^4 \right\}} \right]^{1/2}$$

Finally:

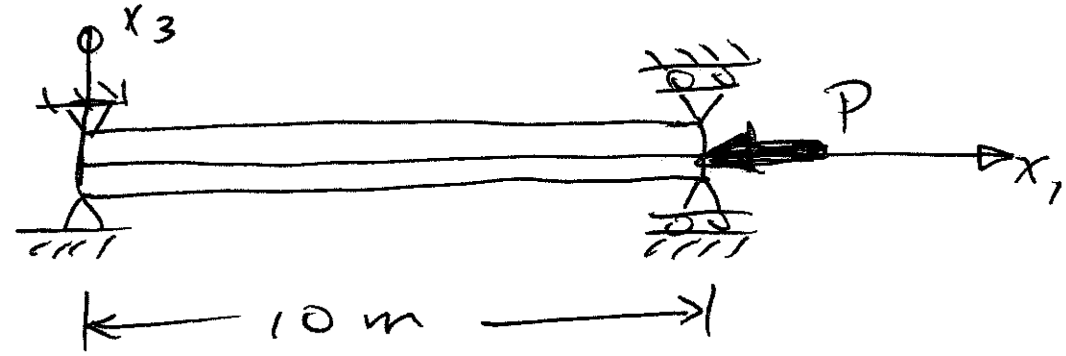
$$\boxed{\lambda = \sqrt{\frac{4 p t (d_o - t)}{E \left[\frac{d_o^4}{16} - \left(\frac{d_o}{2} - t \right)^4 \right]}}}$$

and:

$$\boxed{-\frac{1}{\lambda} \sin \lambda x_1 + L \cos \lambda x_1 - L + x_1 = 0}$$

These two expressions are to be used to find the buckling load (actually the buckling pressure) by finding the eigenvalues that satisfy this now in terms of p. With these value(s) put back in the governing expression, the eigenvectors and thus the buckling modes can be determined.

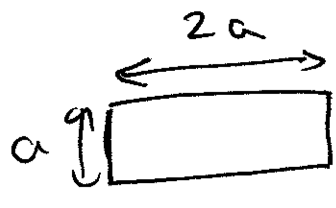
M16 (11.2)



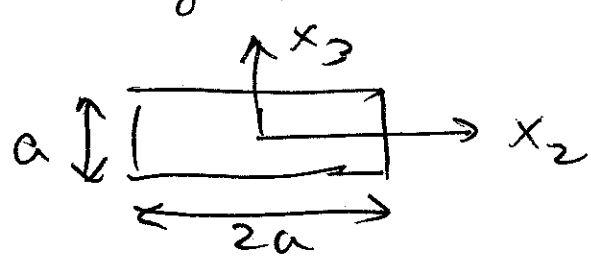
(a) Model this as a simply-supported column. For a simply-supported configuration:

$$P_{cr} = \frac{\pi^2 E I}{L^2}$$

Cross-Section:



Buckling will occur in the direction where the cross-section is "shortest". Thus x_3 is aligned with the dimension, a



for a rectangular cross-section:

$$I = \frac{bh^3}{12}$$

Here: $b = 2a$, $h = a$

$$\Rightarrow I = \frac{2a^4}{12} = \frac{a^4}{6}$$

The value of E for aluminum as per this case is 70 GPa ($= 70 \times 10^9 \frac{N}{m^2}$). The length is 10 m ($= L$).

$\rightarrow$ Use these in the expression for P_{cr}:

$$P_{cr} = \frac{\pi^2 (70 \times 10^9 \frac{N}{m^2})(\frac{a^4}{6})}{(10\ m)^2}$$

Working through this gives:

$$P_{cr} = \left(11.5 \times 10^8 \frac{N}{m^4}\right) a^4$$

P_{cr} in [N]
a in [m]

Normally cross-section dimensions are in centimeters.

Redo these units:

$$100\ cm = 1\ m$$

$$\Rightarrow 11.5\times 10^{8}\ \frac{N}{m^4}\cdot\left(\frac{1\ m}{100\ cm}\right)^4$$

gives

$$\boxed{P_{cr} = 11.5\ a^4}$$

a in [cm]
P in [N]

(b) To determine the squashing load, the material compressive ultimate is needed.

