Unified Propulsion

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UNIFIED PROPULSION: Learning Objectives:

Given the basic geometry and idealized component performance, to be able to estimate the thrust and specific impulse of a gas turbine and a rocket engine from fluid and thermodynamic principles

Measurable outcomes (assessment method):

A. To be able to explain at a level understandable by a high school senior or non-technical person what the various terms are in the integral momentum equation and how jet propulsion works. (quiz, self-assessment)
B. To be able to apply control volume analysis and the integral momentum equation to estimate the forces produced by aerospace propulsion systems (homework, quiz, self-assessment)
C. To be able to describe the principal figures of merit for aircraft engine and rocket motor performance and explain how they are related to vehicle performance. (quiz, self-assessment)
D. Given weight, geometry, and aerodynamic and propulsion system performance information, to be able to estimate the power required for flight, the range, the endurance, and the time-to-climb for an aircraft. (homework, quiz, self-assessment)
E. To be able to explain at a level understandable by a high school senior or non-technical person the arrangement and layout of the major components of gas turbine and rocket engines. (quiz, self-assessment)
F. Given mass fractions, and propulsion system performance information, to be able to estimate the range and velocity of single-stage rockets. (homework and quiz, self-assessment)
G. To be able to describe the principal design parameters and constraints that set the performance of gas turbine engines. (quiz, self-assessment)
H. To be able to apply ideal-cycle analysis to a gas turbine engine to relate thrust and fuel burn to component-level performance parameters and flight conditions. (homework, self-assessment)
I. To be able to explain at a level understandable by a high school senior or non-technical person the energy exchange processes that underlie the workings of multistage compressor or turbine. (quiz, self-assessment)
J. To be able to use velocity triangles and the Euler Turbine Equation to estimate the performance of a compressor or turbine stage. (homework, quiz, self-assessment)

I. Introduction to Propulsion

A. Goal: Create a Force to Propel a Vehicle

Two options:

1. Take mass stored in a vehicle and throw it backwards (*rocket propulsion*).
   Use the reaction force to propel the vehicle.

Propellant	⟶	burn	⟶	expand through nozzle
(chem. energy)		(thermal energy)		(kinetic energy & momentum)

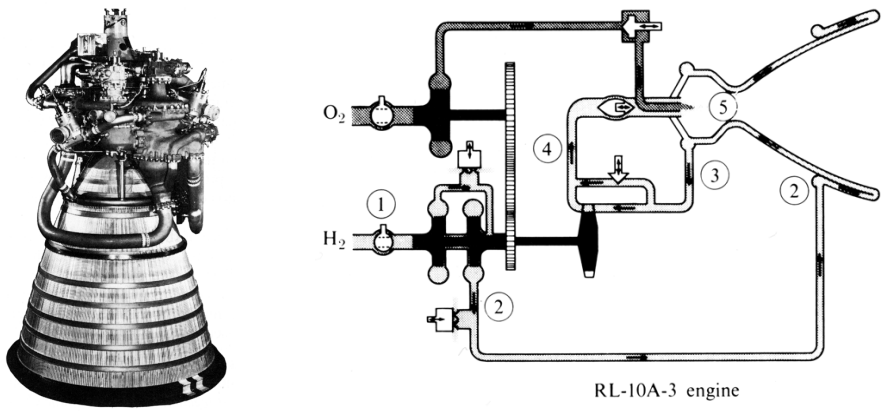


Figure 1.1 Typical liquid propellant rocket motor (Hill and Peterson, 1992).

2. Seize mass from the surroundings and set the mass in motion backwards.
   Use the reaction force to propel vehicle (*air-breathing propulsion*).

Continuously: a) Draw in air.
b) Compress it.
c) Add fuel and burn (convert chemical to thermal energy).
d) Expand through a turbine to drive the compressor (extract work).
e.1) Then expand in a nozzle to convert thermal energy to kinetic energy & momentum (turbojet).
e.2) Or expand in a second turbine (extract work), use this to drive a shaft for a fan (turbofan), or a propeller (turboshaft). The fan or propeller impart k.e. & mom. to the air.

*Remember:

Overall goal: take $\dot{m}$ at V_o (flight speed), throw it out at $V_o + \Delta V$

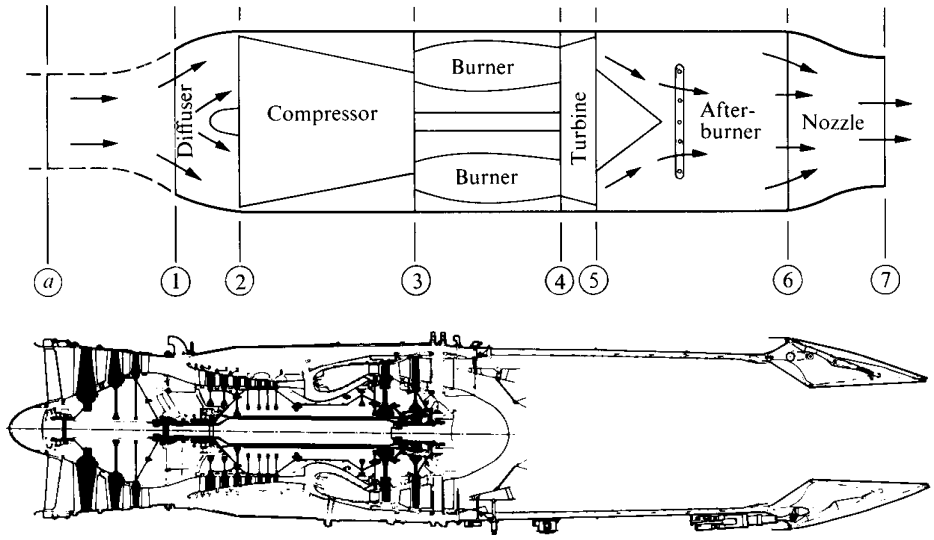


Figure 1.2 Schematics of typical military gas turbine engines. Top: turbojet with afterburning, bottom: GE F404 low bypass ratio turbofan with afterburning (Hill and Peterson, 1992).

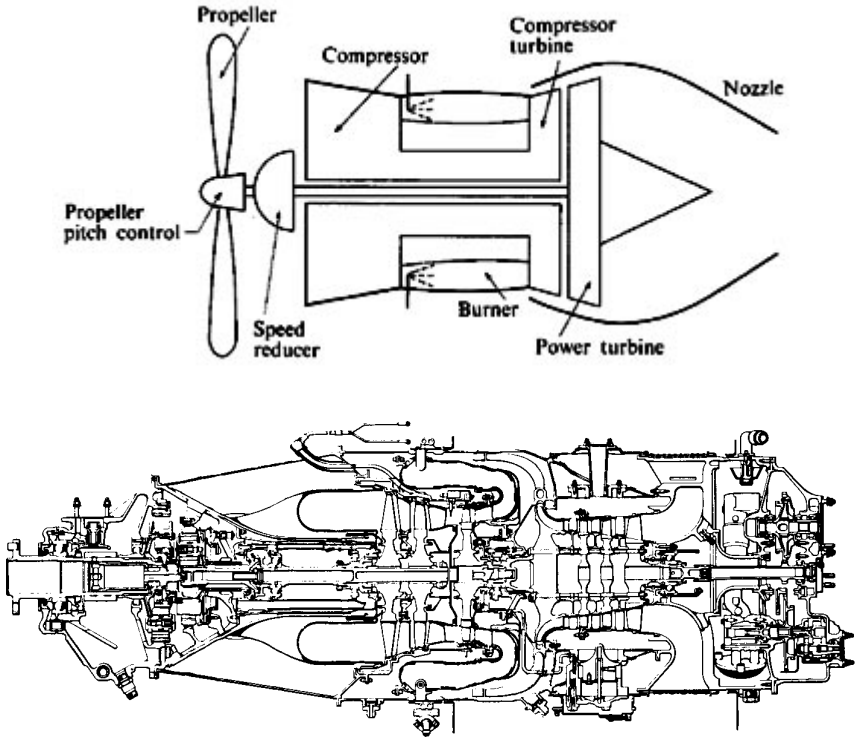
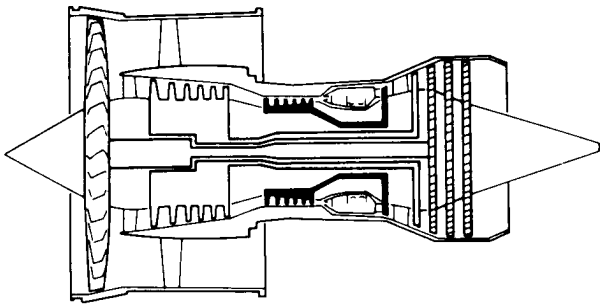


Figure 1.3 Schematics of a PW PT6A-65, a typical turboprop (Hill and Peterson, 1992).



(b) Triple-spool front fan turbojet (High bypass ratio)

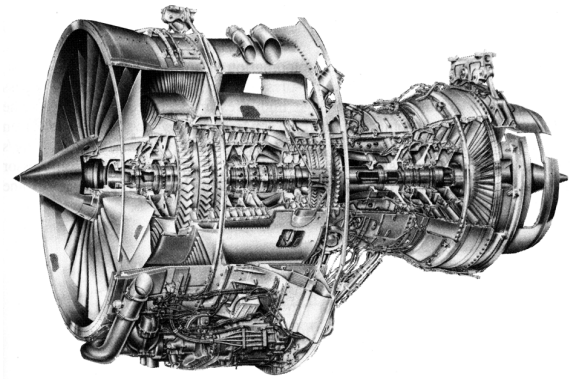


Figure 1.4 The RB211-535E4, a typical high bypass-ratio turbofan (Hill and Peterson, 1992).

B. Performance parameters

The two performance parameters of greatest interest for a propulsion system are the force it produces (*thrust, T*), and the overall efficiency with which it uses energy to produce this force ($\eta_{overall}$). We will begin by looking at the production of thrust using the integral form of the momentum theorem. In the second lecture we will discuss the efficiency of propulsion systems.

C. Propulsion is a systems endeavor

There are a multitude of other factors which a propulsion engineer must take into account when designing a device including weight, cost, manufacturability, safety, environmental effects, etc. Thus propulsion is truly a systems endeavor, requiring knowledge of a variety of disciplines:

$$\underbrace{\text{Fluids} + \text{thermo}} + \text{structures} + \text{dynamics} + \text{controls} + \text{chemistry} + \text{acoustics} + \ldots$$

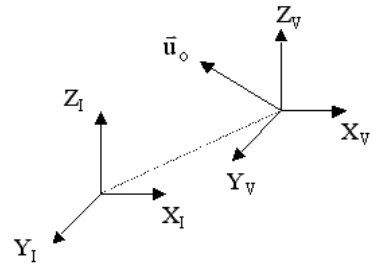
We will focus mostly on these two disciplines in the Unified propulsion lectures.

II. Integral Momentum Theorem

We can learn a great deal about the overall behavior of propulsion systems using the integral form of the momentum equation. The equation is the same as that used in fluid mechanics.

A. An Expression of Newton's 2nd Law (*e.g.* ΣF = d/dt (mv))

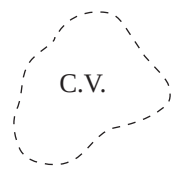
1. Consider two coordinate systems:



   a) Inertial

   b) Fixed to vehicle
      - Moves with velocity $\vec{u}_o$ relative to the inertial coordinate system
      - All velocities relative to the vehicle-fixed coordinate frame are denoted $\vec{u}$

2. Newton's second law for a control volume of fixed mass



$$\Sigma \vec{F} = \int_V \rho \frac{D(\vec{u}_o + \vec{u})}{Dt} dV$$

or

$$\underbrace{\sum \vec{F}}_{\substack{\text{All external}\\ \text{forces on}\\ \text{c.v.}}} = \underbrace{\vec{a}_o \int_V \rho dV}_{\substack{\text{Force due to}\\ \text{change in inertia}\\ \text{for accel. vehicle}\\ \equiv \vec{F}_o}} + \underbrace{\int_V \rho \frac{D\vec{u}}{Dt} dV}_{\substack{\text{Change in momentum}\\ \text{of mass in c.v.}}}$$

- Pressure forces
- Shear forces
- Body forces

To explain the above equation further, consider Figure 2.1

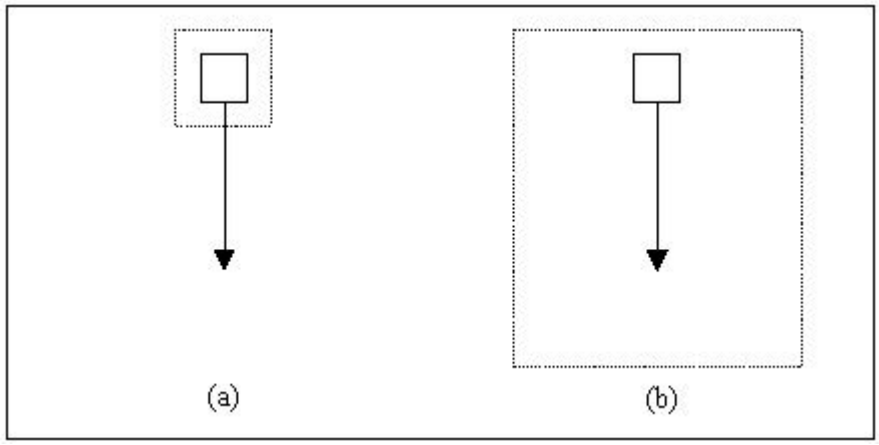


Figure 2.1 Falling blocks.

The falling block labeled (a) has a control volume fixed to it. In this case, the first term of the above equation is nonzero since the control volume is accelerating relative to an inertial reference frame. The second term is zero because the block is not accelerating relative to a coordinate system fixed to the control volume. The opposite is true for the falling block labeled (b), which is falling within a fixed control volume. The first term of the above equation is zero in this case because the control volume is not accelerating relative to an inertial reference frame. The second term is nonzero because the block is moving to a coordinate system fixed to the control volume. The mathematical result of both cases is as follows,

$$\sum \vec{F} = \vec{a}_o \int_V \rho dV + \int_V \rho \frac{D\vec{u}}{Dt} dV$$

(a) $$= m\vec{a}_o + 0$$

$$= m\vec{a}$$

$$\sum \vec{F} = \vec{a}_o \int_V \rho dV + \int_V \rho \frac{D\vec{u}}{Dt} dV$$

(b) $$= 0 + m\frac{D\vec{u}}{Dt}$$

$$= m\vec{a}$$

As expected, the result is the same for both. The integral momentum equation reduces to a familiar form, $\vec{F} = m\vec{a}$.

To continue, the integral momentum equation can be rewritten as follows,

$$\sum \vec{F} - \vec{F}_o = \int_V \rho \left[\frac{\partial \vec{u}}{\partial t} + \vec{u}\vec{\nabla}\vec{u} \right] dV$$

$$= \int_V \left[\rho \frac{\partial \vec{u}}{\partial t} + \vec{u}\frac{\partial \rho}{\partial t} - \vec{u}\frac{\partial \rho}{\partial t} + \rho \vec{u} \cdot \vec{\nabla}\vec{u} \right] dV$$

$$= \int_V \left[\frac{\partial (\rho \vec{u})}{\partial t} - \vec{u}\frac{\partial \rho}{\partial t} + \rho \vec{u} \cdot \vec{\nabla}\vec{u} \right] dV$$

From conservation of mass,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{u}) = 0 \qquad \text{or} \qquad \vec{u}\frac{\partial \rho}{\partial t} + \vec{u}\vec{\nabla} \cdot (\rho \vec{u}) = 0$$

so

$$\sum \vec{F} - \vec{F}_o = \int_V \left[\frac{\partial (\rho \vec{u})}{\partial t} + \vec{u}\vec{\nabla} \cdot (\rho \vec{u}) + \rho \vec{u} \cdot \vec{\nabla}\vec{u} \right] dV$$

<u>NOTE: This is a *vector* equation.</u>

3. Considering only the components in the x-direction

$$\sum F_x - F_{o_x} = \int_V \left[\frac{\partial (\rho u_x)}{dt} + u_x \vec{\nabla} \cdot (\rho \vec{u}) + \rho \vec{u} \cdot \vec{\nabla} u_x \right] dV$$

$$\int_V \left[\frac{\partial (\rho u_x)}{dt} + \left(\rho u_x \frac{\partial (u_x)}{\partial x} i + \rho u_x \frac{\partial (u_y)}{\partial y} j + \rho u_x \frac{\partial (u_z)}{\partial z} k \right) + \left(\rho u_x \frac{\partial (u_x)}{\partial x} i + \rho u_y \frac{\partial (u_x)}{\partial y} j + \rho u_z \frac{\partial (u_x)}{\partial z} k \right) \right] dV$$

$$\int_V \left[\frac{\partial (\rho u_x)}{dt} + \left(\left(\rho u_x \frac{\partial (u_x)}{\partial x} + \rho u_x \frac{\partial (u_x)}{\partial x} \right) i + \left(\rho u_x \frac{\partial (u_y)}{\partial y} + \rho u_y \frac{\partial (u_x)}{\partial y} \right) j + \left(\rho u_x \frac{\partial (u_z)}{\partial z} + \rho u_z \frac{\partial (u_x)}{\partial z} \right) k \right) \right] dV$$

$$= \int_V \left[\frac{\partial (\rho u_x)}{dt} + \vec{\nabla} \cdot (\rho u_x \vec{u}) \right] dV$$

Then, by the divergence theorem,

$$\int_V \vec{\nabla} \cdot \vec{A} dV = \int_S \vec{n} \cdot \vec{A} ds$$ where $\vec{n}$ is outward unit normal vector.

so

$$\boxed{\underbrace{\sum F_x - F_{o_x}}_{\substack{\text{sum of forces}\\ \text{on C.V.}}} = \underbrace{\int_V \left[\frac{\partial(\rho u_x)}{\partial t}\right] dV}_{\substack{\text{change in momentum of}\\ \text{mass contained in C.V.}\\ \text{over time}}} + \underbrace{\int_S u_x(\rho\vec{u}) \cdot \vec{n} ds}_{\substack{\text{change in momentum}\\ \text{flux across surface of C.V.}}}}$$

<u>where the forces acting on the control volume may be composed of **pressure forces**, **body forces**, and **skin friction**</u>

4. For steady flow, with no acceleration of the vehicle then

$$\boxed{\sum F_x = \int_S u_x \rho \vec{u} \cdot \vec{n} ds}$$

This is the form we will use most frequently in this class.

