Unified Engineering Problem Set 9

Week 15  Spring 2009

SOLUTIONS

M15 (M11.1)

General configuration:

\[ x_3 \]

\[ L \]

Note: \( x_1 - x_3 \) axes placed as in building
work done in lecture

Consider the cross-section:
Use a general structural property characterization:

- Cross-Sectional Area $= A$
- Moment of inertia $= I$
- Modulus $= E$
- End Load $= P$ ($= p A$)
- Pressure

With this characterization, the buckling behavior can be characterized in terms of cross-section representations of $I$ and $A$, along with the overall load $P$. The specifics of the cross-section can then be used.

Start with the basic governing equation:
\[
\frac{d^2u_3}{dx_1^2} + \frac{P}{EI} u_3 = 0
\]  

(1)

with the general homogeneous solution,

\[u_3 = A \sin \left( \sqrt{\frac{P}{EI}} x_1 \right) + B \cos \left( \sqrt{\frac{P}{EI}} x_1 \right) + C + D x_1,\]

(2)

Look at the boundary conditions for this configuration

- At the clamped end \((x_1 = 0)\):
  \[u_3 = 0\]
  \[\frac{du_3}{dx_1} = 0\]

- At the roller-supported end with applied load \((x_1 = L)\):
  \[u_3 = 0\]
  \[M = 0 \Rightarrow \frac{d^2u_3}{dx_1^2} = 0\]

To facilitate writing the expression for the solution, represent:

\[\lambda = \sqrt{\frac{P}{EI}}\]

So:
\[
\frac{d^2u_3}{dx_1^2} + \lambda^2 u_3 = 0
\]

(1')
and: 
\[ u_3 = A \sin \lambda x + B \cos \lambda x + C + Dx \quad (2') \]

take the first two derivatives:
\[ \frac{du_3}{dx} = \lambda A \cos \lambda x - \lambda B \sin \lambda x + D \]
\[ \frac{d^2u_3}{dx^2} = -\lambda^2 A \sin \lambda x - \lambda^2 B \cos \lambda x, \]

→ Now apply each of the 4 Boundary Conditions to get 4 equations:

@ \( x = 0 \), \( u_3 = 0 \) \( \Rightarrow B + C = 0 \quad (3) \)

@ \( x = 0 \), \( \frac{du_3}{dx} = 0 \) \( \Rightarrow \lambda A + D = 0 \quad (4) \)

@ \( x = L \), \( u_3 = 0 \) \( \Rightarrow A \sin \lambda L + B \cos \lambda L + C + DL = 0 \quad (5) \)

@ \( x = L \), \( \frac{d^2u_3}{dx^2} = 0 \) \( \Rightarrow -\lambda^2 A \sin \lambda L + \lambda^2 B \cos \lambda L = 0 \)

yielding: \( A \sin \lambda L + B \cos \lambda L = 0 \quad (6) \)

→ Manipulate these resulting equations...

close (6) with (5)
\[ \Rightarrow C + DL = 0 \]

Solving: \( C = -DL \)
Use this result with (3) to get:
\[
B = DL
\]

Work directly with (4) to get:
\[
A = -D/\lambda
\]

→ All constants are now in terms of one constant D. Use these expressions for the overall solution (2) :
\[
\begin{align*}
\psi_3 &= -\frac{D}{\lambda} \sin \lambda x, + DL \cos \lambda x, - DL + D x, \\
\Rightarrow \quad \psi_3 &= D \left( -\frac{1}{\lambda} \sin \lambda x, + L \cos \lambda x, - L + x,\right)
\end{align*}
\]

This gives two possible solutions:
- \( D = 0 \) (trivial)
- or

\[-\frac{1}{\lambda} \sin \lambda x, + L \cos \lambda x, - L + x, = 0\]