For the aluminium: $\sigma_{cu} = 425\ MPa$
$= 425\times 10^{6}\ \frac{N}{m^2}$

Have:

$$\frac{P_{squash}}{A} = \sigma_{cu}$$

For this configuration: $A = (2a)a = 2a^2$

So:

$$P_{sq} = \left(850\ \frac{N}{m^2}\times 10^{6}\right)a^2$$

a in [m]
P in [N]

make the same unit change for the cross-section dimensions:

$$P_{sq} = 850 \times 10^6 \frac{N}{m^2} \left(\frac{1\,m}{100\,cm}\right)^2 a^2$$

$$\Rightarrow \boxed{P_{sq} = 85{,}000\, a^2}$$ a in [cm], P in [N]

→ It is also important to determine the start of the "transition" zone via:

$$\frac{P_{transition}}{A} = \sigma_{cy}$$

here:

$$\sigma_{cy} = 370\ MPa = 370 \frac{N}{m^2}$$

using the same procedure yields:

$$\boxed{\underset{(yield)}{P_{tran}} = 74{,}000\, a^2}$$ a in [cm], P in [N]

(c) The key to drawing the design chart is to determine the points (P and a) where the mode of failure goes from "buckling" to "transition" to "crushing/squashing". Do this by equating the buckling case with the latter two, solving for a, and substituting the result to get P. Then plot each curve.

→ Summarizing:

$$\left.\begin{array}{ll} \text{(A) Buckling:} & P_{cr} = 11.5a^4 \\ \text{(B) Transition (yielding):} & P_{trans} = 74{,}000a^2 \\ \text{(C) Squashing:} & P_{sq} = 85{,}000a^2 \end{array}\right\} \begin{array}{l} \text{All with:} \\ a \text{ in [cm]} \\ P \text{ in [N]} \end{array}$$

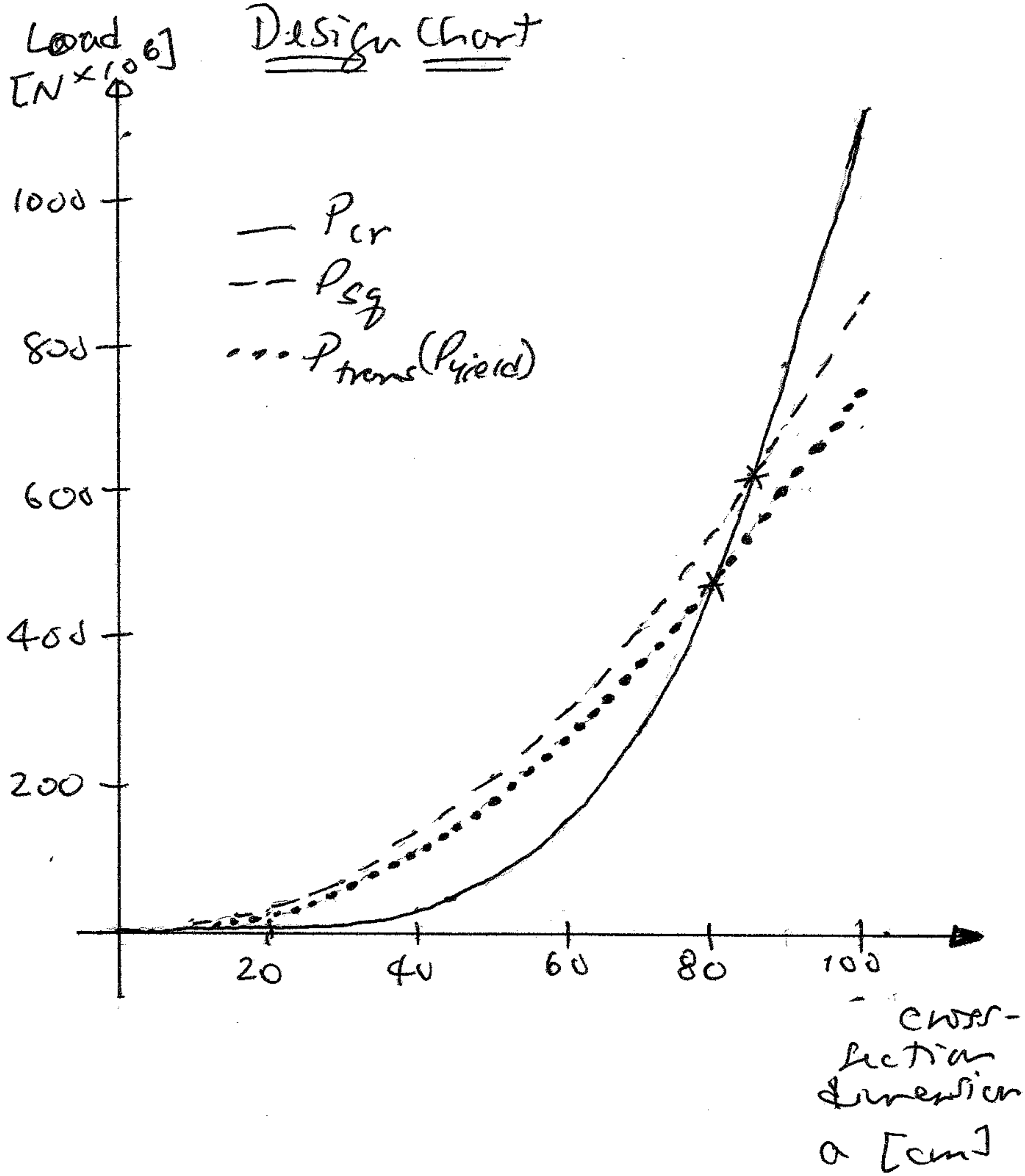
- going from (A) to (B):