B. Application of the Integral Momentum Equation to Rockets

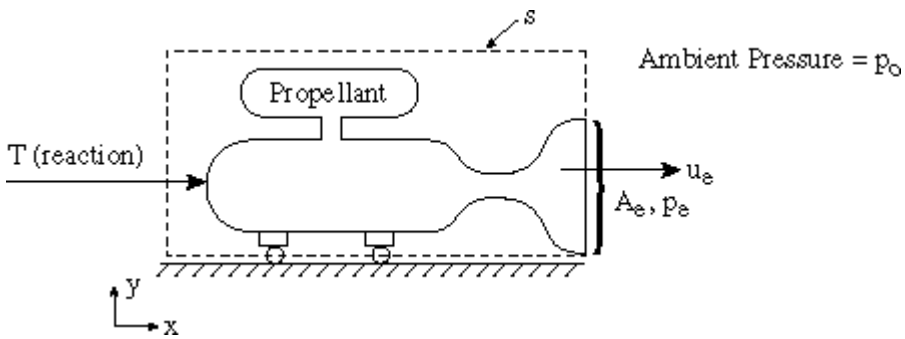


Figure 2.2 Control volume for application of momentum theorem to a rocket.

$$\sum F_x = T + A_e(p_o - p_e) \ , \quad \vec{u}\cdot\vec{n} = 0 \ \text{ everywhere except } A_e$$

$$T + A_e(p_o - p_e) = \rho u_e u_e A_e = \dot{m} u_e$$

$$\boxed{T = \dot{m} u_e + A_e(p_e - p_o)}$$ Static thrust for a rocket engine

C. Application of the Momentum Equation to an Aircraft Engine

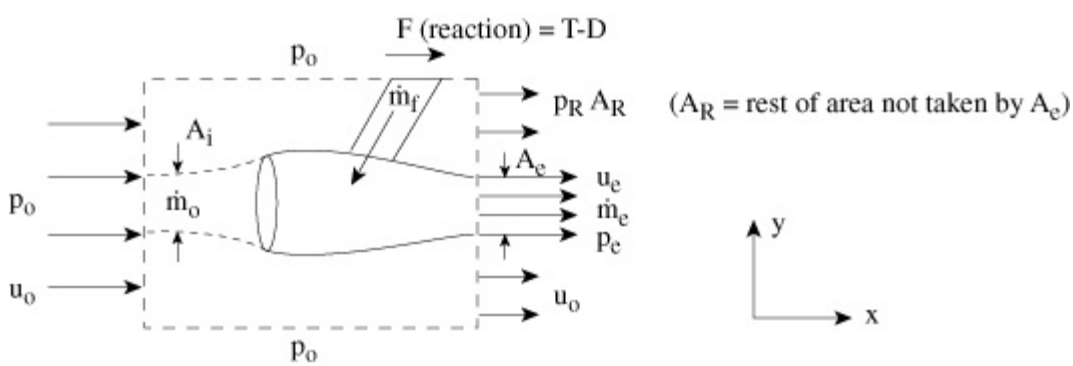


Figure 2.3 Control volume for application of the momentum theorem to a gas turbine engine.

$$\int_s \rho u_x (\vec{u}\cdot\vec{n})dA = T - D + \sum \text{PressureForces}$$

$$\int_s \rho u_x (\vec{u}\cdot\vec{n})dA = T - D + \left[-P_e A_e + P_o A_e - \int_{A_R} (P_R - P_o) dA_R \right]$$

$$\int_s \rho u_x (\vec{u}\cdot\vec{n})dA = \underbrace{\rho_e u_e A_e}_{\dot{m}_e} u_e - \underbrace{\rho_o u_o A_o}_{\dot{m}_o} u_o + \int_{C_s - A_o - A_e} \rho u_x \vec{u}\cdot\vec{n} dA$$

$$= \dot{m}_e u_e - \dot{m}_o u_o + \int_{C_s - A_o - A_e} \rho u_x \vec{u}\cdot\vec{n} dA$$

So we have:

$$T - D - (P_e - P_o)A_e - \int_{A_R} (P_R - P_o) dA_R = \dot{m}_e u_e - \dot{m}_o u_o + \int_{C_s - A_e - A_o} \rho u_x \vec{u}\cdot\vec{n} dA$$

Everything that relates to flow through the engine is conventionally called <u>thrust</u>. Everything that relates to the flow on the outside of the engine is conventionally call <u>drag</u>. Therefore, gathering only those terms that relate to the fluid that passes through the engine, we have:

$$\boxed{T = \dot{m}_e u_e - \dot{m}_o u_o + (P_e - P_o)A_e}$$

The thrust is largely composed of the net change in momentum of the air entering and leaving the engine, with a typically small adjustment for the differences in pressure between the inlet and the exit. We could have arrived at the same equation by considering only the streamtube that passes through the engine as shown below:

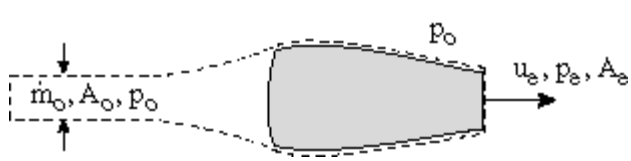


Figure 2.4 Control volume for flow through engine.

III. Efficiencies of A/C Engines

In the first lecture we arrived at general expressions that related the thrust of a propulsion system to the net changes in momentum, pressure forces, etc. Now we will look at how efficiently the propulsion system converts one form of energy to another on its way to producing thrust.

A. Overall Efficiency

$$\textit{overall efficiency} = \frac{\text{what you get}}{\text{what you pay for}} = \frac{\text{propulsive power}}{\text{fuel power}}$$

$$\textit{propulsive power} = \text{thrust} \cdot \text{flight velocity} = Tu_0$$

$$\textit{fuel power} = \text{fuel mass flow rate} \cdot \text{fuel energy per unit mass} = \dot{m}_f h$$

Thus

$$\boxed{\eta_{\text{overall}} = \frac{Tu_0}{\dot{m}_f h}}$$

B. Thermal and Propulsive Efficiency

It is often convenient to break the overall efficiency into two parts: thermal efficiency and propulsive efficiency where

$$\eta_{\text{thermal}} = \frac{\text{rate of production of kinetic energy}}{\text{fuel power}} = \frac{\left(\frac{\dot{m}_e u_e^2}{2} - \frac{\dot{m}_o u_o^2}{2}\right)}{\dot{m}_f h}$$

$$\eta_{\text{prop}} = \frac{\text{propulsive power}}{\text{rate of production of propulsive kinetic energy}} = \frac{Tu_0}{\left(\frac{\dot{m}_e u_e^2}{2} - \frac{\dot{m}_o u_o^2}{2}\right)}$$

such that

$$\eta_{\text{overall}} = \eta_{\text{thermal}} \cdot \eta_{\text{prop}}$$

The thermal efficiency in this expression is the same as that which we used extensively in Thermodynamics during Fall term. For an ideal Brayton cycle it is a function of the temperature ratio across the compressor.

$$\eta_{\text{th-ideal Brayton cycle}} = \frac{W_{net}}{Q_{in}} = 1 - \frac{T_1}{T_2}$$

Note that we can use our expression for thrust to rewrite the equation for propulsive efficiency in a more convenient form

$$T \approx m(u_e - u_o) \qquad (\text{since } \dot{m}_e \approx \dot{m}_o)$$

Then

$$\eta_p = \frac{\dot{m}u_o(u_e - u_o)}{\frac{\dot{m}}{2}\left(u_e^2 - u_o^2\right)} = \frac{2u_o}{u_o + u_e} = \frac{2}{1 + \frac{u_e}{u_o}}$$

C. Implications of propulsive efficiency for engine design

If we consider our expressions for thrust and propulsive efficiency together

$$F = \dot{m}(u_e - u_o) \qquad \text{and} \qquad \eta_{prop} = \frac{2}{1 + \frac{u_e}{u_o}}$$

we see that

as $\frac{u_e}{u_o} \uparrow$ $\frac{F}{\dot{m}} \uparrow$ but $\eta_{prop} \downarrow$

and as $\frac{u_e}{u_o} \to 1$ $\frac{F}{\dot{m}} \downarrow$ and $\eta_{prop} \uparrow$

Also note that for $\frac{F}{\dot{m}u_o} \uparrow$ $A_{inlet} \downarrow$ $Drag \downarrow$

The balance between propulsive efficiency and specific thrust (~ thrust per unit mass flow) is shown in Figure 3.1.

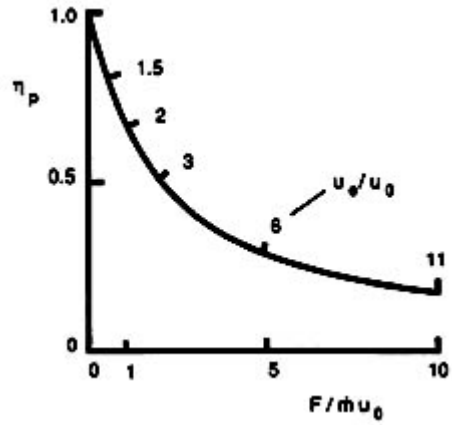


Figure 3.1 Propulsive efficiency and specific thrust as a function of exhaust velocity (Kerrebrock, 1991).

For fighter aircraft that need high thrust/weight and fly at high speed, it is typical to employ engines with smaller inlet areas and higher thrust per unit mass flow

$$\therefore \frac{u_e}{u_o} \uparrow \qquad \text{and} \qquad \eta_{prop} \downarrow$$

Figure 3.2 The F-22 Raptor (Copyright 1999 by Lockheed Martin).

However, transport aircraft that require higher efficiency and fly at lower speeds usually employ engines with relatively larger inlet areas and lower thrust per unit mass flow

$$\therefore \frac{u_e}{u_o} \rightarrow 1 \qquad \text{and} \qquad \eta_{prop} \uparrow$$

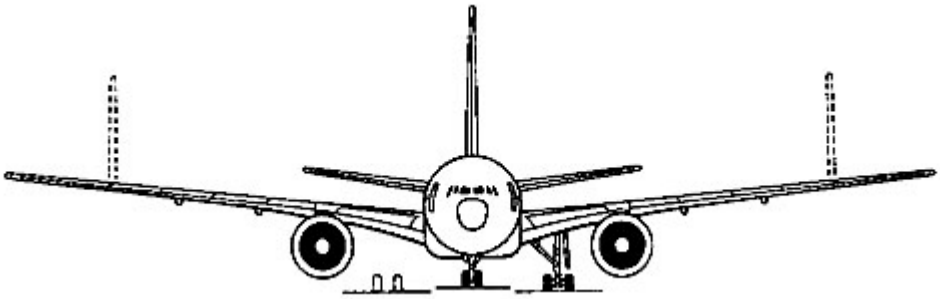


Figure 3.3 The Boeing 777-200 (Janes, 1998-9).

At low flight velocities, the highest propulsive efficiency is typically obtained with a propeller or an unducted fan.

Figure 3.4 A propeller gives a relatively small impulse (Δu) to a relatively large mass flow (Boeing, 2000)

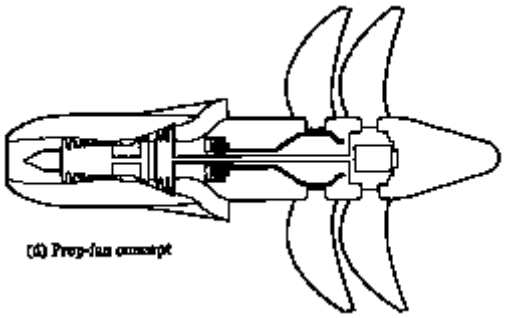


Figure 3.5 An advanced, contour-rotating, unducted fan concept (Rolls-Royce, 1992).

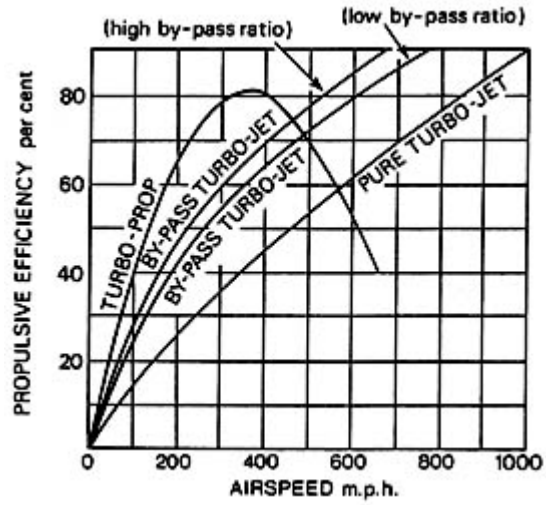


Figure 3.6 Propulsive efficiency comparison for various gas turbine engine configurations (Rolls-Royce, 1992).

D. Other expressions for efficiency

Sometimes the overall efficiency of aircraft engines is expressed in alternative parameters: specific impulse, I, and thrust specific fuel consumption, TSFC or just SFC. Both of these parameters have <u>dimensions</u>.

Specific Impulse (I or I_{sp}):

$$I = \frac{\text{thrust}}{\text{fuel weight flow rate}} \qquad \text{(units of seconds)}$$

Specific Fuel Consumption (SFC or TSFC):

$$SFC = \frac{\text{mass flow rate of fuel}}{\text{thrust}}$$

(lbm/hr/lbf or kg/s/N)

E. Trends in thermal and propulsive efficiency

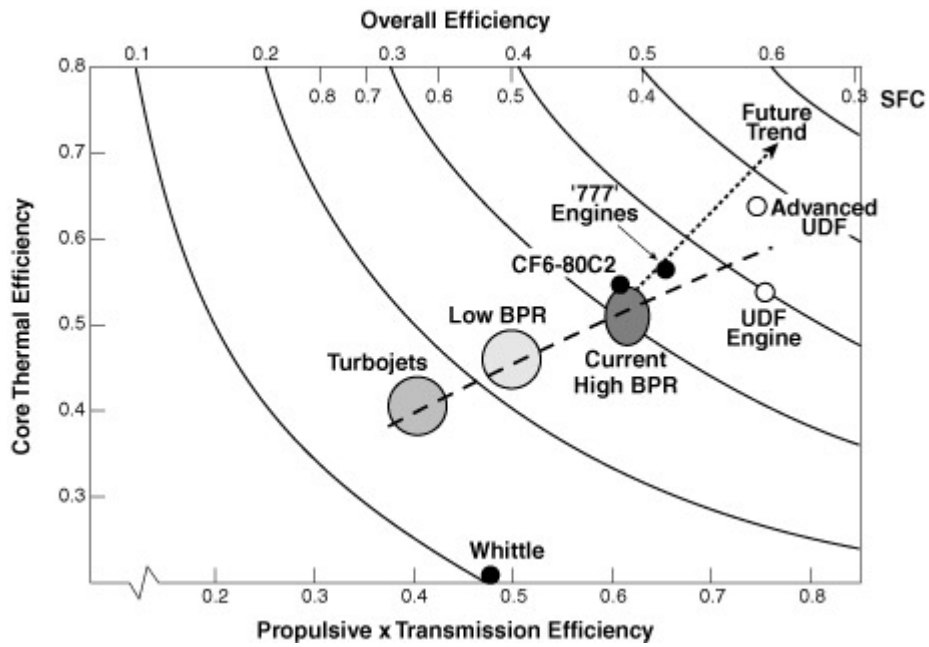


Figure 3.7 Trends in aircraft engine efficiency (after Pratt & Whitney).

Trends in *thermal efficiency* are driven by increasing compression ratios and corresponding increases in turbine inlet temperature as shown in Figures 3.8 and 3.9. Whereas trends in propulsive efficiency are due to generally higher bypass ratio engines.