Bringing back \( \lambda = \sqrt{\frac{P}{EI}} \) yields:
\[-\sqrt{\frac{EI}{P}} \sin \sqrt{\frac{P}{EI}} x, + L \cos (\sqrt{\frac{P}{EI}} x)-L+x, = 0\]

→ Go to the cross-section equations to incorporate \( p, d, \) and \( t \) into \( A \)
With a tube, area and moment of inertia can be determined by subtracting the inner portion from the outer portion.

\[ \text{with } d_i = d_o - 2t \]

From:

\[ r_o = \frac{d_o}{2}, \quad r_i = \frac{d_i}{2} = \frac{d_o}{2} - t \]

\[ = r_o - t \]

Circumference:

\[ A = \pi \left( \left( \frac{d_o}{2} \right)^2 - (\frac{d_o}{2} - t)^2 \right) \]

\[ = \pi \left[ \frac{d_o^2}{4} - \frac{d_o^2}{4} + d_o t - t^2 \right] \]

\[ = \pi t (d_o - t) \]

Circular moment of inertia:

\[ \frac{\pi r^4}{4} \]

\[ I = \pi \left[ \left( \frac{d_o}{2} \right)^4 - \frac{1}{4} (\frac{d_o}{2} - t)^4 \right] \]

\[ = \frac{\pi}{4} \left[ \frac{d_o^4}{16} - (\frac{d_o}{2} - t)^4 \right] \]
\[
\lambda = \frac{\sqrt{P}}{\pi l}
\]

\[
\Rightarrow \lambda = \left( \frac{P \pi^2 \varepsilon (d_0 - t)}{E \pi^2 \left( \frac{d_0}{16} - \left( \frac{d_0}{2} - t \right)^2 \right)} \right)^{1/2}
\]

Finally:

\[
\lambda = \frac{\sqrt{4p \varepsilon (d_0 - t)}}{\sqrt{E \left[ \frac{d_0}{16} - \left( \frac{d_0}{2} - t \right)^2 \right]}},
\]

and:

\[
- \frac{1}{\lambda} \sin \lambda x_1 + L \cos \lambda x_1, \quad -L + x_1 = 0
\]

These two expressions are to be used to find the buckling load (actually the buckling pressure) by finding the eigenvalues that satisfy this now in terms of \( P \). With these values put back in the governing expression, the eigenvectors and thus the buckling modes can be determined.
(a) Model this as a simply-supported column. For a simply-supported configuration:

\[ P_{cr} = \frac{\pi^2EI}{L^2} \]

**Cross-Section:**

Buckling will occur in the direction where the cross-section is "sharpest." Thus \( x_3 \) is aligned with the dimension \( a \).
for a rectangular cross-section:
\[
I = \frac{bh^3}{12}
\]

Here: \( b = 2a, h = a \)

\[
\Rightarrow I = \frac{2a^4}{12} = \frac{a^4}{6}
\]

The value of \( E \) for aluminium as per this case is 70 GPa (= 70 \( \times 10^9 \) N/m\(^2\)). The length is 10 m (= \( L \)).

\[ \text{Use these in the expression for } P_{cr}. \]

\[
P_{cr} = \frac{\pi^2 \left( 70 \times 10^9 \ \text{N/m}^2 \right) \left( \frac{a^4}{6} \right)}{(10 \ \text{m})^2}
\]

Working through this gives:

\[
P_{cr} = (11.5 \times 10^8 \ \text{N/m}^4) a^4
\]

\[ P_{cr} \text{ in MN} \]

\[ a \text{ in m} \]

Normally cross-section dimensions are in centimeters.
2. do these units:

\[ 100 \text{ cm} = 1 \text{ m} \]

\[ \Rightarrow 11.5 \times 10^8 \frac{N}{m^2} \cdot \left( \frac{1 \text{ m}}{100 \text{ cm}} \right)^4 \]

This gives:

\[ P_{cu} = 11.5 a^4 \quad \text{a in [cm]} \]

\[ P \text{ in [N]} \]

(5) To determine the squashing load, the material compressive ultimate is needed.

