Happy is he who gets to know the reasons for things.

VIRGIL (70–19 B.C.)

chapter 11 Accelerated Frames and Inertial Forces[1]

11.1 Introduction

Almost everyone has had the experience of riding in an automobile that turns a corner rapidly, and of being "thrown" toward the outer edge of the curve. To the passengers, this force seems very real—in fact, the passengers may even be "thrown" against the side of the car with a fairly large impact. Similarly, passengers in an elevator accelerating upward experience a force that apparently pushes them harder against the floor. And riders in a moving automobile that is braked rapidly believe there is a force propelling them toward the front of the car so violently that, in the absence of seatbelts, they brace themselves against the dashboard to counteract this force. As a final example, it has been proposed that spaceships on long voyages be given a rotation to provide astronauts with an "artificial gravity," or an outward "centrifugal force," as it is sometimes called. Experiments have shown that such an artificial gravity force is important for reasons of health as well as for comfort, since muscle tissue and bones tend to deteriorate if they are not exercised by working against this artificial gravity, or by doing some other exercise.

Each of these examples describes the same special type of force: a force that has no physical source or external object causing it. Its origin lies in the fact that *the observer is in an accelerated frame of reference*. To understand these forces requires a reorientation in our usual way of thinking. Although this may seem hard at first, if you follow the discussion step by step it will not be difficult.

11.2 Inertial Forces

In previous chapters, all the forces discussed may be traced to a "source"; that is, some other object or physical system causes the force. We will call these forces "real" forces, signifying that in each case the origin of the force may be traced to some other object. A few examples are listed below:

(a) Tensions in springs, strings, and so on.

(b) Forces of contact in which one object pushes or pulls on another because of physical contact between the objects.

[1] A portion of this chapter is adapted from one of the authors' (AMH) discussion in *Physics—A New Introductory Course*, Science Teaching Center, M.I.T. 1965.

(c) Gravitational forces of attraction between masses.
(d) Electrical forces between objects that have a net electric charge.
(e) Magnetic forces (ultimately traceable to forces between electric charges in motion).
(f) Nuclear forces.

As we delve deeper into the nature of forces, we discover that what we conveniently call forces of contact, tensions in strings, cohesion forces between atoms and molecules, and so on, are ultimately electromagnetic in origin; in other words, they are forces between charges on electrons and protons. So our list of real forces reduces to the following basic classifications:

Gravitational forces	(between masses)
Electromagnetic forces	(between charged particles)
Nuclear forces	(between certain fundamental particles, only significant at distances comparable to nuclear dimensions)

The kinds of forces listed above are involved in the analysis of phenomena viewed from an inertial frame of reference. Analysis by an observer in an *accelerated* frame of reference involves an additional class of forces that are evident only to such an accelerated observer. Because these forces are associated with the inertial property of matter (rather than being "caused" by another object or physical system), they are called **inertial forces.**[2] A phenomenon such as a ball bouncing in an accelerating freight car may be analyzed either by an observer in the inertial frame of the railway station or by an observer in the accelerated frame of the freight car. To the observer in the station, no inertial forces are present, but to the observer in the car, inertial forces must be considered in the analysis of motion. The choice of a frame of reference for the analysis is entirely a matter of convenience.

11.3 Linearly Accelerated Frames of Reference

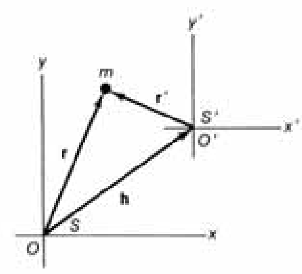


Figure 11–1
Observers in the S and S' frames of reference designate the location of the mass m by the position vectors $\mathbf{r}$ and $\mathbf{r}'$, respectively. The vector $\mathbf{h}$ locates the origin of the S' frame relative to the origin of the S frame.

Let us establish a notation to describe the motion of a mass point m as seen[3] in two different frames of reference. Consider a coordinate system S that is at rest with respect to an inertial frame of reference. Another coordinate system S' has arbitrary translational motion with respect to the S system (see Figure 11–1). Measurements made in the S' system will be designated by primed symbols. (For convenience, we position the x', y', and z' axes of S' so that they are parallel with the corresponding axes of S.) The position vector $\mathbf{h}$ locates the origin O' with respect to O.