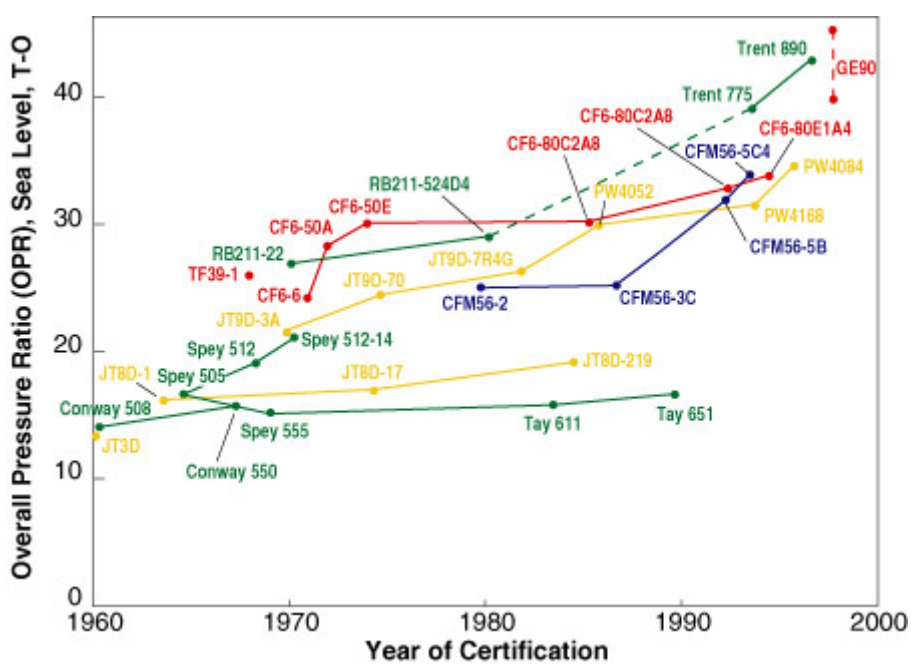


Figure 3.8 Pressure ratio trends for commercial transport engines (Epstein, 1998).

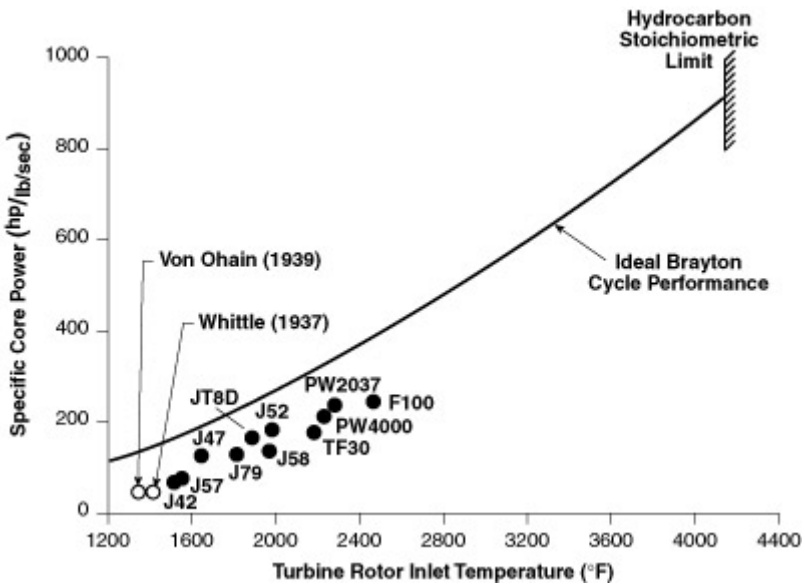


Figure 3.9 Trends in turbine inlet temperature (Koff, 1991).

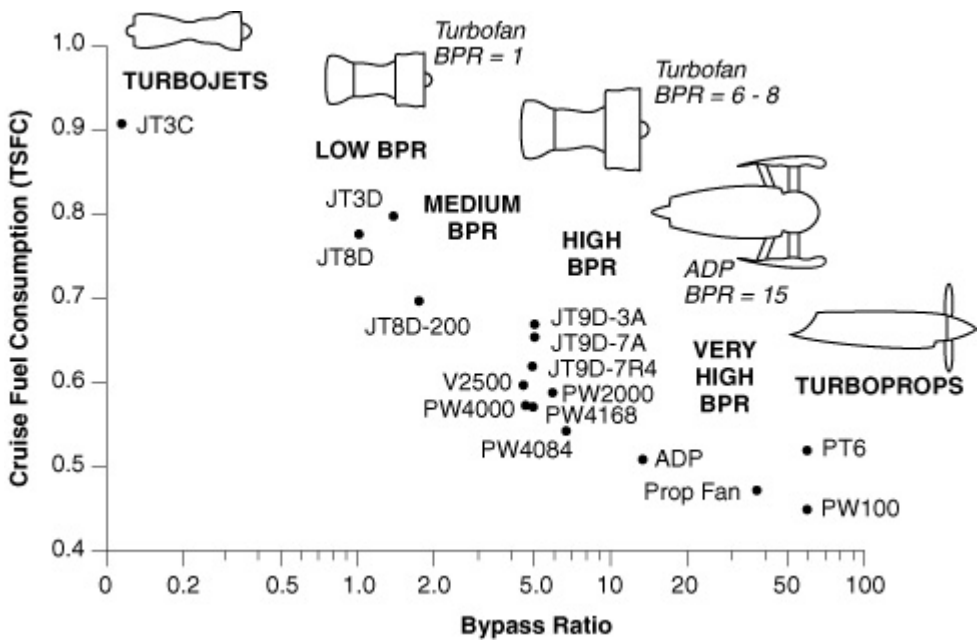


Figure 3.10 Trends in engine bypass ratio (Epstein, 1998).

IV. Aircraft Performance

In this lecture we will make the connections between aircraft performance and propulsion system performance.

For a vehicle in steady, level flight, the thrust force is equal to the drag force, and lift is equal to weight. Any thrust available in excess of that required to overcome the drag can be applied to accelerate the vehicle (increasing kinetic energy) or to cause the vehicle to climb (increasing potential energy).

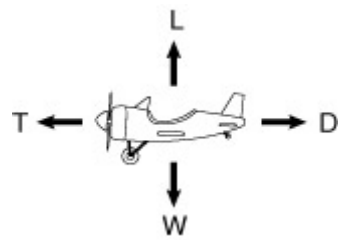


Figure 4.1 Force balance for aircraft in steady level flight.

$$T = D, \quad L = W \qquad\qquad W = L = D\frac{L}{D} = T\left(\frac{L}{D}\right)$$

A. Vehicle Drag

Recall from fluids that drag takes the form shown below, being composed of a part termed *parasitic drag* that increases with the square of the flight velocity, and a part called *induced drag*, or drag due to lift, that decreases in proportion to the inverse of the flight velocity.

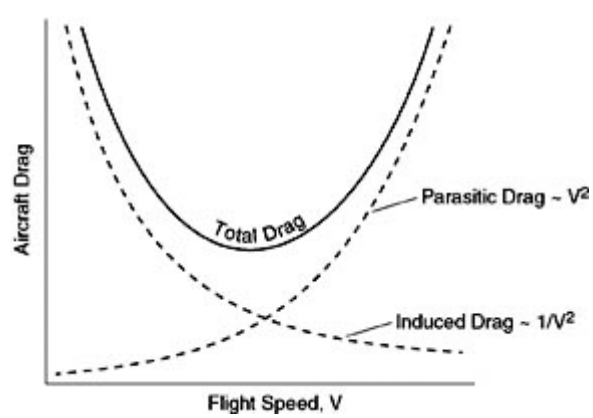


Figure 4.2 Components of vehicle drag.

$$C_D = C_{D_o} + \frac{C_L{}^2}{\pi e AR} \qquad \text{where} \quad L = \frac{1}{2}\rho V^2 S C_L \quad \text{and } D = \frac{1}{2}\rho V^2 S C_D$$

Thus

$$D = \frac{1}{2}\rho V^2 S C_{D_o} + \frac{L^2}{\frac{1}{2}\rho V^2 S}\left(\frac{1}{\pi e AR}\right) \qquad \text{or}$$

$$D = \frac{1}{2}\rho V^2 S C_{D_o} + \frac{W^2}{\frac{1}{2}\rho V^2 S}\left(\frac{1}{\pi e AR}\right)$$

The **minimum drag** is a condition of interest. We can see that for a given weight, it occurs at the condition of maximum lift-to-drag ratio

$$D = L\frac{D}{L} = W\left(\frac{D}{L}\right) = W\left(\frac{C_D}{C_L}\right)$$

We can find a relationship for the maximum lift-to-drag ratio by setting

$$\frac{d}{dC_L}\left(\frac{C_{D_o} + \frac{C_L{}^2}{\pi e AR}}{C_L}\right) = 0$$

from which we find that

$$C_{L\,\text{min drag}} = \sqrt{\pi e AR C_{D_o}} \qquad \text{and} \qquad C_{D\,\text{min drag}} = 2C_{D_o}$$

$$\left(\frac{C_L}{C_D}\right)_{max} = \frac{1}{2}\sqrt{\frac{\pi e AR}{C_{D_o}}} \qquad \text{and}$$

$$V_{\text{min drag}} = \sqrt{\frac{W}{\frac{1}{2}\rho S C_{L\,\text{min drag}}}} = \left[4\left(\frac{W}{S}\right)^2 \frac{1}{\rho^2}\frac{1}{C_{D_o}}\left(\frac{1}{\pi e AR}\right)\right]^{1/4}$$

B. Power Required

Now we can look at the propulsion system requirements to maintain steady level flight since

$$T_{req} = D \qquad \text{and} \qquad P_{req} = T_{req} V = DV$$

$$P_{req} = \frac{1}{2}\rho V^3 S C_{D_o} + \frac{W^2}{\frac{1}{2}\rho V S}\left(\frac{1}{\pi e AR}\right)$$

Thus the power required (for steady level flight) takes the form

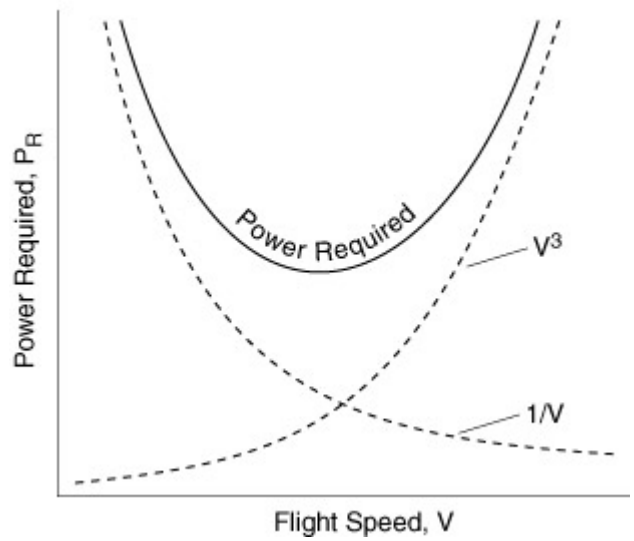


Figure 4.3 Typical power required curve for an aircraft.

The velocity for minimum power is obtained by taking the derivative of the equation for P_{req} with respect to V and setting it equal to zero.

$$V_{\text{minimum power}} = \left[\frac{4}{3}\left(\frac{W}{S}\right)^2 \frac{1}{\rho^2}\frac{1}{C_{D_o}}\left(\frac{1}{\pi e AR}\right)\right]^{1/4}$$

As we will see shortly, maximum *endurance* (time aloft) occurs when the minimum power is used to maintain steady level flight. Maximum *range* (distance traveled) is obtained when the aircraft is flown at the most aerodynamically efficient condition (maximum C_L/C_D).

C. Aircraft Range, the **Breguet Range Equation**

Again, for steady, level flight,

$$T = D, \quad L = W \qquad \text{or} \qquad W = L = D\frac{L}{D} = T\left(\frac{L}{D}\right)$$

The weight of the aircraft changes in response to the fuel burned

$$\frac{dW}{dt} = \dot{m}_f \cdot g = \frac{-W}{\left(\frac{L}{D}\right) Isp}$$

or

$$\frac{dW}{W} = \frac{-dt}{\left(\frac{L}{D}\right) Isp} \qquad \Rightarrow \qquad \ell n W = \text{constant} - \frac{t}{\left(\frac{L}{D}\right) Isp}$$

applying the initial conditions, at $t = 0$ $W = W_{initial}$ $\therefore$ const. $= \ell n\, W_{initial}$

$$\therefore t = \frac{-L}{D}\, Isp\, \ell n \frac{W}{W_{initial}}$$

the time the aircraft has flown corresponds to the amount of fuel burned, therefore

$$t_{final} = \frac{-L}{D}\, Isp\, \ell n \frac{W_{final}}{W_{initial}}$$

then multiplying by the flight velocity we arrive at the <u>Breguet Range Equation</u> which applies for situations where Isp and flight velocity are constant over the flight (**In the text:** For more information about the Breguet Range Equation, see the notes on it from the Fall).

$$\boxed{\text{Range} = u_o \quad \frac{L}{D} \quad Isp \quad \ell n \frac{W_{initial}}{W_{final}}}$$

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This can be re-written in other forms:

$$u_o Isp = u_o \frac{F}{\dot{m}_f g} = \frac{F u_o}{\dot{m}_f h} \frac{h}{g}$$

$$\frac{F u_o}{\dot{m}_f h} = \eta_{overall} \qquad \text{where} \qquad \eta_{overall} = \frac{g}{h} u_o\, Isp$$

$$\boxed{\text{Range} = \frac{h}{g} \eta_{overall} \frac{L}{D}\, \ell n \frac{W_{initial}}{W_{final}}} \qquad \text{or} \qquad \text{Range} = \frac{u_o}{g} \frac{1}{SFC} \frac{L}{D}\, \ell n \frac{W_{initial}}{W_{final}}$$

D. Aircraft Endurance

For a given amount of available fuel energy (Joules), the maximum endurance (time aloft) is obtained at a flight condition corresponding to the minimum rate of energy expenditure (Joules/second), or $Preq_{min}$, as shown in Figure 4.3.

We can determine the aerodynamic configuration which provides the minimum energy expenditure:

$$D = W\frac{D}{L} = W\left(\frac{C_D}{C_L}\right) \qquad \text{so} \qquad P = W\left(\frac{C_D}{C_L}\right)\cdot V$$

where

$$V = \sqrt{\frac{W}{\frac{1}{2}\rho S C_L}}$$

Then

$$P = \sqrt{\frac{W^3}{\frac{1}{2}\rho S}}\left(\frac{C_D}{C_L^{3/2}}\right)$$

So the minimum power required (**maximum endurance**) occurs when $\frac{C_L^{3/2}}{C_D}$ is a maximum.

With a little algebra we can arrive at an expression for the maximum endurance. Setting

$$\frac{d}{dC_L}\left(\frac{C_{D_o} + \frac{C_L^2}{\pi e AR}}{C_L^{3/2}}\right) = 0$$

we find that

$$C_{L_{min\ power}} = \sqrt{3\pi e AR C_{D_o}} \quad \text{and} \quad C_{D_{min\ power}} = 4C_{D_o}$$

$$\left(\frac{C_L}{C_D}\right)_{min\ power} = \sqrt{\frac{3\pi e AR}{16C_{D_o}}} \quad \text{and}$$

$$V_{min\ power} = \sqrt{\frac{W}{\frac{1}{2}\rho S C_{L_{min\ power}}}} = \left[\frac{4}{3}\left(\frac{W}{S}\right)^2 \frac{1}{\rho^2}\frac{1}{C_{D_o}}\left(\frac{1}{\pi e AR}\right)\right]^{1/4}$$

Thus the minimum power (maximum endurance) condition occurs at a speed which is $3^{-1/4}$ =

76% of the minimum drag (maximum range) condition. The corresponding lift-to-drag ratio is 86.6% of the maximum lift-to-drag ratio.

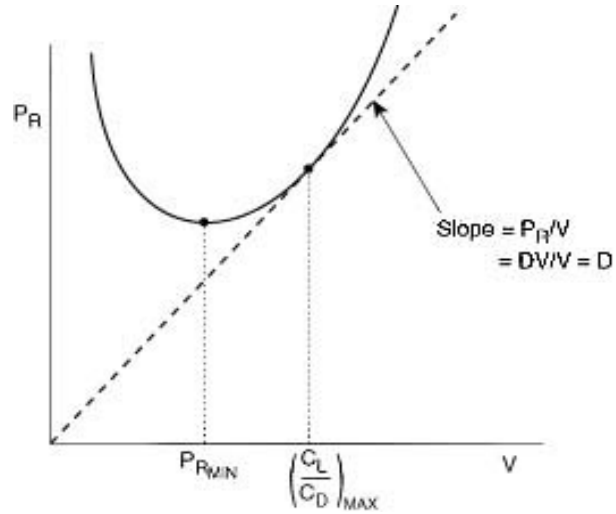

Figure 4.4 Relationship between condition for maximum endurance and maximum range.

Continuing

$$D_{\text{min imum power}} = W\left[\frac{16}{3}\frac{C_{D_o}}{\pi e AR}\right]^{1/2}$$

which can be substituted into

$$\frac{dW}{dt} = \dot{m}_f \cdot g = \frac{-T\dot{m}_f \cdot g}{T} = \frac{-T}{Isp} = \frac{-D}{Isp}$$

Such that, for maximum endurance

$$\frac{dW}{dt} = \frac{-W}{Isp}\left[\frac{16}{3}\frac{C_{D_o}}{\pi e AR}\right]^{1/2}$$

which can be integrated (assuming constant Isp) to yield

$$\boxed{t_{max} = Isp\left[\frac{16}{3}\frac{C_{D_o}}{\pi e AR}\right]^{-1/2} \ln\left(\frac{W_{initial}}{W_{final}}\right)}$$

E. Climbing Flight

Any excess in power beyond that required to overcome drag will cause the vehicle increase kinetic or potential energy. We consider this case by resolving forces about the direction of flight and equating these with accelerations.