In the aluminum: \( \sigma_{cu} = 425 \text{ MPa} \)

\[ = 425 \times 10^6 \frac{N}{m^2} \]

Have:

\[ \frac{P_{\text{squash}}}{A} = \sigma_{cu} \]

For this configuration: \( A = (2a) a = 2a^2 \)

So:

\[ P_{\text{sq}} = \left( 850 \frac{N}{m^2} \times 10^6 \right) a^2 \quad \text{a in [cm]} \]

\[ P \text{ in [N]} \]
make the same unit change for the cross-section dimensions:

\[ P_{eq} = 850 \times 10^6 \frac{N}{m^2} \left(\frac{1 m}{100 cm}\right)^2 a^2 \]

\[ \Rightarrow P_{eq} = 85,000 \ a^2 \quad 0 \text{ in } [cm] \quad P \text{ in } [N] \]

\[ \Rightarrow \text{It is also important to determine the start of the "transition" zone via:} \]

\[ \frac{P_{\text{transition}}}{A} = \sigma_{cy} \]

here:

\[ \sigma_{cy} = 370 \text{ MPa} = 370 \frac{N}{m^2} \]

using the same procedure yields:

\[ P_{\text{Yield}} = 74,000 \ a^2 \quad a \text{ in } [cm] \quad P \text{ in } [N] \]
(c) The key to drawing the design chart is to determine the points (P and a) where the mode of failure goes from "buckling" to "transition" to "crushing/squashing". Do this by equating the buckling curve with the latter two, solving for a, and substituting the result to get P. Then plot each curve.

→ Summarizing:

(A) Buckling: $P_{cr} = 11.5 \times 10^4$ N ($a = \frac{1}{2} \sqrt{\frac{E}{K'}}$) $a = \frac{1}{2} \text{cm}$ $P_{cr} = \text{N}$

(B) Transition: $P_{tr} = 74,000 a^2$

(C) Squashing: $P_{sq} = 85,000 a^2$

- Going from (A) to (B):

$$11.5 \times 10^4 = 74,000 \times a^2$$

$$\Rightarrow a^2 = 6.535$$

$$\Rightarrow a = 80.2 \text{ cm} \quad \text{giving } P = 47.6 \times 10^6 \text{ N}$$

- Going from (A) to (C):

$$11.5 \times 10^4 = 85,000 \times a^2$$

$$\Rightarrow a^2 = 73.91$$

$$\Rightarrow a = 86.0 \text{ cm} \quad \text{giving } P = 6.29 \times 10^6 \text{ N}$$
Draw the plots of each curve and label these key points:

- $P_{cr}$
- $P_{eq}$
- $P_{trans}$ (yield)

Load $[N \times 10^6]$
\[ M_{17}(M_{11.3}) \]

(Note: Buckling occurs in "shutter" direction along \( x_3 \))

(a) The maximum load is the limit placed by the buckling load. This does not change due to the eccentric loading. This is a simply-supported configuration so:
\[ P_{cr} = \frac{\pi^2 EI}{L^2} \]

Here for Titanium: \( E = 16.5 \text{ MSi} \)
\[ = 16.5 \times 10^6 \frac{16}{\text{in}^2} \]

\[ \Rightarrow \text{Need moment of inertia, } \\ I = \frac{6h^3}{12} \]

Here, \( h = 2\text{ in} \) and \( b = 4\text{in} \)
\[ \Rightarrow I = \frac{(4\text{in})(2\text{in})^3}{12} = 2.67\text{ in}^4 \]

\[ \Rightarrow \text{ Also change } L \text{ ft to inches} \\ L = 4\text{ ft} = 48\text{ in} \]

\[ \Rightarrow \text{ All units in expression for } P_{cr} \text{ given:} \\ P_{cr} = \frac{\pi^2 (16.5 \times 10^6 \frac{16}{\text{in}^2})(2.67\text{ in}^4)}{(48\text{ in})^2} \]

\[ \Rightarrow P_{cr} = 211,600 \text{ lb} \]
\[ \sigma_c = \frac{P_{cr}}{A} = \frac{211,600 \text{ lb}}{(4 \text{ in})(2\text{ in})} = 26,450 \frac{\text{lb}}{\text{in}^2} \]