Observers in the S and S' frames describe the location of the same mass point m by the position vectors $\mathbf{r}$ and $\mathbf{r}'$, respectively. These vectors are related by the usual rule of vector addition:

$$\mathbf{r} = \mathbf{r}' + \mathbf{h} \tag{11–1}$$

[2] Other common names for inertial forces are *fictitious* or *pseudo* forces. We prefer to avoid such terms since they may imply that the forces are somewhat imaginary. As we shall see, inertial forces are just as verifiable as the force of gravity.

[3] The phrase "as *seen* in a given frame of reference" is not to be interpreted literally. "Viewing" or "observing" the motion of an object in a frame of reference here means "*making space and time measurements of the object* relative to that particular frame of reference."

If both m and S' move, then the velocity of m as measured in the two frames of reference is found by taking derivatives of the position vectors with respect to time:

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}'}{dt} + \frac{d\mathbf{h}}{dt}$$

or

$$\mathbf{v} = \mathbf{v}' + \mathbf{v}_h \tag{11–2}$$

where $\mathbf{v}_h$ denotes the velocity of O' relative to O. Similarly, if m has acceleration, we differentiate again to obtain

$$\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}'}{dt} + \frac{d\mathbf{v}_h}{dt}$$

or

$$\mathbf{a} = \mathbf{a}' + \mathbf{a}_h \tag{11–3}$$

where $\mathbf{a}_h$ represents any accelerations of O' relative to O. Equation (11–3) is a general statement applicable in all cases.

We shall first examine the special case of S' moving *uniformly* (no acceleration or rotation) with respect to S, where the acceleration $\mathbf{a}_h$ is therefore equal to zero. Since Newton's laws are valid in an inertial frame (S), if m is subject to a (real) external force $\mathbf{F}$, we have

$$\textit{As seen in } S\text{:} \quad \Sigma\mathbf{F}_{\text{real}} = m\mathbf{a} \tag{11–4}$$

Multiplying Equation (11–3) by m, we obtain

$$m\mathbf{a} = m\mathbf{a}' + m\mathbf{a}_h \tag{11–5}$$

But since $\mathbf{a}_h = 0$, this reduces to

$$m\mathbf{a} = m\mathbf{a}'$$

or, by Equation (11–4):

$$\textit{As seen in } S'\text{:} \quad \Sigma\mathbf{F}_{\text{real}} = m\mathbf{a}' \quad \text{(when } S' \text{ has uniform translation and no acceleration)} \tag{11–6}$$

Thus Newton's first and second laws are valid for a uniformly moving frame of reference, that is, a frame with no acceleration. We call all such frames **inertial frames** because Newton's first law—the law of inertia—is correct in these frames.

Let us now consider the more interesting case where S' accelerates uniformly relative to S, with constant linear acceleration $\mathbf{a}_h$. Rearranging Equation (11–5), we have

$$m\mathbf{a} - m\mathbf{a}_h = m\mathbf{a}' \tag{11–7}$$

Since $\Sigma\mathbf{F}_{\text{real}} = m\mathbf{a}$, we may write:

$$\Sigma\mathbf{F}_{\text{real}} - m\mathbf{a}_h = m\mathbf{a}' \tag{11–8}$$

Here lies a crucial step in the derivation. In the S' frame, the mass m is observed to have the acceleration $\mathbf{a}'$. In terms of Newtonian mechanics, an object accelerates because a net force acts on it. Newton's way of thinking is deeply ingrained in our common sense, and in order to retain the form of Newton's law, we consider the quantity $\mathbf{F} - m\mathbf{a}_h$ as the net force $\Sigma\mathbf{F}'$ acting on m to give it the acceleration $\mathbf{a}'$ as observed in S'. This net force $\Sigma\mathbf{F}'$ is made up of two parts: a real force $\Sigma\mathbf{F}_{\text{real}}$ and an inertial force $(-m\mathbf{a}_h)$. Thus, in the accelerated frame