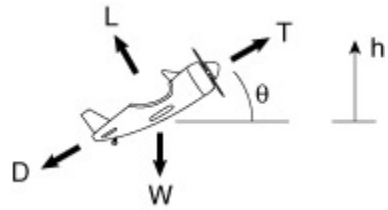

Figure 4.5 Force balance for an aircraft in climbing flight.

$$L - W\cos\theta = \frac{W}{g} V \frac{d\theta}{dt}$$ where $V\frac{d\theta}{dt}$ is the accel. normal to the flight path

$$T - D - W\sin\theta = \frac{W}{g}\frac{dV}{dt}$$ where $\frac{dV}{dt}$ is the accel. tangent to the flight path

So the change in height of the vehicle (the **rate of climb**, **R/C**) is:

$$R/C = \frac{dh}{dt} = V\sin\theta = V\left(\frac{T-D}{W}\right) - \frac{V}{g}\frac{dV}{dt}$$

which is instructive to rewrite in the form

$$TV - DV = W\frac{dh}{dt} + \frac{d}{dt}\left(\frac{1}{2}\frac{W}{g}V^2\right)$$

or

$$P_{available} - P_{required} = W\frac{dh}{dt} + \frac{d}{dt}\left(\frac{1}{2}\frac{W}{g}V^2\right)$$

in words:

excess power = change in potential energy + change in kinetic energy

<u>For steady climbing flight</u>,

$$R/C = V\left(\frac{T-D}{W}\right) = \frac{P_{avail} - P_{req}}{W}$$

and the time-to-climb is

$$t = \int_{h_1}^{h_2} \frac{dh}{R/C}$$

where

$$P_{available} = \eta_{prop} P_{shaft} \quad \text{for example, and} \quad P_{required} = DV$$

The power available is a function of the propulsion system, the flight velocity, altitude, etc. Typically it takes a form such as that shown in Figure 4.6. The shortest time-to-climb occurs at the flight velocity where $P_{avail} - P_{req}$ is a maximum.

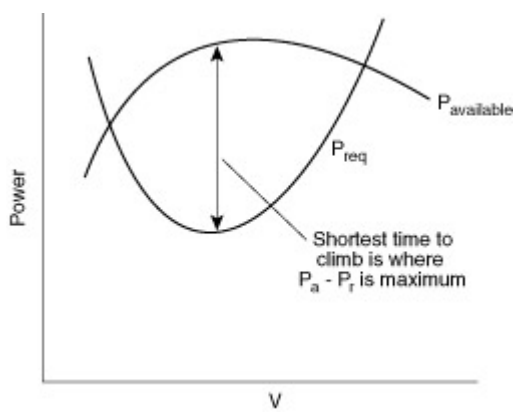

Figure 4.6 Typical behavior of power available as a function of flight velocity.

Figure 4.7 Lockheed Martin F-16 performing a vertical accelerated climb.

V. Rocket Performance

A. Thrust and Specific Impulse for Rockets

During the Fall semester thermodynamics lectures we used the *steady flow energy equation* to relate the exhaust velocity of a rocket motor to the conditions in the combustion chamber and the exit pressure

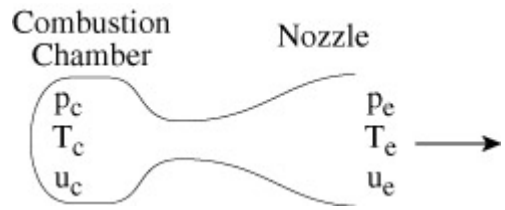

Figure 5.1 Schematic of rocket nozzle and combustion chamber

The steady flow energy equation

$$q_{1-2} - W_{s_{1-2}} = h_{T_2} - h_{T_1}$$

then with no heat transfer or shaft work

$$h_{T_2} = h_{T_1} \qquad \text{or} \qquad h_{T_c} - h_{T_e}$$

which can be written as

$$C_p T_c + \frac{u_c^2}{2} = C_p T_e + \frac{u_e^2}{2}$$

and manipulated to obtain

$$u_e = \sqrt{2C_p T_c \left[1 - \left(\frac{P_e}{P_c}\right)^{\frac{\gamma-1}{\gamma}}\right]} \qquad \text{using} \qquad \frac{T_e}{T_c} = \left(\frac{P_e}{P_c}\right)^{\frac{\gamma-1}{\gamma}}$$

Then considering the relationship we derived for thrust

$$\tau = \dot{m}u_e + A_e(p_e - p_o) \qquad \text{and assuming } p_e = p_o = 0$$

then $u_e = \sqrt{2C_p T_c}$, $\tau = \dot{m}u_e$

and $\quad Isp = \dfrac{\tau}{\dot{W}} \equiv \dfrac{\text{thrust}}{\text{fuel weight flow rate}}$

so $\quad \dot{m}u_e = \dot{W}Isp$

Thus the specific impulse can be directly related to the exhaust velocity leaving the rocket

$$\boxed{Isp = \frac{u_e}{g}}$$

B. The Rocket Equation

We can now look at the role of specific impulse in setting the performance of a rocket. A large fraction (typically 90%) of the mass of a rocket is propellant, thus it is important to consider the change in mass of the vehicle as it accelerates.

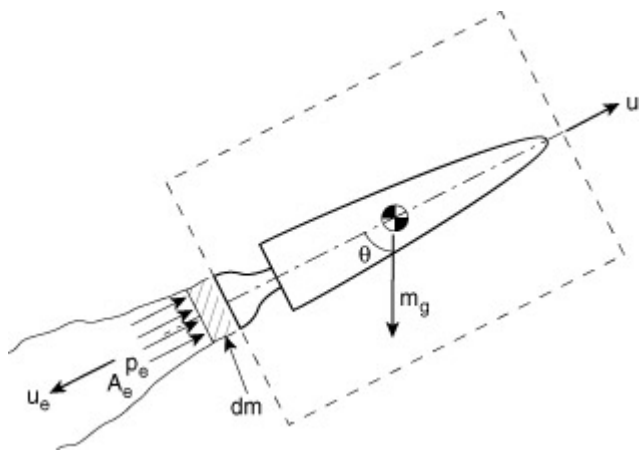

Figure 5.2 Schematic for application of the momentum theorem.

There are several ways to do this through applying conservation of momentum. Here we will apply the momentum theorem differentially by considering a small mass, dm, expelled from the rocket during time dt.

The initial momentum of the mass in the control volume (the vehicle) is

$$m_v u$$

The final momentum of mass in the control volume (the vehicle and the mass expelled, dm) is

$$(m_v - dm)(u + du) + dm(u - u_e)$$

$$= m_v u + m_v\, du - udm - dudm + udm - u_e dm$$

The change in momentum during the interval dt is

$= \text{momentum}_{final} - \text{momentum}_{initial}$

$= m_v\, du - u_e dm$ (since dudm is a higher order term)

Now consider the forces acting on the system which is composed of the masses m (the rocket), and dm (the small amount of propellant expelled from the rocket during time dt):

$$\sum F = (p_e - p_o)A_e - D - mg\cos\theta$$

Applying conservation of momentum, the resulting impulse, ΣFdt, must balance the change in momentum of the system.

$$m_v du - dmu_e = [(p_e - p_o)A_e - D - m_v g\cos\theta]dt$$

then since $dm = \dot{m}dt = -\frac{dm_v}{dt}dt$ where $\dot{m}$ = propellant mass flow rate

we have

$$m_v du = [(p_e - p_o)A_e + \dot{m}u_e - D - m_v g\cos\theta]dt$$

or for $p_e = p_o$

$$\boxed{du = -\frac{u_e dm_v}{m_v} - \frac{D}{m_v}dt - g\cos\theta dt}$$

<u>The Rocket Equation</u>

The above can be integrated as a function of time to determine the velocity of the rocket.

If we set u_e = constant, and assume that at t=0, u=0, and neglect drag, and set θ = 0, then we arrive at

$$du = -u_e\frac{dm_v}{m_v} - gdt$$

which can be integrated to give

$$u = -u_e\,\ell n\left(\frac{m_v}{m_{v_o}}\right) - gt$$

where m_{vo} is the initial mass of the rocket

or

$$\boxed{u = g\left[Isp\,\ell n\left(\frac{m_{v_o}}{m_v}\right) - t\right]}$$

We can view this equation as being similar to the Breguet Range Equation for aircraft. It presents the overall dependence of the principal performance parameter for a rocket (velocity, u), on the efficiency of the propulsion system (Isp), and the structural design (ratio of total mass to structural mass – since the initial mass is the fuel mass plus the structural mass and the final mass. is only the structural mass).

Assuming the rate of fuel consumption is constant, the mass of the rocket varies over time as

$$m_v(t) = m_{vo} - (m_{vo} - m_{vfinal})t/t_b$$

where t_b is the time at which all of the propellant is used. This expression can be substituted into the equation for velocity and then integrated to find the height at the end of burnout:

$$h_b = \int_0^{t_b} u\,dt$$

which for a single stage sounding rocket with no drag and constant gravity yields

$$h_b = g\left[-t_b Isp \frac{\ell n\left(\frac{m_{v_o}}{m_{vfinal}}\right)}{\left(\frac{m_{v_o}}{m_{vfinal}}\right) - 1} + t_b Isp - \frac{1}{2} t_b^{\,2} \right]$$

the final height of the rocket can then be determined by equating the kinetic energy of the vehicle at burnout with its change in potential energy between that point and the maximum height. This is left as an exercise for the reader.

Figure 5.3 The Saturn V rocket stood 365 feet tall and had 5 stages. It produced over 7.5 million pounds of thrust at liftoff (NASA, 1969).

VI. Rocket Nozzles: Connection of Flow to Geometry

We have considered the overall performance of a rocket and seen that is directly dependent on the exit velocity of the propellant. Further, we have used the steady flow energy equation to determine the exhaust velocity using the combustion chamber conditions and the nozzle exit pressure. In this brief section, we will apply concepts from thermodynamics and fluids to relate geometrical (design) parameters for a rocket nozzle to the exhaust velocity.

We will make the following assumptions:

1) The propellant gas obeys the perfect gas law
2) The specific heat is constant
3) The flow in the nozzle is one-dimensional
4) There are no losses due to friction
5) There is no heat transfer
6) The flow velocity in the combustion chamber is negligible (zero)
7) The flow is steady

A. Quasi-one-dimensional compressible flow in a variable area duct

$$p = \rho RT \qquad \text{(ideal gas)}$$

$$\frac{p}{p_c} = \left(\frac{T}{T_c}\right)^{\frac{\gamma}{\gamma-1}} \qquad \text{(isentropic flow)}$$

$$\frac{T_c}{T} = 1 + \frac{u^2}{2C_p T} = 1 + \frac{\gamma-1}{2}M^2 \qquad \text{(energy equation)}$$

imply that

$$\frac{P_c}{P} = \left[1 + \frac{\gamma-1}{2}M^2\right]^{\frac{\gamma}{\gamma-1}}$$

Then from conservation of mass

$$\rho u A = \dot{m} \qquad \text{(cons. of mass)}$$

$$\frac{P}{RT}uA = \dot{m} \qquad \frac{P_c\sqrt{\gamma}}{\sqrt{RT_c}}\frac{M}{\left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma+1}{2(\gamma-1)}}}A = \dot{m}$$

The above equation relates the flow area, the mass flow, the Mach number and the stagnation conditions. It is frequently rewritten in a non-dimensional form by dividing through by the value at M=1 (where the area at M=1 is A*):

$$\frac{A^*}{A} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

which takes a form something like that shown in Figure 6.1 below

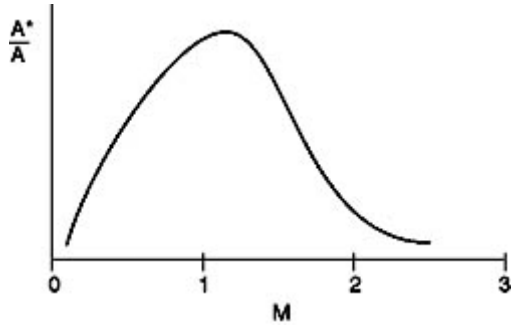

Figure 6.1 General form of relationship between flow area and Mach number.

B. Thrust in terms of nozzle geometry

We can use these equations to rewrite our expression for rocket thrust in terms of nozzle geometry (throat area = A*, and exit area A_e).

$$\tau = \dot{m}u_e + A_e(p_e - p_o)$$

$$\dot{m} = \frac{p}{RT}uA \quad \leftarrow \text{ evaluate at } M = 1 \quad \text{(throat)}$$

$$\dot{m} = \left(\frac{p}{RT}uA\right)_{M=1} = \sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\frac{p_c}{\sqrt{RT_c}}A^*$$

$$u_e = M_e\sqrt{\gamma RT_e} = M_e\sqrt{\gamma RT_c\left(\frac{1}{1+\frac{\gamma-1}{2}M_e^2}\right)}$$

$$\frac{P_e}{P_c} = \frac{1}{\left[1+\frac{\gamma-1}{2}M_e^{\,2}\right]^{\frac{\gamma}{\gamma-1}}}$$

We can now specify geometry (A* and A_e) to determine M_e. Then use M_e with the combustion chamber conditions to determine thrust and I_{sp}.

VII. Production of Thrust with a Propeller

A. Overview of propeller performance

Each propeller blade is a rotating airfoil which produces lift and drag, and because of a (complex helical) trailing vortex system has an induced upwash and an induced downwash.

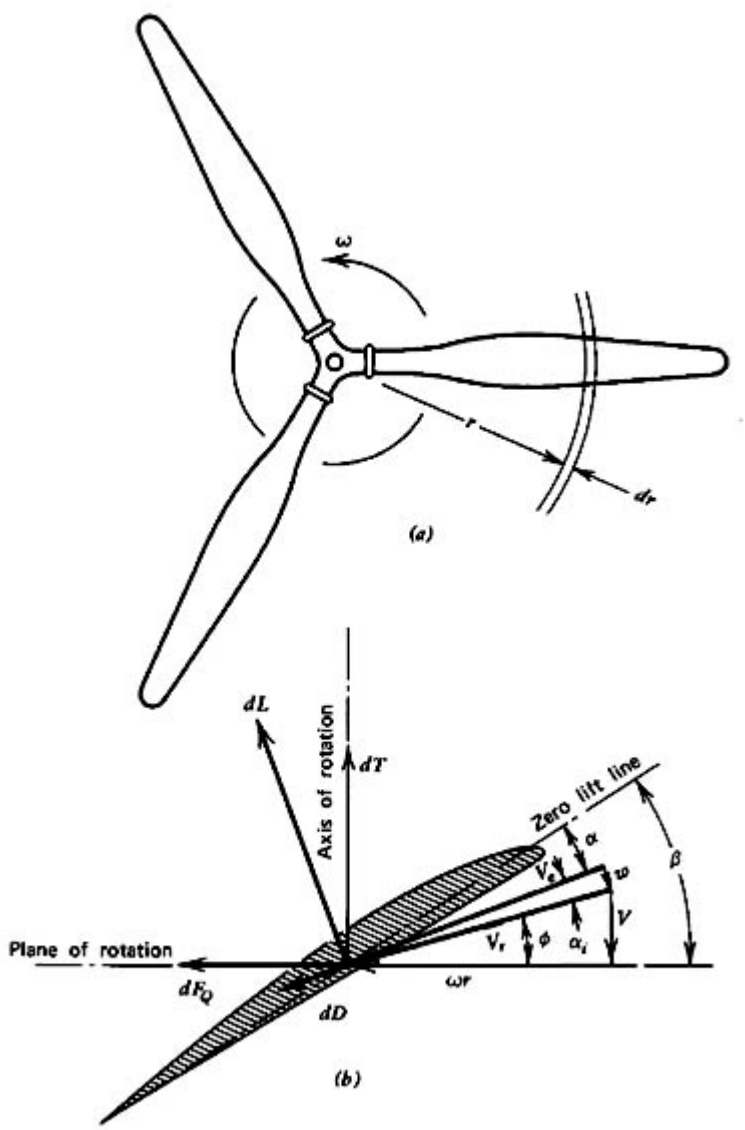

Figure 7.1 Schematic of propeller (McCormick, 1979)

Figure 7.2 Cessna Skyhawk single engine propeller plane (Cessna, 2000)

The two quantities of interest are the thrust (T) and the torque (Q). We can write expressions for these for a small radial element (dr) on one of the blades:

$$dT = dL\cos(\phi + \alpha_i) - dD\sin(\phi + \alpha_i)$$

$$dQ = r\left[dL\sin(\phi + \alpha_i) + dD\cos(\phi + \alpha_i)\right]$$

where

$$dL = \frac{1}{2}\rho V_e^2 c\, C_\ell dr \qquad \text{and} \qquad dD = \frac{1}{2}\rho V_e^2 c\, C_d dr$$

It is possible to integrate the relationships as a function of r with the appropriate lift and drag coefficients for the local airfoil shape, but determining the induced upwash (α_i) is difficult because of the complex helical nature of the trailing vortex system. In order to learn about the details of propeller design, it is necessary to do this. However, for our purposes, we can learn a about the overall performance features using the integral momentum theorem, some further approximations called "actuator disk theory", and dimensional analysis.