This is well below the yield and ultimate stress of 98 ksi and 150 ksi:

\[ (\text{ksi} = 10^3 \frac{\text{Psi}}{\text{in}^2}) \]

(b) for the case of a simply-supported configuration loaded eccentrically, the governing equation is:

\[ u_3 = e \left[ \frac{(1 - \cos \sqrt{\frac{P}{EI}} \cdot L)}{\sin \sqrt{\frac{P}{EI}} \cdot L} \cdot \sin \sqrt{\frac{P}{EI}} \cdot x_1 \right. \]

\[ \left. + \cos \sqrt{\frac{P}{EI}} \cdot x_1 - 1 \right] \]

\[ \rightarrow \text{Use the pertinent values of } P_{cr}, E, \text{ and } I, \text{ along with } L \text{ to determine the deflection at the column center } (x_1 = \frac{L}{2} = 24 \text{ in}). \]
Normalize that deflection by the length and normalize the applied load by the critical load.
To do this...

- multiply \( P \) by \( \frac{P_{cr}}{P} = \frac{\pi^2 EI}{P_{cr} L^2} \)

\[
\sqrt{\frac{P}{EI}} = \sqrt{\frac{P}{EI} \cdot \frac{\pi^2 EI}{P_{cr} L^2}} = \sqrt{\frac{P}{P_{cr}} \cdot \frac{\pi^2}{L^2}}
\]

So:
\[
\sqrt{\frac{P}{EI}} = \frac{\pi}{L} \sqrt{\frac{P}{P_{cr}}}
\]

Put this back into the earlier equation to get:

\[
u_3 = e \left[ \frac{1 - \cos \left( \frac{\pi}{L} \sqrt{\frac{P}{P_{cr}}} y \right)}{\sin \left( \frac{\pi}{L} \sqrt{\frac{P}{P_{cr}}} L \right)} \sin \left( \frac{\pi}{L} \sqrt{\frac{P}{P_{cr}}} x \right) \right]
\]

\[
+ \cos \left( \frac{\pi}{L} \sqrt{\frac{P}{P_{cr}}} x \right) - 1
\]

Continuing on and dividing through by \( e \):
\[
\frac{U_3}{L} = \frac{e}{L} \left[ \frac{1 - \cos \left( \pi \sqrt{\frac{P}{P_{cr}}} \right)}{\sin \pi \sqrt{\frac{P}{P_{cr}}} \sin \left( \pi \sqrt{\frac{P}{P_{cr}}} \frac{x_1}{L} \right)} \sin \left( \pi \sqrt{\frac{P}{P_{cr}}} \frac{x_1}{L} \right) + \cos \left( \pi \sqrt{\frac{P}{P_{cr}}} \frac{x_1}{L} \right) - 1 \right]
\]

→ And at the center: \( \frac{x_1}{L} = 0.5 \), giving:

\[
\frac{U_3}{L} = \frac{e}{L} \left[ \frac{1 - \cos \left( \pi \sqrt{\frac{P}{P_{cr}}} \right)}{\sin \pi \sqrt{\frac{P}{P_{cr}}} \sin \left( \frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} \right) + \cos \left( \frac{\pi}{2} \sqrt{\frac{P}{P_{cr}}} \right) - 1 \right]
\]

Same for all cases with \( e \) normalized by \( L \)

(c) Use this relationship to make plots for the five cases of:

\[ \frac{e}{L} = 0, 0.01, 0.02, 0.05, 0.1 \]