$$11.5a^4 = 74{,}000a^2$$
$$\Rightarrow a^2 = 6435$$
$$\Rightarrow a = 80.2\text{ cm} \quad \text{giving } P = 476\times10^6\text{ N}$$

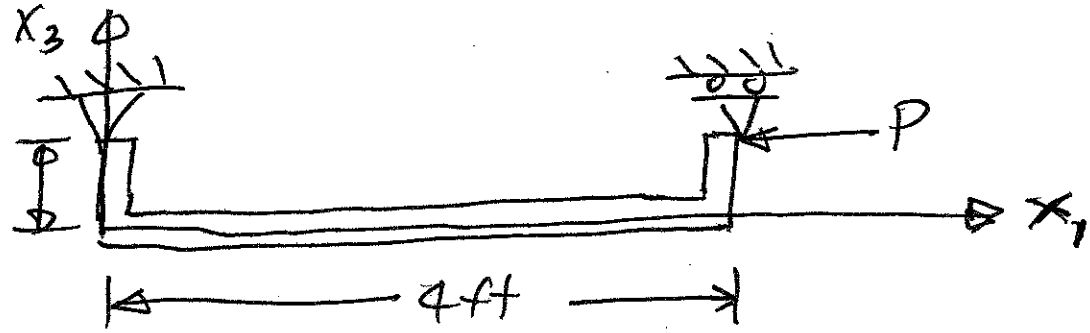
- going from (A) to (C):

$$11.5a^4 = 85{,}000a^2$$
$$\Rightarrow a^2 = 7391$$
$$\Rightarrow a = 86.0\text{ cm} \quad \text{giving } P = 629\times10^6\text{ N}$$

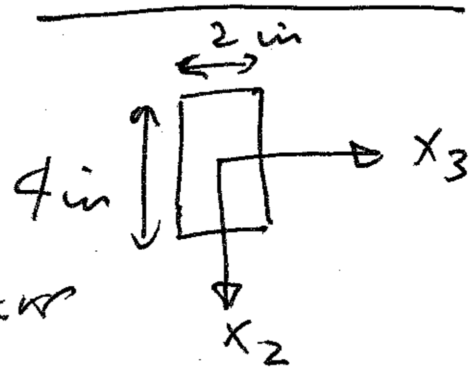
→ Draw the plots of each curve and label these key points:



M17 (M11.3)



<u>Cross Section</u>



(Note: Buckling occurs in "shorter" direction ⇒ along x_3)

(a) The maximum load is the limit placed by the buckling load. This does <u>not</u> change due to the eccentric loading. This is a simply-supported configuration, so:

$$P_{cr} = \frac{\pi^2 E I}{L^2}$$

Have for titanium: $E = 18.5 \text{ Msi}$
$= 18.5 \times 10^6 \frac{lb}{in^2}$

→ Need moment of inertia:

$$I = \frac{b h^3}{12}$$

here, $h = 2$ in and $b = 4$ in

$$\Rightarrow I = \frac{(4 \text{ in})(2 \text{ in})^3}{12} = 2.67 \text{ in}^4$$

→ Also change L in feet to inches

$$L = 4 \text{ ft} = 48 \text{ in}$$

→ All this in expression for P_{cr} gives:

$$P_{cr} = \frac{\pi^2 (18.5 \times 10^6 \frac{lb}{in^2})(2.67 \text{ in}^4)}{(48 \text{ in})^2}$$

$$\Rightarrow \boxed{P_{cr} = 211{,}600 \text{ lb}}$$

→ check σ_{cr} to see if it is σ_{cy} and σ_{cu}:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{211{,}600\ lb}{(4\ in)(2\ in)} = 26{,}450 \frac{lb}{in^2}$$

this is well below the yield and ultimate stress of 98 ksi and 150 ksi

($ksi = 10^3\ lb/in^2$)

(b) for the case of a simply-supported configuration loaded eccentrically, the governing equation is:

$$u_3 = e \left[\frac{\left(1 - \cos\sqrt{\frac{P}{EI}}\,L\right)}{\sin\sqrt{\frac{P}{EI}}\,L} \sin\sqrt{\frac{P}{EI}}\,x_1 + \cos\sqrt{\frac{P}{EI}}\,x_1 - 1 \right]$$

→ use the pertinent values of P_{cr}, E, and I, along with L to determine the deflection at the column center ($x_1 = L/2 = 24\ in$).

→ Normalize that deflection by the length and normalize the applied load by the critical load.

To do this....

- multiply P by $\frac{P_{cr}}{P_{cr}} = \frac{\pi^2 EI}{P_{cr} L^2}$

$$\Rightarrow \sqrt{\frac{P}{EI}} = \sqrt{\frac{P}{EI} \cdot \frac{\pi^2 EI}{P_{cr} L^2}} = \sqrt{\frac{P}{P_{cr}} \frac{\pi^2}{L^2}}$$

$$\text{so: } \sqrt{\frac{P}{EI}} = \frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}$$

Put this back into the earlier equation to get:

$$u_3 = e\left[\frac{1-\cos\left(\frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}\, L\right)}{\sin\left(\frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}\, L\right)} \sin\left(\frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}\, x_1\right) + \cos\left(\frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}\, x_1\right) - 1\right]$$

Continuing on and dividing through by e:

$$\frac{u_3}{L} = \frac{e}{L}\left[\frac{1-\cos\pi\sqrt{\frac{P}{P_{cr}}}}{\sin\pi\sqrt{\frac{P}{P_{cr}}}}\sin\left(\pi\sqrt{\frac{P}{P_{cr}}}\frac{x_1}{L}\right)\right.$$

$$\left. + \cos\left(\pi\sqrt{\frac{P}{P_{cr}}}\frac{x_1}{L}\right) - 1\right]$$

→ and at the center: $\frac{x_1}{L} = 0.5$, giving...

$$\boxed{\frac{u_3}{L} = \frac{e}{L}\left[\frac{1-\cos\pi\sqrt{\frac{P}{P_{cr}}}}{\sin\pi\sqrt{\frac{P}{P_{cr}}}}\sin\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)+\cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)-1\right]}$$

<u>same</u> for all cases with e normalized by L

(c) Use this relationship to make plots for the five cases of:

$\frac{e}{L} = 0,\ 0.01,\ 0.02,\ 0.05,\ 0.1$

Normalized Load vs. Normalized Center Deflection