Figure 7.3 The V-22 Osprey utilizes tiltrotor technology (Boeing, 2000)

B. Application of the Integral Momentum Theorem to Propellers

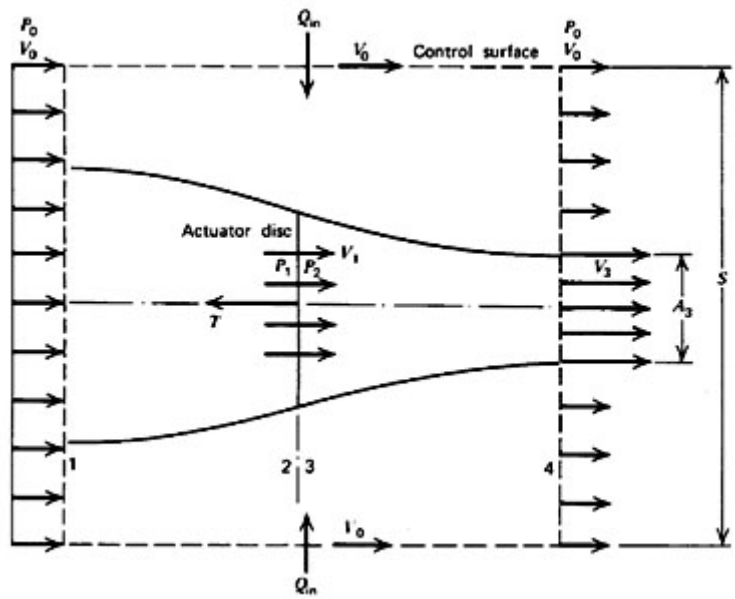

Figure 7.4 Control volume for analysis of a propeller (McCormick, 1979)

The control volume shown in Figure 7.2 has been drawn far enough from the device so that the pressure is everywhere equal to a constant. This is not required, but it makes it more convenient to apply the integral momentum theorem. We will also assume that the flow outside of the propeller streamtube does not have any change in total pressure. Then since the flow is steady we apply:

$$\sum F_X = \int_s u_X \rho \vec{u} \cdot \vec{n} ds$$

Since the pressure forces everywhere are balanced, then the only force on the control volume is due to the change in momentum flux across its boundaries. Thus by inspection, we can say that

$$\boxed{T = \dot{m}(u_e - u_o)}$$

Or we can arrive at the same result in a step-by-step manner as we did for the jet engine example in Section II:

$$T = \int_s \rho u_x (\vec{u} \cdot \vec{n}) dA$$

$$\int_s \rho u_x \vec{u}\cdot\vec{n}dA = \underbrace{\rho_e u_e A_e}_{\dot{m}} u_e - \underbrace{\rho_o u_o A_o}_{\dot{m}} u_o + \int_{C_s - A_o - A_e} \rho u_x \vec{u}\cdot\vec{n}dA$$

$$= \dot{m}u_e - \dot{m}u_o + \vec{u}_o \int_{C_s - A_o - A_e} \rho\vec{u}\cdot\vec{n}dA$$

Note that the last term is identically equal to zero by conservation of mass. If the mass flow in and out of the propeller streamtube are the same (as we have defined), then the net mass flux into the rest of the control volume must also be zero.

So we have:

$$T = \dot{m}(u_e - u_o)$$

as we reasoned before.

The power expended is equal to the power imparted to the fluid which is the change in kinetic energy of the flow as it passes through the propeller

$$\text{power imparted to the fluid} = \dot{m}\left(\frac{u_e^2}{2} - \frac{u_o^2}{2}\right)$$

The propulsive power is the rate at which useful work is done which is the thrust multiplied by the flight velocity

$$\textit{propulsive power} = \text{thrust}\cdot\text{flight velocity} = Tu_o$$

The propulsive efficiency is then the ratio of these two:

$$\eta_{prop} = \frac{2}{1 + \frac{u_e}{u_o}}$$

Which is the same expression as we arrived at before for the jet engine (as you might have expected).

C. Actuator Disk Theory

To understand more about the performance of propellers, and to relate this performance to simple design parameters, we will apply actuator disk theory. We model the flow through the propeller as shown in Figure 7.3 and make the following assumptions:

- Neglect rotation imparted to the flow.
- Assume the Mach number is low so that the flow behaves as an incompressible fluid.
- Assume the flow outside the propeller streamtube has constant stagnation pressure (no work is imparted to it).
- Assume that the flow is steady. Smear out the moving blades so they are one thin steady disk that has approximately the same effect on the flow as the moving blades (the "actuator disk").
- Across the actuator disk, assume that the pressure changes discontinuously, but the velocity varies in a continuous manner.

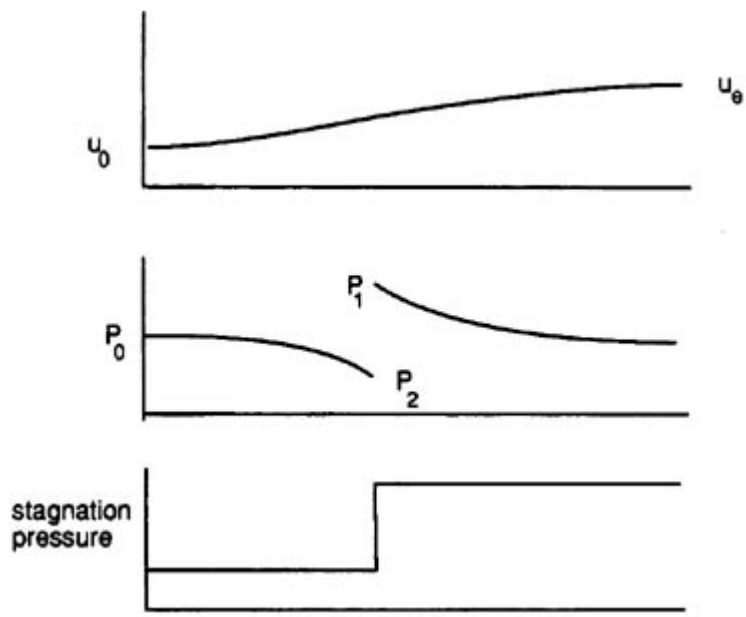

Figure 7.5 Schematic of actuator disk model (Kerrebrock).

We then take a control volume around the disk as shown in Figure 7.4

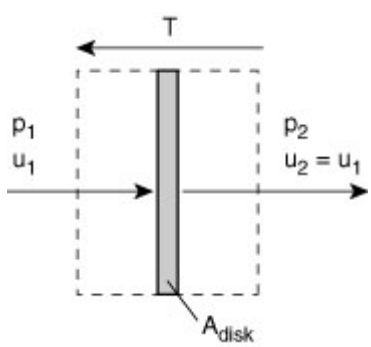

Figure 7.6 Control volume around actuator disk.

The force, T, on the disk is

$$T = A_{disk}(p_2 - p_1)$$

So the power is

$$\text{Power} = Fu_{disk} = A_{disk}(p_2 - p_1)u_{disk} = \dot{m}(u_e - u_o)u_{disk}$$

We also know that the power is

$$\text{Power} = \dot{m}\left(\frac{u_e^2 - u_o^2}{2}\right) = \dot{m}(u_e - u_o)\frac{(u_e + u_o)}{2}$$

Thus we see that the velocity at the disk is

$$u_{disk} = \frac{(u_e + u_o)}{2}$$

Half of the axial velocity change occurs upstream of the disk and half occurs downstream of the disk.

We can now find the pressure upstream and downstream of the disk by applying the Bernoulli equation in the regions of the flow where the pressure and velocity are varying continuously.

$$p_1 + \frac{1}{2}\rho u_{disk}^2 = p_o + \frac{1}{2}\rho u_o^2 \qquad \text{and} \qquad p_2 + \frac{1}{2}\rho u_{disk}^2 = p_o + \frac{1}{2}\rho u_e^2$$

From which we can determine

$$p_1 - p_2 = \frac{1}{2}\rho\left(u_e^2 - u_o^2\right)$$

We generally don't measure or control u_{disk} directly. Therefore, it is more useful to write our expressions in terms of flight velocity u_o, thrust, T, (which must equal drag for steady level flight) and propeller disk area, A_{disk}.

$$\dot{m} = \rho u_{disk} A_{disk} = \rho A_{disk}\frac{(u_e + u_o)}{2}$$

So

$$T = \rho A_{disk} \frac{(u_e + u_o)}{2}(u_e - u_o) = \rho A_{disk} \frac{(u_e^2 - u_o^2)}{2}$$

From which we can obtain an expression for the exit velocity in terms of thrust and flight velocity which are vehicle parameters

$$\left(\frac{u_e}{u_o}\right)^2 = \frac{T}{A_{disk} u_o^2 \frac{\rho}{2}} + 1$$

The other parameters of interest become

$$\frac{u_{disk}}{u_o} = \frac{1}{2}\left[\frac{T}{A_{disk} u_o^2 \frac{\rho}{2}} + 1\right]^{1/2} + \frac{1}{2}$$

$$\boxed{\text{Power} = T u_{disk} = \frac{1}{2} T u_o \left\{\left[\frac{T}{A_{disk} u_o^2 \frac{\rho}{2}} + 1\right]^{1/2} + 1\right\}}$$

This is the **ideal (minimum) power required to drive the propeller**. In general , the actual power required would be about 15% greater than this.

$$\boxed{\eta_{propulsive} = \frac{2}{1 + \left[\frac{T}{A_{disk} u_o^2 \frac{\rho}{2}} + 1\right]^{1/2}}}$$

There are several important trends that are apparent upon consideration of these equations. We see that the propulsive efficiency is zero when the flight velocity is zero (no useful work, just a force), and tends towards one when the flight velocity increases. In practice, the propulsive efficiency typically peaks at a level of around 0.8 for a propeller before various aerodynamic effects act to decay its performance as will be shown in the following section.

D. Dimensional Analysis

We will now use dimensional analysis to arrive at a few important parameters for the design and choice of a propeller. Dimensional analysis leads to a number of coefficients which are useful for presenting performance data for propellers.

Parameter	Symbol	Units
propeller diameter	D	m
propeller speed	n	rev/s
torque	Q	Nm
thrust	T	N
fluid density	ρ	kg/m^3
fluid viscosity	ν	m^2/s
fluid bulk elasticity modulus	K	N/m^2
flight velocity	u_o	m/s

1. Thrust Coefficient

$$T = f(D; n; \rho; \nu; K; u_o) = \text{Const. } D^a n^b \rho^c \nu^d K^e u_o^f$$

Then putting this in dimensional form

$$[MLT^{-2}] = [(L)^a(T)^{-b}(ML^{-3})^c(L^2T^{-1})^d(ML^{-1}T^{-2})^e(LT^{-1})^f]$$

Which implies

(M) $1 = c + e$
(L) $1 = a - 3c + 2d - e + f$
(T) $2 = b + d + 2e + f$

So

$a = 4 - 2e - 2d - f$
$b = 2 - d - 2e - f$
$c = 1 - e$

$$T = \text{Const. } D^{4-2e-2d-f} n^{2-d-2e-f} \rho^{1-e} \nu^d K^e u_o^f$$

$$T = (Const.) \cdot \rho n^2 D^4 \cdot func.\left[\left(\frac{\nu}{D^2 n}\right)^d; \left(\frac{K}{\rho D^2 n^2}\right)^e; \left(\frac{u_o}{Dn}\right)^f\right]$$

We can now consider the three terms in the square brackets

$\frac{\nu}{D^2 n}$: Dn is proportional to the tip speed, so this term is like $\frac{\nu}{\text{length} \cdot \text{velocity}} \propto \frac{1}{\text{Re}}$

$\frac{K}{\rho D^2 n^2}$: $K/\rho = a^2$ where a is the speed of sound, this is like $\frac{a^2}{\text{tip speed}^2} \propto \frac{1}{M_{tip}{}^2}$

$\frac{u_o}{Dn}$: u_o/n is the distance advanced by the propeller in one revolution, here non-dimensionalized by the propeller diameter.

This last coefficient is typically called the ***advance ratio*** and given the symbol **J**.

Thus we see that the thrust may be written as

$$T = (Const.) \cdot \rho n^2 D^4 \cdot func.\left[\text{Re}; M_{tip}; J\right]$$

which is often expressed as

$$\boxed{T = k_T \rho n^2 D^4}$$

where k_T is called the thrust coefficient and in general is a function of propeller design, Re, M_{tip} and J.

2. Torque Coefficient

We can follow the same steps to arrive at a relevant expression and functional dependence for the torque or apply physical reasoning. Since torque is a force multiplied by a length, it follows that

$$\boxed{Q = k_Q \rho n^2 D^5}$$

where k_Q is called the thrust coefficient and in general is a function of propeller design, Re, M_{tip} and J.

3. Efficiency

The power supplied to the propeller is P_{in} where

$$P_{in} = 2\pi n Q$$

The useful power output is P_{out} where

$$P_{out} = T u_o$$

Therefore the efficiency is given by

$$\boxed{\eta_{prop} = \frac{Tu_o}{2\pi nQ} = \frac{k_T \rho n^2 D^4 u_o}{k_Q \rho n^2 D^5 2\pi n} = \frac{1}{2\pi}\frac{k_T}{k_Q} J}$$

4. Power Coefficient

The power required to drive the propeller is

$$P_{in} = 2\pi nQ = 2\pi n\left(k_Q \rho n^2 D^5\right) = 2\pi \rho k_Q n^3 D^5$$

which is often written using a power coefficient $C_p = 2\pi k_Q$

$$\boxed{P_{in} = C_p \rho n^3 D^5} \qquad \text{then} \qquad \boxed{\eta_{prop} = J\left(\frac{k_T}{C_p}\right)}$$

E. Typical propeller performance

Typical propeller performance curves are shown in the following figures.

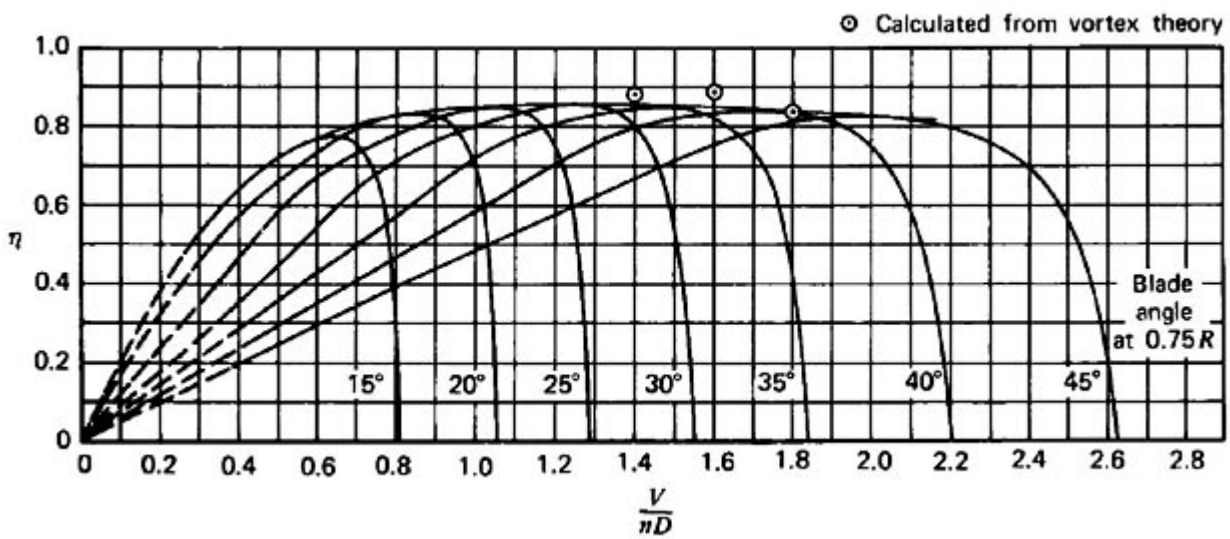

Figure 7.7 Typical propeller efficiency curves as a function of advance ratio ($J=u_o/nD$) and blade angle (McCormick, 1979).

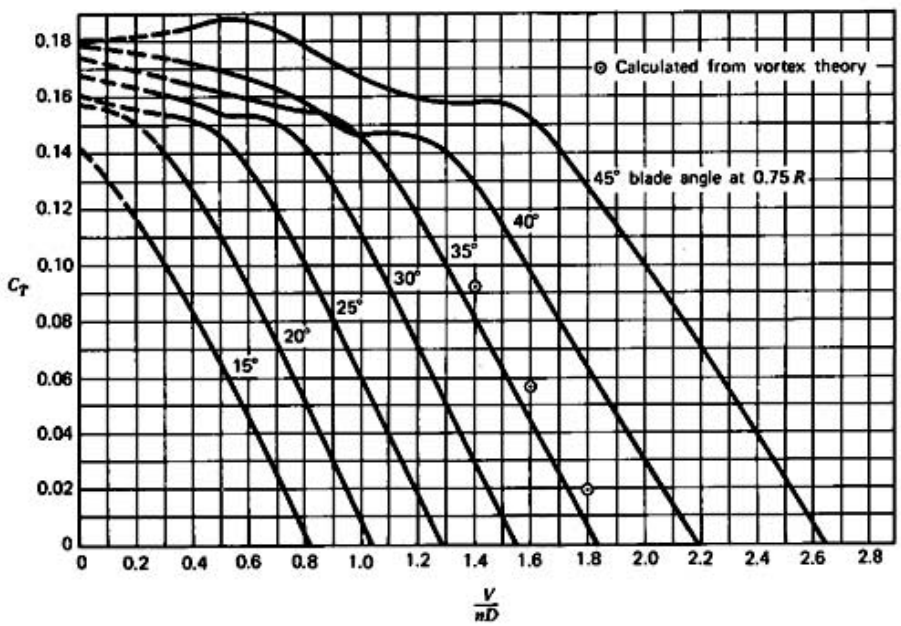

Figure 7.8 Typical propeller thrust curves as a function of advance ratio ($J=u_o/nD$) and blade angle (McCormick, 1979).