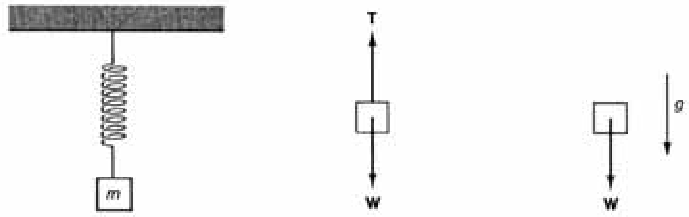


Figure 11–2
These analyses are in an inertial frame of reference at rest in the presence of a gravitational field **g**.

(**a**) A mass on a spring.

(**b**) A free-body diagram for a mass hanging on a spring.

(**c**) A freely falling mass has downward acceleration equal to *g*.

we write Newton's second law as follows:

NEWTON'S SECOND LAW (in a frame with linear acceleration $\mathbf{a}_h$)

$$\overbrace{\underbrace{\Sigma\mathbf{F}_{\text{real}}}_{\substack{\text{Sum of all the}\\ \text{real forces}}} + \underbrace{(-m\mathbf{a}_h)}_{\substack{\text{Inertial}\\ \text{force}}}}^{\Sigma\mathbf{F}'} = m\mathbf{a}' \qquad (11\text{–}9)$$

This is not merely a mathematical trick. As you will see, observers in the accelerated frame can experimentally verify the presence of the force $(-m\mathbf{a}_h)$ just as they can verify the presence of a gravitational force.

We need to make a special comment regarding signs. In equations, we will always associate the minus sign with this inertial force: $\mathbf{F}_{\text{inertial}} = (-m\mathbf{a}_h)$. Note that this is *opposite* to the direction of the acceleration of the frame $\mathbf{a}_h$. This procedure is consistent with our interpretation of the net force on an object as the *sum* of all the forces acting on it. Thus, in the accelerated frame the net force $\Sigma\mathbf{F}'$ is the *sum* of the real plus the inertial force:

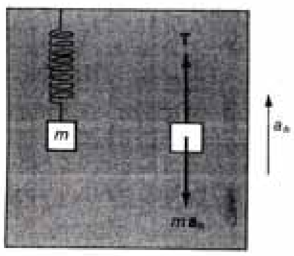
$$\Sigma\mathbf{F}' = \Sigma\mathbf{F}_{\text{real}} + \mathbf{F}_{\text{inertial}}$$

where $\mathbf{F}_{\text{inertial}} = -m\mathbf{a}_h$.

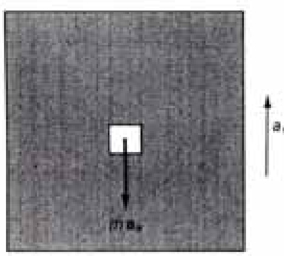
A note of caution: In drawing vector diagrams, we follow the custom of never using a minus sign with the symbol on the arrow. The diagram itself displays the proper directions for the forces, independent of the particular direction designated "minus" by the choice of coordinate system. Thus, having determined the correct direction of the inertial force to be opposite to $\mathbf{a}_h$, in sketching the vector we label it simply with its magnitude ma_h. We use a minus sign with a vector arrow only when it specifically represents the negative of a vector.

We cannot trace the origin of the term $(-m\mathbf{a}_h)$ to any other physical system or object. In this sense it is "fictitious." But to observers in the S' frame, it is really "there." Let us discuss this point a bit further. Consider how we detect the force of gravity in an inertial frame of reference. If we hang a mass m on the end of a spring [see Figure 11–2(a)], the spring elongates to exert an upward spring force **T**, exactly balancing the downward force of gravity **W**. In fact, we infer the downward force precisely because the spring elongates. When we combine **T** with the inferred force **W**, there is zero net force on m, and the mass remains at rest. If the spring breaks, the mass accelerates downward, which provides further evidence of a gravitational force. Applying $\Sigma\mathbf{F} = m\mathbf{a}$, we say that the single gravitational force **W** produces the downward acceleration **g**: $\mathbf{W} = m\mathbf{g}$.