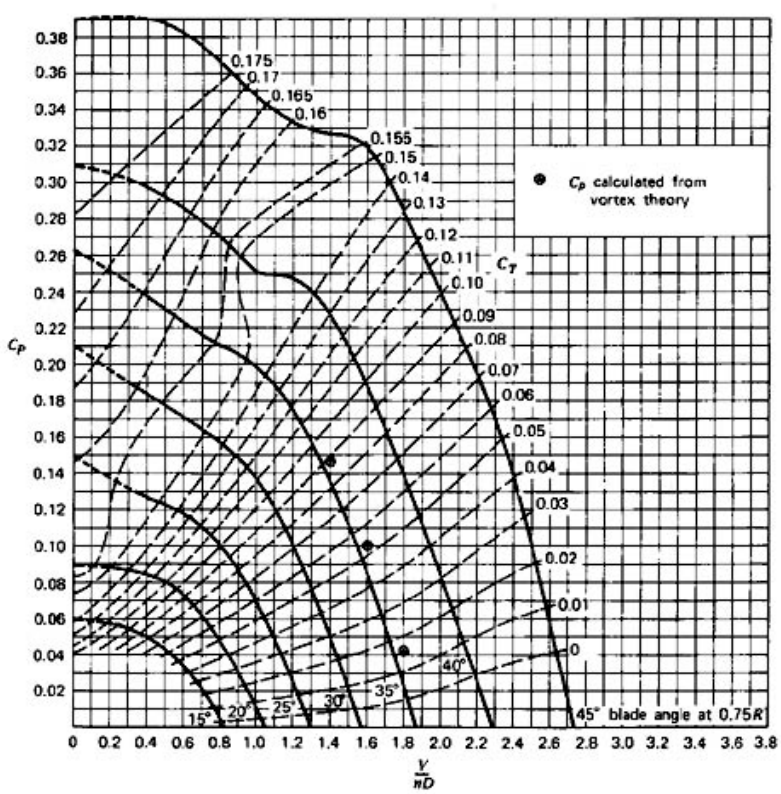

Figure 7.9 Typical propeller power curves as a function of advance ratio ($J=u_o/nD$) and blade angle (McCormick, 1979).

VIII. Ideal Cycle Analysis of Aircraft Gas Turbine Engines

A. Introduction

In thermodynamics we represented a gas turbine engine using a Brayton cycle, as shown in Figure 8.1, and derived expressions for efficiency and work as functions of the temperature at various points in the cycle. In this section we will perform "ideal cycle analysis", which is a method for expressing the thrust and thermal efficiency of engines in terms of *useful design variables*. If you take a further course in propulsion, this ideal cycle analysis will be extended to take account of various inefficiencies in the different components of the engine–that type of analysis is called non-ideal cycle analysis.

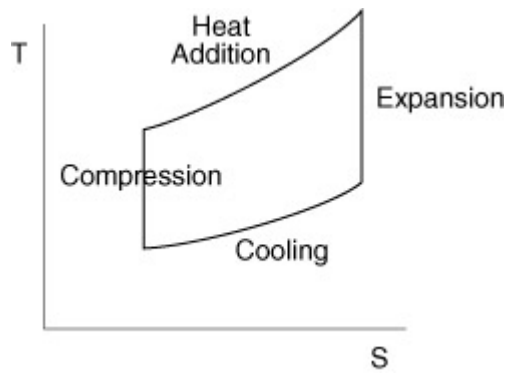

Figure 8.1 Schematic of a Brayton cycle.

1. Objective of ideal cycle analysis

Our objective is to express thrust, T, and thermal efficiency, η (or alternatively I) as functions of 1) typical design limiters, 2) flight conditions, and 3) design choices so that we can analyze the performance of various engines. The expressions will allow us to define a particular mission and then determine the optimum component characteristics (e.g. compressor, combustor, turbine) for an engine for a given mission. Note that ideal cycle analysis addresses only the thermodynamics of the airflow within the engine. It does not describe the details of the components (the blading, the rotational speed, etc.), but only the results the various components produce (e.g. pressure ratios, temperature ratios). In Section IX we will look in greater detail at how some of the components (the turbine and the compressor) produce these effects.

2. Notation and station numbering

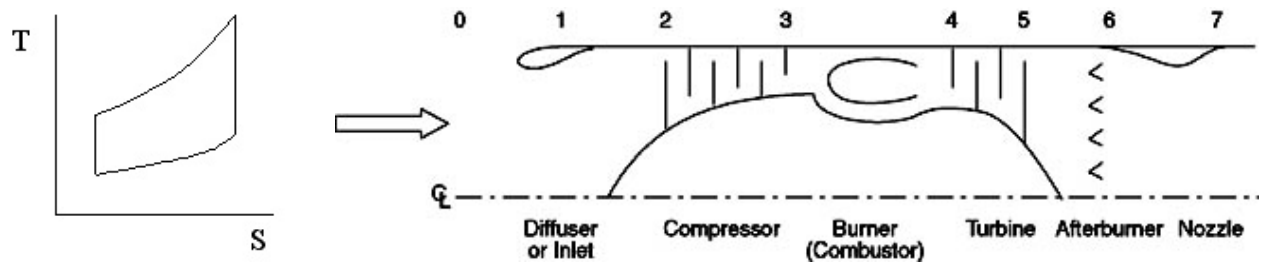

Figure 8.2 Gas turbine engine station numbering.

Notation:

$$T_T = T\left(1+\frac{\gamma-1}{2}M^2\right), \quad P_T = P\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_{T_0}}{T_0} \equiv \theta_0 \qquad \frac{T_{T_t}}{T_0} \equiv \theta_t$$

$$\frac{P_{T_0}}{P_0} \equiv \delta_0 \qquad (\therefore\ \delta_0 = \theta_0^{\gamma/\gamma-1})$$

Stagnation properties, T_T & P_T, are more easily measured quantities than static properties (T and p). Thus, it is standard convention to express the performance of various components in terms of stagnation pressure and temperature ratios:

$\pi \equiv$ total or stagnation pressure ratio across component (d, c, b, t, a, n)

$\tau \equiv$ total or stagnation temperature ratio across component (d, c, b, t, a, n)

where d= diffuser (or inlet), c= compressor, b= burner (or combustor), t= turbine, a= afterburner, and n=nozzle.

3. Ideal Assumptions

1) Inlet/Diffuser: $\pi_d = 1$, $\tau_d = 1$ (adiabatic, isentropic)
2) Compressor or fan: $\tau_c = \pi_c^{\gamma-1/\gamma}$, $\tau_f = \pi_f^{\gamma-1/\gamma}$
3) Combustor/burner or afterburner: $\pi_b = 1$, $\pi_a = 1$
4) Turbine: $\tau_t = \pi_t^{\gamma-1/\gamma}$

5) Nozzle: $\pi_n = 1$, $\pi_n = 1$

B. Ideal Cycle Analysis Example: Turbojet Engine

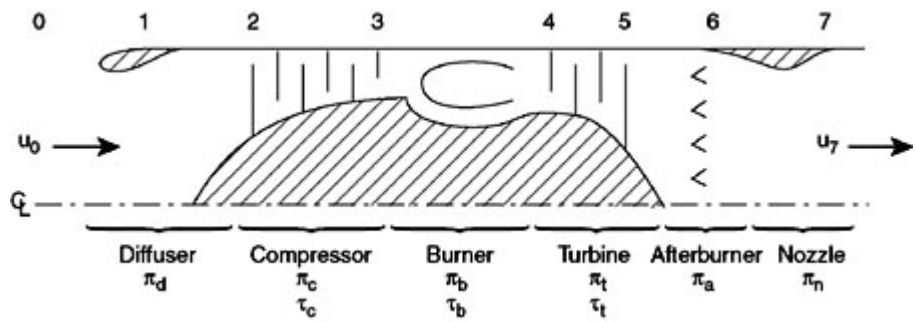

Figure 8.3 Schematic with appropriate component notations added.

<u>Methodology:</u>

1) Find thrust by finding u_{exit}/u_o in terms of θ_o, temperature ratios, etc.
2) Use a power balance to relate turbine parameters to compressor parameters
3) Use an energy balance across the combustor to relate the combustor temperature rise to the fuel flow rate and fuel energy content.

First write-out the expressions for thrust and I:

$$T = \dot{m}[(1+f)u_7 - u_0] + (p_7 - p_0)A_7$$ where f is the fuel/air mass flow ratio

$$\boxed{T = \dot{m}[u_7 - u_0] \Rightarrow \frac{T}{\dot{m}a_0} = M_0\left[\frac{u_7}{u_0} - 1\right]}$$ (can neglect fuel)

$$I = \frac{F}{\dot{m}_f g} = \frac{F}{g\dot{m}f}$$

That is the easy part! Now we have to do a little algebra to manipulate these expressions into more useful forms.

First we write an expression for the exit velocity:

$$\frac{u_7}{u_0} = \frac{M_7}{M_0}\sqrt{\frac{\gamma R T_7}{\gamma R T_0}} \cong \frac{M_7}{M_0}\sqrt{\frac{T_7}{T_0}}$$

Noting that

$$\frac{T_{T_7}}{T_7} = T\left(1 + \frac{\gamma - 1}{2} M_7^2\right)$$

We can write

$$T_{T_7} = T_{T_0}\left(\frac{T_{T_2}}{T_{T_0}}\right)\left(\frac{T_{T_3}}{T_{T_2}}\right)\left(\frac{T_{T_4}}{T_{T_3}}\right)\left(\frac{T_{T_5}}{T_{T_4}}\right)\left(\frac{T_{T_7}}{T_{T_5}}\right)$$

$$= T_{T_0}(\tau_d\ \tau_c\ \tau_b\ \tau_t\ \tau_n)$$

$$= T_0\left(1 + \frac{\gamma - 1}{2} M_0^2\right)\tau_c\ \tau_b\ \tau_t$$

Thus

$$T_{T_7} = T_0\,\theta_0\,\tau_c\,\tau_b\,\tau_t \quad \textbf{(*)}$$

Which expresses the exit temperature as a function of the inlet temperature, the Mach number, and the temperature changes across each component. Since we will use this expression again later we will mark it with an asterisk (*).

We now write the pressure at the exit in a similar manner:

$$P_{T_7} = P_0\left(1 + \frac{\gamma - 1}{2} M_0^2\right)^{\frac{\gamma}{\gamma - 1}}\pi_d\ \pi_c\ \pi_b\ \pi_t\ \pi_n$$

$$P_{T_7} = p_0\,\delta_0\ \pi_c\ \pi_t = p_7\left(1 + \frac{\gamma - 1}{2} M_7^2\right)^{\frac{\gamma}{\gamma - 1}}$$

$$\left(1 + \frac{\gamma - 1}{2} M_7^2\right)^{\frac{\gamma}{\gamma - 1}} = \delta_0\ \pi_c\ \pi_t$$

and then equate this to our expression for the temperature (*)

$$1+\frac{\gamma-1}{2}M_7^2=\delta_0^{\frac{\gamma-1}{2}}\ \pi_c^{\frac{\gamma-1}{2}}\ \pi_t^{\frac{\gamma-1}{2}}=\theta_0\,\tau_c\,\tau_t\left(=\frac{T_{T7}}{T_7}\right)\quad \textbf{(**)}$$

and again label it (**) for use later as we do the following expression:

$$M_7=\sqrt{\frac{2}{\gamma-1}}(\theta_0\,\tau_c\,\tau_t-1)^{1/2}\quad \textbf{(***)}$$

We now continue on the path to our expression for u_7/u_o.

$$\frac{T_7}{T_0}=\frac{T_7}{T_{T7}}\frac{T_{T7}}{T_0}=\frac{\theta_0\,\tau_c\,\tau_b\,\tau_t}{\theta_0\,\tau_c\,\tau_t}=\tau_b$$

$$\frac{u_7}{u_0}=\frac{M_7}{M_0}\sqrt{\frac{T_7}{T_0}}=\frac{\sqrt{\frac{2}{\gamma-1}}}{M_0}(\theta_0\,\tau_c\,\tau_t-1)^{1/2}\sqrt{\tau_b}$$

$$\theta_0=1+\frac{\gamma-1}{2}M_0^2\quad\Rightarrow\quad M_0^2=\frac{2}{\gamma-1}(\theta_0-1)$$

Therefore

$$\boxed{\frac{u_7}{u_0}=\sqrt{\frac{(\theta_0\,\tau_c\,\tau_t-1)\tau_b}{\theta_0-1}}}$$

Now we have <u>two steps left</u>. First we write τ_c in terms of τ_t, by noting that they are related by the condition that the power used by the compressor is equal to the power extracted by the turbine. Second, we put the burner temperature ratio in terms of the exit temperature of the burner, (T_{T4} or more specifically $\theta_t=T_{T4}/T_o$) since this is the hottest point in the engine and is a frequent benchmark used for judging various designs.

The steady flow energy equation tells us that

$$\dot{m}\Delta h_T=\dot{q}-\dot{w}_s$$

Assuming that the compressor and turbine are adiabatic, then

$\dot{m}\,\Delta h_T$ = - rate of shaft work done <u>by</u> the system = rate of shaft work done <u>on</u> the system

Since the turbine shaft is connected to the compressor shaft

$$\dot{m}C_p\left(T_{T_3} - T_{T_2}\right) = \dot{m}C_p\left(T_{T_4} - T_{T_5}\right) \quad \text{assuming } \dot{m} \text{ and } C_p \text{ are the same}$$

This can be rewritten as

$$\left(\frac{T_{T3}}{T_{T2}} - 1\right)\frac{T_{T2}}{T_0} = \left(\frac{T_{T4}}{T_0}\right)\left(1 - \frac{T_{T5}}{T_{T4}}\right) \qquad \text{where} \qquad \frac{T_{T2}}{T_0} = \tau_d\theta_0 = \theta_0$$

so

$$(\tau_c - 1)\theta_0 = \theta_t(1 - \tau_t) \qquad \text{or} \qquad \tau_t = 1 - \frac{\theta_0}{\theta_t}(\tau_c - 1)$$

That was the first step – relating the temperature rise across the turbine to that across the compressor. The remaining step is to write the temperature rise across the combustor in terms of $\theta_t = T_{T4}/T_o$.

$$\tau_b = \frac{\theta_t}{\theta_0\tau_c}$$

and for an engine with an afterburner

$$\tau_a = \frac{\theta_a}{\theta_t\tau_t}$$

Now substituting our expressions for τ_b, and τ_t into our expression for u_7/u_o, and finally into the first expression we wrote for thrust, we get:

$$\boxed{\frac{T}{\dot{m}a_0} = M_0\left[\left\{\left(\frac{\theta_0}{\theta_0 - 1}\right)\left(\frac{\theta_t}{\theta_0\tau_c} - 1\right)(\tau_c - 1) + \frac{\theta_t}{\theta_0\tau_c}\right\}^{1/2} - 1\right]}$$

<u>Specific thrust for a turbojet</u>

which is what we were seeking, an expression for thrust in terms of important design parameters and flight parameters:

$$\frac{T}{\dot{m}a_0} = \text{fnc.}\left(M_0, \tau_c, \theta_t\right)$$

With algebra $\left[\text{add \& substract } \frac{2\theta}{\gamma - 1}\left(\frac{\theta_t}{\theta_0\tau_c}\right)\right]$

We may write this write in another form which is often used

Reminder:

τ = temperature ratio across compressor

$\theta_t = \dfrac{\text{stagnation temperature at turbine inlet}}{\text{atmospheric temperature}}$

$\theta_0 = \dfrac{\text{atmospheric stagnation temperature}}{\text{atmospheric static temperature}}$

a_0 = speed of sound

T = thrust

$$\frac{T}{\dot{m}a_0} = \sqrt{\frac{2\theta_0}{\gamma-1}\left(\frac{\theta_t}{\theta_0\tau_c}-1\right)(\tau_c-1)+\frac{\theta_t M_0^2}{\theta_0\tau_c}} - M_0$$

Our final step involves writing the <u>specific impulse and other measures of efficiency</u> in terms of these same parameters. We begin by writing the First Law across the combustor to relate the fuel flow rate and heating value of the fuel to the total enthalpy rise.

$$\dot{m}_f h = \dot{m}C_p\left(T_{T_4} - T_{T_3}\right)$$

and

$$f = \frac{\dot{m}_f}{\dot{m}} = \frac{C_p T_0}{h}\left(\theta_t - \tau_c\theta_0\right)$$ where again, f is the fuel/air mass flow ratio

So the specific impulse becomes

$$\boxed{I = \frac{T}{gf\dot{m}} = \frac{a_0 h}{gC_p T_0}\frac{\left(\frac{T}{\dot{m}a_0}\right)}{\left(\theta_t - \tau_c\theta_0\right)}}$$ <u>Specific Impulse for an ideal turbojet</u>

where I is expressed in terms of typical design parameters, flight conditions, and physical constants

$$I = \text{fcn.}\ (M_0, \tau_c, \theta_t, a_0, T_0, h, C_p)$$

(flight condition) (design) (materials/design) (fuel and atmospheric properties)

Similarly, we can write our <u>overall efficiency</u>, $\eta_{overall}$

$$\eta_{overall} = \frac{Tu_o}{\dot{m}_f h}$$

$$\eta_{overall} = \frac{a_0^{\ 2}}{C_p T_0}\frac{M_0\left(\frac{T}{\dot{m}a_0}\right)}{\left(\theta_t - \tau_c\theta_0\right)}$$ or $$\boxed{\eta_{overall} = \frac{M_0(\gamma-1)\left(\frac{T}{\dot{m}a_0}\right)}{\left(\theta_t - \tau_c\theta_0\right)}}$$

The ideal thermal efficiency is

$$\boxed{\eta_{thermal} = 1 - \frac{1}{\theta_0 \tau_c}}$$

and the propulsive efficiency can be found from $\eta_{prop} = \eta_{overall} / \eta_{thermal}$

We can now use these equations to better understand the performance of a simple turbojet engine. We will use the following parameters (with γ=1.4):

Mach number	Altitude	Ambient Temp.	Speed of sound
0	Sea level	288K	340m/s
0.85	12km	217K	295m/s
2.0	18km	217K	295m/s

Note it is more typical to work with the compressor pressure ratio (π_c) rather than the temperature ratio (τ_c) so we will substitute the isentropic relationship:

$$\pi_c = \tau_c^{\gamma/\gamma-1}$$

into the equations before plotting the results.

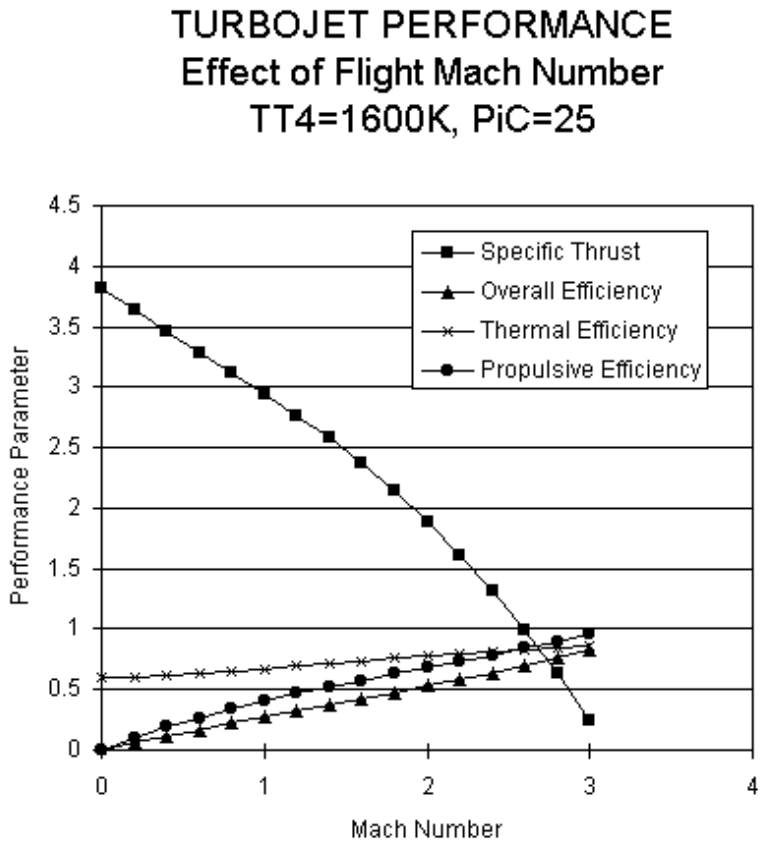

Figure 8.4 Performance of an ideal turbojet engine as a function of flight Mach number.

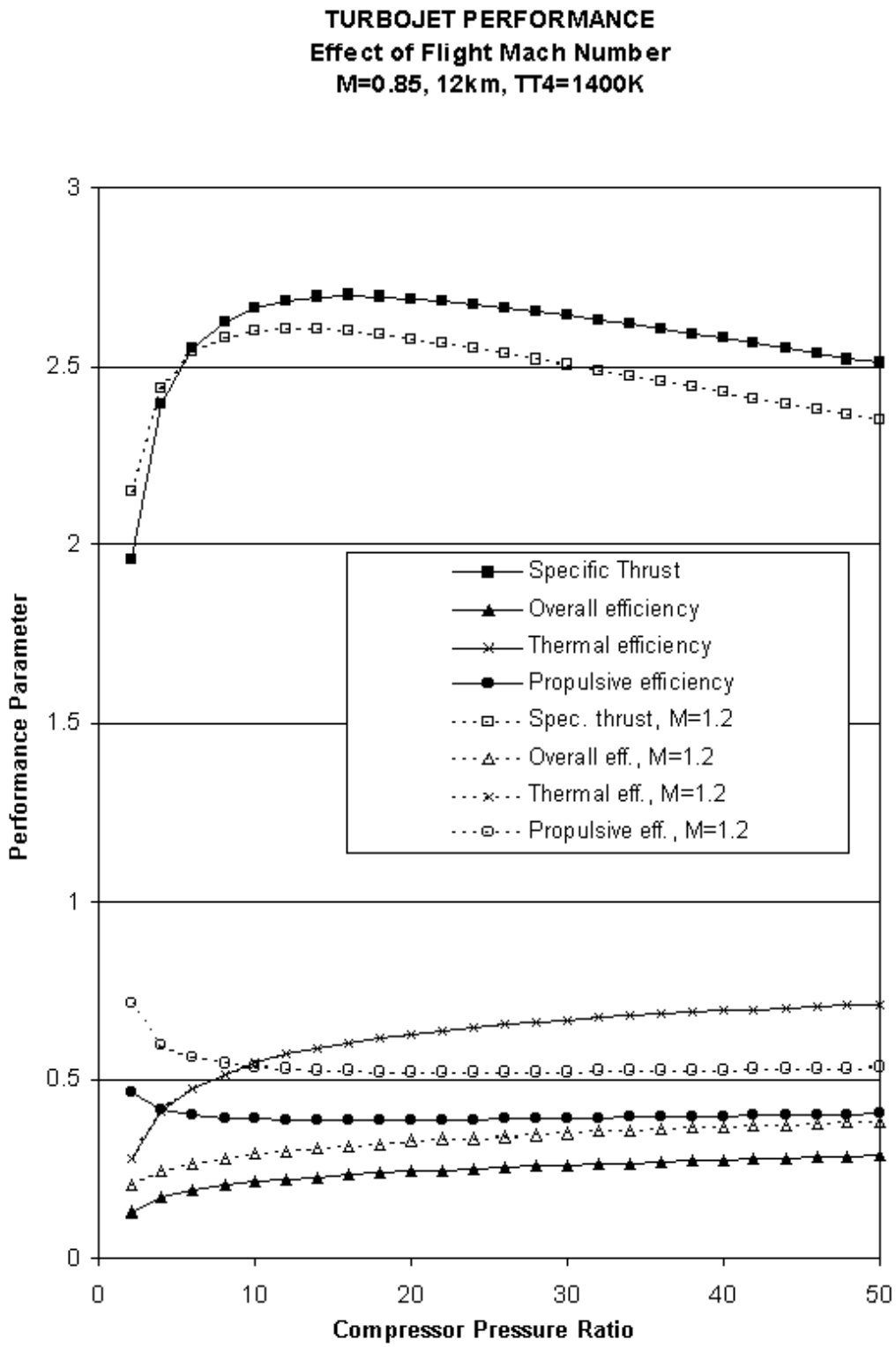

Figure 8.5 Performance of an ideal turbojet engine as a function of compressor pressure ratio and flight Mach number.

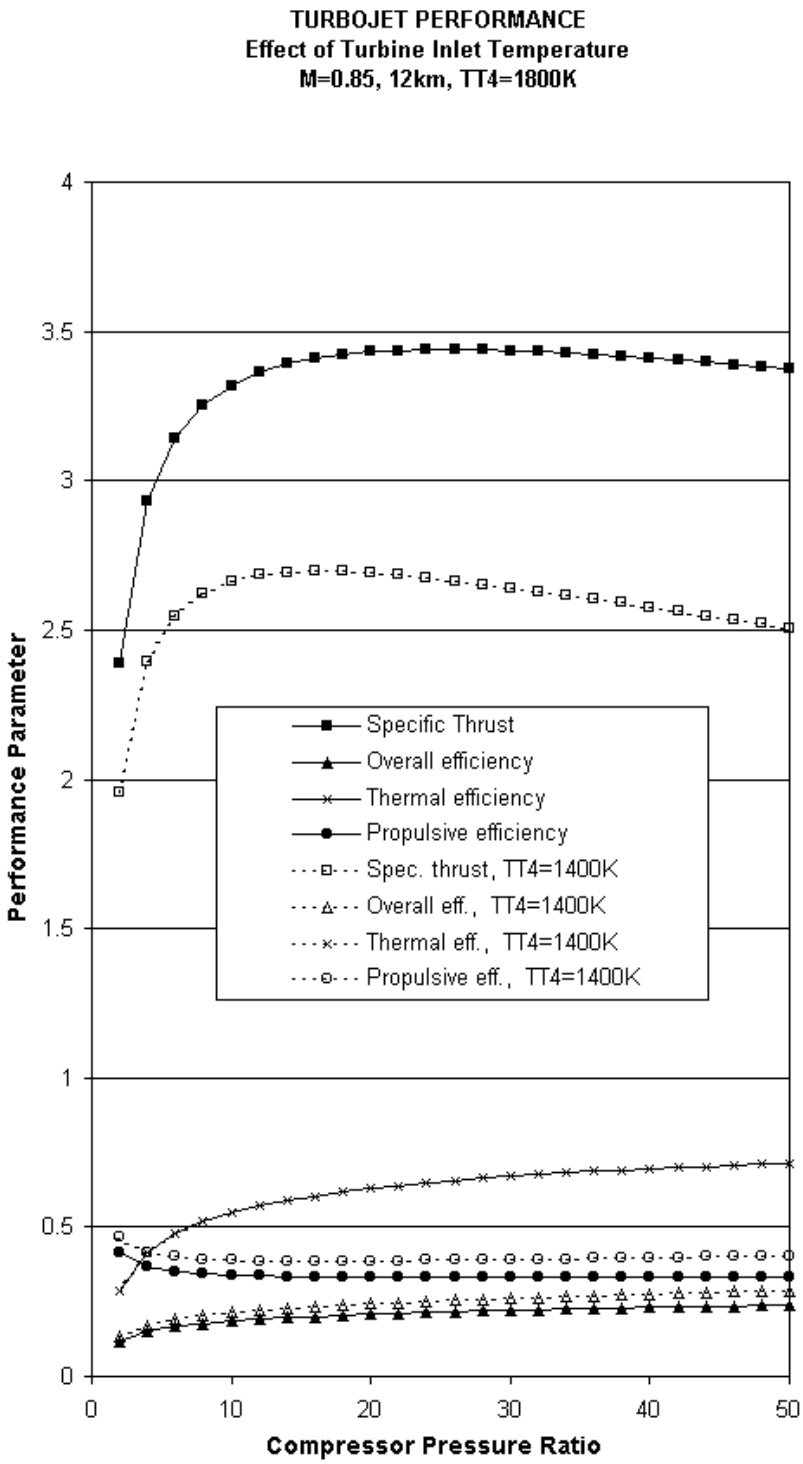

Figure 8.6 Performance of an ideal turbojet engine as a function of compressor pressure ratio and turbine inlet temperature.

IX. Energy Exchange with Moving Blades

A. Introduction

So far we have only looked at the thermodynamic results of compressors and turbines (π's and τ's). Now we will look in more detail at how the components of a gas turbine engine produce these effects. You will learn later (in 16.50 for example) that without heat transfer, it is only possible to change the total enthalpy of a fluid with an unsteady process (e.g. moving blades). Still we will use many of the steady flow tools that we have discussed in thermodynamics and propulsion by considering the steady flow in and out of a component as shown in Figure 8.1.

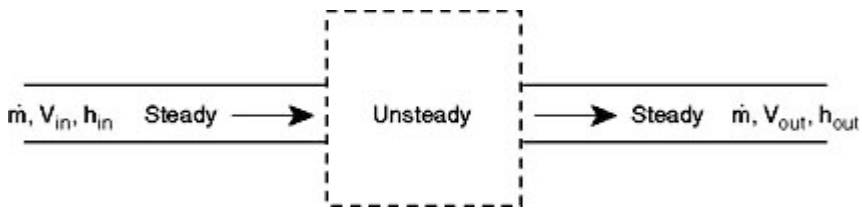

Figure 9.1 Control volume around compressor or turbine.

B. The Euler Turbine Equation

The Euler turbine equation relates the power added to or removed from the flow, to characteristics of a rotating blade row. The equation is based on the concepts of **conservation of angular momentum** and **conservation of energy**. We will work with the following model of the blade row:

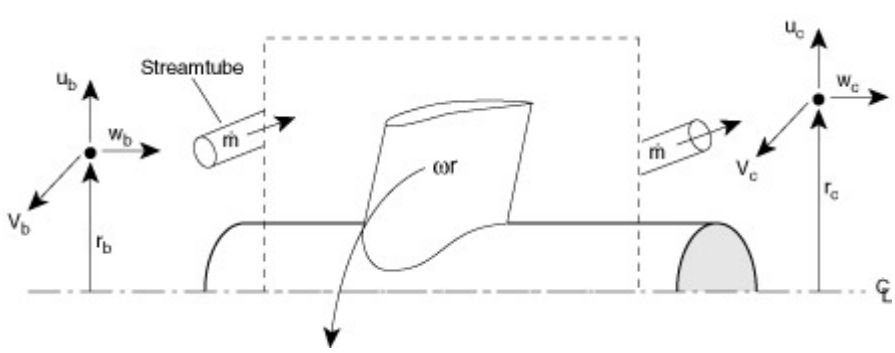

Figure 9.2 Control volume for Euler Turbine Equation.

Applying conservation of angular momentum, we note that the torque, **T**, must be equal to the time rate of change of angular momentum in a streamtube that flows through the device (**In the text:** For more information about angular momentum and rotational energy, see pages 246 and 558 in Hibbler).

$$\mathbf{T} = \dot{m}(v_c r_c - v_b r_b)$$

This is true whether the blade row is rotating or not. The sign matters (i.e. angular momentum is a vector -- positive means it is spinning in one direction, negative means it is spinning in the other direction). So depending on how things are defined, there can be positive and negative torques, and positive and negative angular momentum. In Figure 9.2, torque is positive when $V_{tangential\ out} > V_{tangential\ in}$ ---- the same sense as the angular velocity.

If the blade row is moving, then work is done on/by the fluid. The work per unit time, or power, ***P***, is the torque multiplied by the angular velocity, ω

$$\mathbf{P} = \mathbf{T}\omega = \omega\dot{m}(v_c r_c - v_b r_b)$$

If torque and angular velocity are of like sign, work is being done on the fluid (a compressor). If torque and angular velocity are of opposite sign work is being extracted from the fluid (a turbine). Here is another approach to the same idea:

- If the tangential velocity increases across a blade row (where positive tangential velocity is defined in the same direction as the rotor motion) then work is added to the flow (a compressor).

- If the tangential velocity decreases across a blade row (where positive tangential velocity is defined in the same direction as the rotor motion) then work is removed from the flow (a turbine).

From the steady flow energy equation

$$\dot{q} - \dot{w}_s = \dot{m}\Delta h_T \qquad \text{with} \qquad \dot{q} = 0 \quad \text{and} \quad -\dot{w}_s = P$$

$$\mathbf{P} = m(h_{T_c} - h_{T_b})$$

Then equating this expression of conservation of energy with our expression from conservation of angular momentum, we arrive at:

$$\boxed{\begin{array}{l} h_{T_c} - h_{T_b} = \omega\,(r_c v_c - r_b v_b) \\ \\ \text{or for a perfect gas with } C_p\text{=constant} \\ \\ C_p\,(T_{T_c} - T_{T_b}) = \omega\,(r_c v_c - r_b v_b) \end{array}}$$

Which is called the **<u>Euler Turbine Equation</u>**. It relates the temperature ratio (and hence the pressure ratio) across a turbine or compressor to the rotational speed and the change in

momentum per unit mass. Note that the velocities used in this equation are what we will later call absolute frame velocities (as opposed to relative frame velocities).

- If angular momentum increases across a blade row, then $T_{Tc} > T_{Tb}$ and work was done on the fluid (a compressor).
- If angular momentum decreases across a blade row, then $T_{Tc} < T_{Tb}$ and work was done by the fluid (a turbine).

C. Multistage Axial Compressors

An axial compressor is typically made up of many alternating rows of rotating and stationary blades called rotors and stators, respectively, as shown in Figures 9.3 and 9.4. The first stationary row (which comes in front of the rotor) is typically called the inlet guide vanes or IGV Each successive rotor-stator pair is called a compressor stage. Hence compressors with many blade rows are termed multistage compressors.