Suppose we perform the same experiment in a gravity-free region but in a frame of reference S' which has linear acceleration $\mathbf{a}_h$ as shown in Figure 11–3. Observers in that frame note an elongation of the spring, signifying an upward spring force on m. Since the mass is at rest in this frame, however, they infer



(**a**) Observers in the accelerated box believe there are two forces acting on m to produce equilibrium conditions. Thus the observers explain why m remains at rest in their frame of reference.



(**b**) If the spring breaks, observers see the mass m accelerate toward the floor of the box in response to the unbalanced force $m\mathbf{a}_h$ downward.

Figure 11–3
These analyses are in a frame of reference attached to the box, which has upward acceleration a_h in a region where there is no gravitational field. It is therefore an accelerated frame of reference

the presence of a second "downward" force $-m\mathbf{a}_h$ on the mass to make it remain at rest in S'. Furthermore, if the spring breaks, they observe the mass to move "downward" with an acceleration $-\mathbf{a}_h$ in response to the unbalanced force $-m\mathbf{a}_h$.

Thus, observers in the accelerated frame S' infer the presence of the inertial force $-m\mathbf{a}_h$ in exactly the same way we infer the force of gravity, and the inertial force is just as real to them as the force of gravity is to us. There are some novel features, however. Though we assign the cause of gravity to be the presence of a nearby mass (the earth), the origin of the inertial force cannot be traced to any other object or system. Since no *agent* exerts it, it does not obey Newton's third law. *The inertial force is a consequence of analyzing the situation from an accelerated frame of reference and represents a desire to preserve the form of Newton's second law.* We cling strongly to the idea that an object accelerates because there is an unbalanced force acting on it.

Is gravity an inertial force? It has much in common with inertial forces. For example, in a stationary frame in the presence of gravity, all freely falling masses have the same acceleration g. Similarly, in an accelerated frame (with no gravity), all free masses undergo the acceleration $-\mathbf{a}_h$. So, like gravity, the inertial force is proportional to the mass m. Furthermore, just as we may consider all gravitational forces to be acting on a single point (the center of mass of an object), inertial forces may be analyzed as if they acted on the center of mass of the object.

To pursue this connection further, hold your arm out horizontally. Can you sense the downward force of gravity acting on your arm? Most people will answer, "Yes, of course I can feel the force of gravity pulling my arm downward." But think again. You actually do *not* have any sensation of a gravitational force acting downward on each atom of your arm. You "feel" only the muscular tension you must exert to apply an *upward* force on your arm to hold it in equilibrium. *The downward force of gravity itself creates no physical sensation at all in your arm.* You are aware only of forces you exert to *oppose* the force of gravity.

Another example may emphasize the fact that we do not experience directly the force of gravity. An astronaut circling the earth in a satellite will note that any object left to itself will float freely inside the space capsule. Indeed, if not strapped down, he himself will float. The astronaut feels no net force on himself, yet an appreciable gravitational force is acting on him. (For a space station 300 mi above the earth's surface, the force of gravity is about 86% of its value at the earth's surface.) Surprising as it may seem, we cannot "feel" the force of gravity.

□ □ □

Let us consider a few examples of motion as observed in a linearly accelerated frame of reference. To clarify the analysis, we shall present parallel treatments as seen in the S and S' frames. In all the examples of this chapter, we will assume that observers in S and S' know which way is "up" and that both use the same gravitational force W on objects. The only difference between the two descriptions that the observers give for the same occurrence is the presence or absence of inertial forces.

EXAMPLE 11–1

Consider a mass m suspended by a string from the ceiling of a railroad car. The car has constant acceleration as shown in Figure 11–4, causing the mass to hang at a steady angle θ with the vertical. Find the angle θ in terms of the other symbols given.

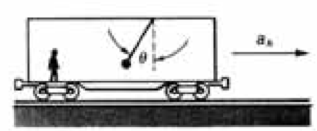


Figure 11–4
Example 11–1