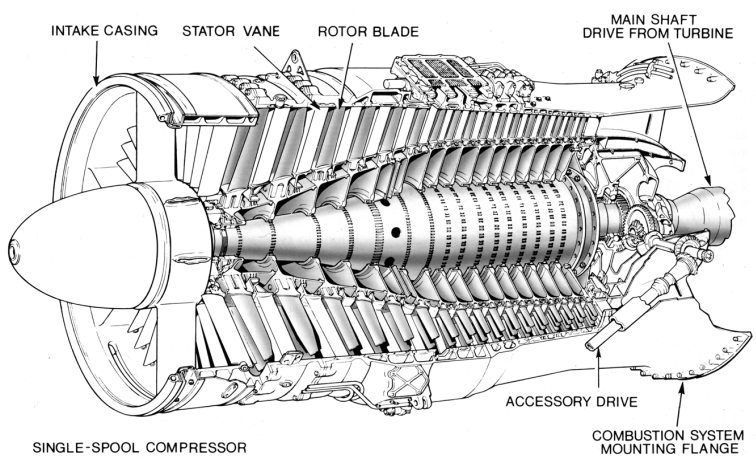

Figure 9.3 A typical multistage axial flow compressor (Rolls-Royce, 1992).

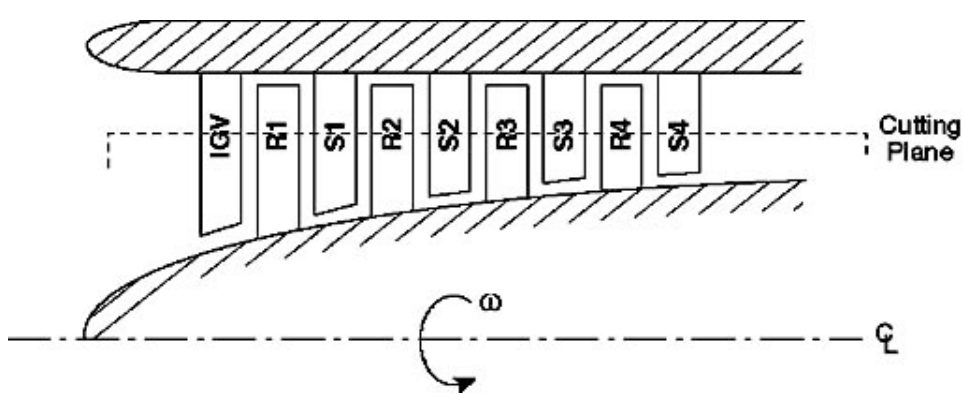

Figure 9.4 Schematic representation of an axial flow compressor.

One way to understand the workings of a compressor is to consider energy exchanges. We can get an approximate picture of this using the Bernoulli Equation, where P_T is the stagnation pressure, a measure of the total energy carried in the flow, p is the static pressure a measure of the internal energy, and the velocity terms are a measure of the kinetic energy associated with each component of velocity (u is radial, v is tangential, w is axial).

$$P_T = p + \frac{1}{2}\rho\left(u^2 + v^2 + w^2\right)$$

The rotor adds swirl to the flow, thus increasing the total energy carried in the flow by increasing the angular momentum (adding to the kinetic energy associated with the tangential or swirl velocity, $1/2\rho v^2$).

The stator removes swirl from the flow, but it is not a moving blade row and thus cannot add any net energy to the flow. Rather, the stator rather converts the kinetic energy associated with swirl to internal energy (raising the static pressure of the flow). Thus typical velocity and pressure profiles through a multistage axial compressor look like those shown in Figure 9.5.

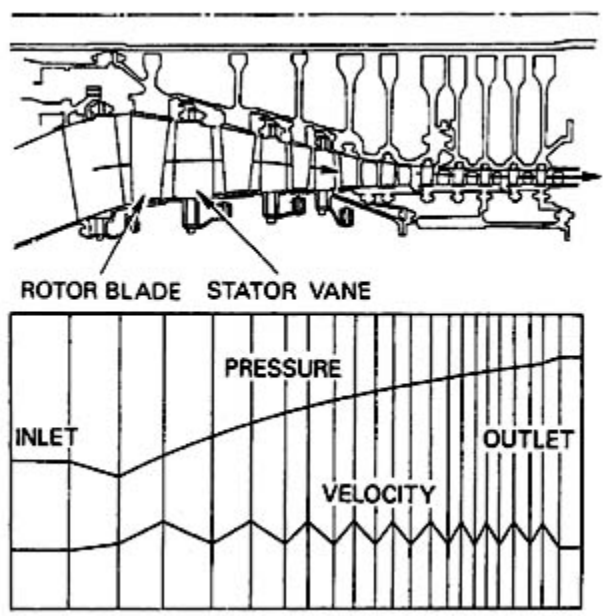

Figure 9.5 Pressure and velocity profiles through a multi-stage axial compressor (Rolls-Royce, 1992).

Note that the IGV also adds no energy to the flow. It is designed to add swirl in the direction of rotor motion to lower the Mach number of the flow relative to the rotor blades, and thus improve the aerodynamic performance of the rotor.

D. Velocity Triangles for an Axial Compressor Stage

Velocity triangles are typically used to relate the flow properties and blade design parameters in the **relative frame** (rotating with the moving blades), to the properties in the stationary or **absolute frame**.

We begin by "unwrapping" the compressor. That is, we take a cutting plane at a particular radius (e.g. as shown in Figure 9.3) and unwrap it azimuthally to arrive at the diagrams shown in Figure 9.6. Here we have assumed that the area of the annulus through which the flow passes is nearly constant and the density changes are small so that the axial velocity is approximately constant.

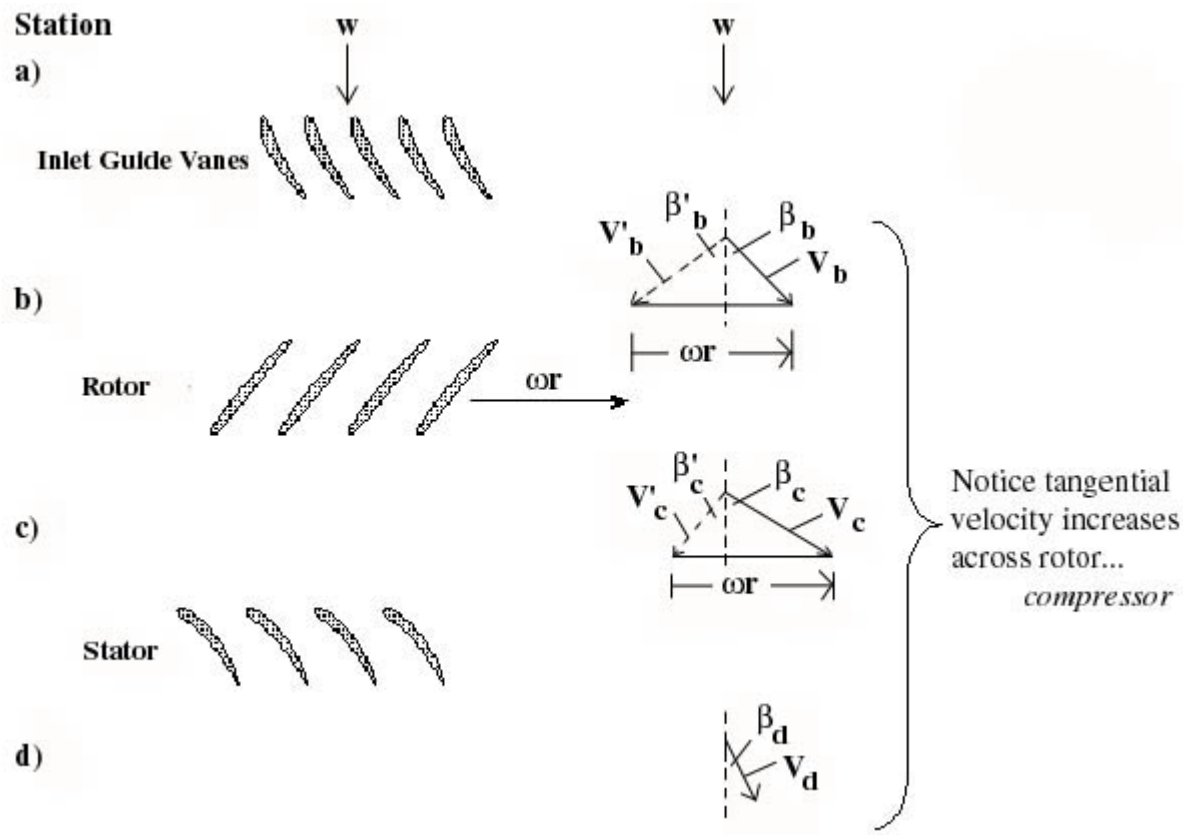

Figure 9.6 Velocity triangles for an axial compressor stage. Primed quantities are in the relative frame, unprimed quantities are in the absolute frame.

In drawing these velocity diagrams it is important to note that the flow typically leaves the trailing edges of the blades at approximately the trailing edge angle in the coordinate frame attached to the blade (i.e. relative frame for the rotor, absolute frame for the stator).

We will now write the Euler Turbine Equation in terms of stage design parameters: ω, the rotational speed, and β_b and β_c' the leaving angles of the blades.

$$C_p\,(T_{T_c} - T_{T_b}) = \omega\,(r_c v_c - r_b v_b)$$

From geometry,

$$v_b = w_b \tan \beta_b \qquad \text{and} \qquad v_c = w_c \tan \beta_c = \omega r_c - w_c \tan \beta'_c$$

so

$$C_p \left(T_{T_c} - T_{T_b}\right) = \omega\left(\omega r_c^2 - w_c r_c \tan\beta'_c - r_b w_b \tan\beta_b\right)$$

or

$$\frac{T_{T_c}}{T_{T_b}} = 1 + \frac{(\omega r_c)^2}{C_p T_{T_b}}\left[1 - \frac{w_c}{\omega r_c}\left(\tan\beta'_c + \frac{r_b w_b}{r w_c}\tan\beta_b\right)\right]$$

Stagnation temperature change across stage

Proportional to tip Mach number squared

Proportional to flow velocity through compressor

Set by blade design

So we see that the total or stagnation temperature rise across the stage increases with the tip Mach number squared, and for fixed positive blade angles, decreases with increasing mass flow. This behavior is represented schematically in Figure 9.7.

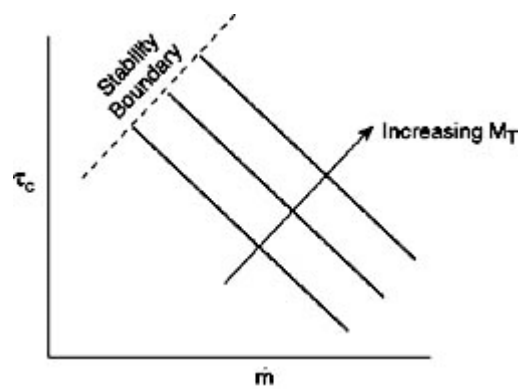

Figure 9.7 Compressor behavior

E. Velocity Triangles for an Axial Flow Turbine Stage

We can apply the same analysis techniques to a turbine. The stator, again does no work. It adds swirl to the flow, converting internal energy into kinetic energy. The turbine rotor then extracts work from the flow by removing the kinetic energy associated with the swirl velocity.

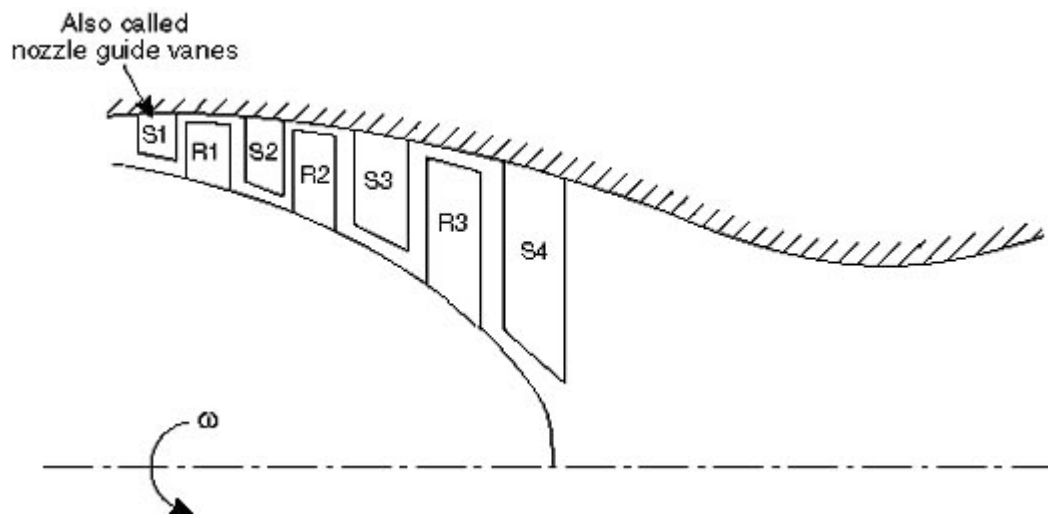

Figure 9.8 Schematic of an axial flow turbine.

The appropriate velocity triangles are shown in Figure 9.9, where again the axial velocity was assumed to be constant for purposes of illustration.

As we did for the compressor, we can write the Euler Turbine Equation in terms of useful design variables:

$$1-\frac{T_{T_c}}{T_{T_b}}=\frac{(\omega r)^2}{C_p T_{T_b}}\left[\frac{w_b}{\omega r}\tan\beta_b+\left(\frac{w_c}{\omega r}\tan\beta_c'-1\right)\right]$$

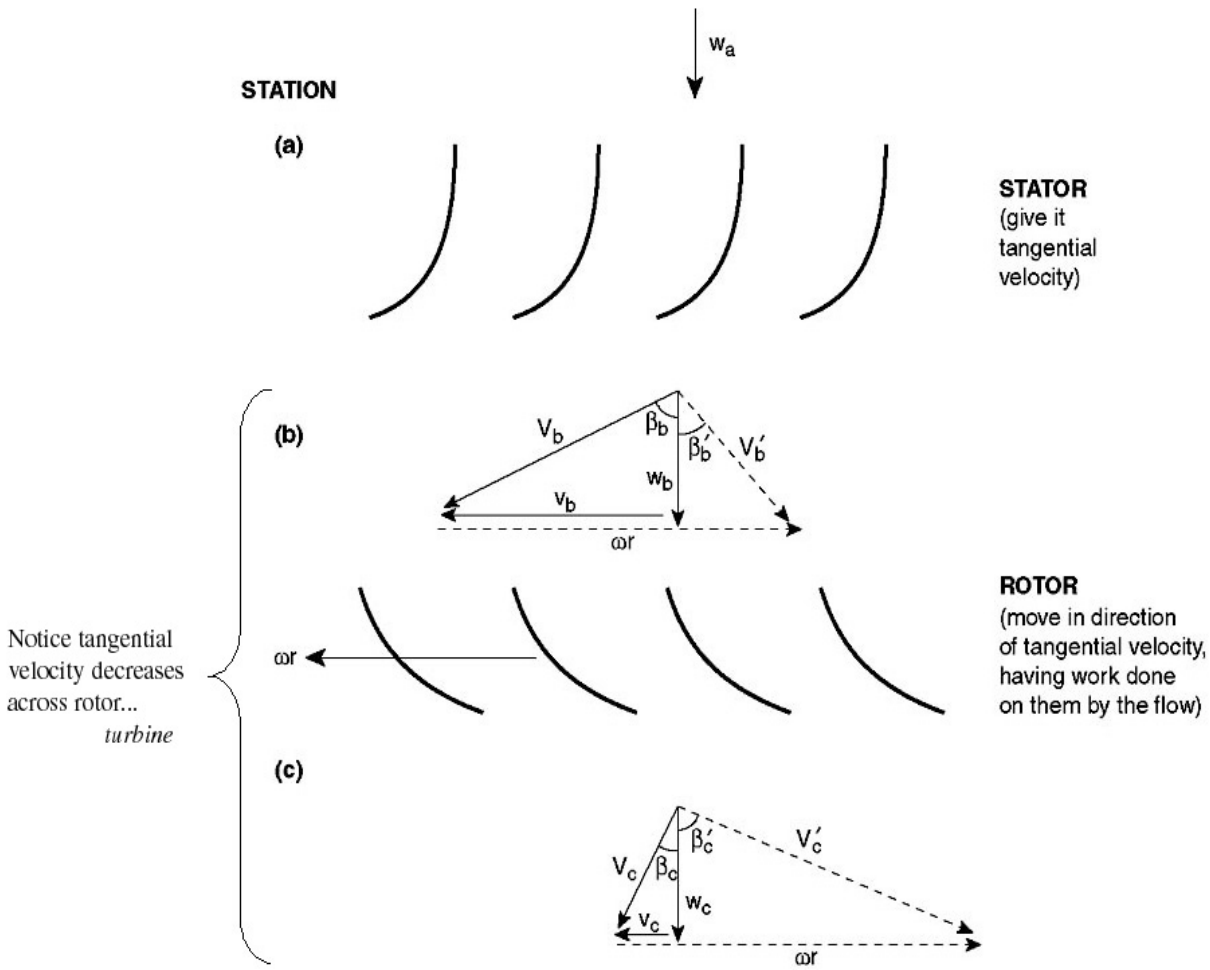

Figure 9.9 Velocity triangles for an axial flow turbine stage.