Name: SOLUTION

MIT ID (last four digits):

Unified Quiz S1

April 15, 2009

S-Portion

- Put the last four digits of your MIT ID # on each page of the exam.
- Do all work on that question on the page(s) provided. Use back of the page(s) if necessary.
- Show all your work, especially intermediate results. Partial credit cannot be given without intermediate results.
- Show the logical path of your work. Explain clearly your reasoning and what you are doing.
- Please be neat. It will be easier to identify correct or partially correct responses when the response is neat.
- Be sure to show the appropriate units throughout. Answers are not correct without the units.
- Box your final answers.

EXAM SCORING

Question	Scores
1	/ 24
2	/ 26
3	/ 30
4	/ 20
TOTAL	/ 100

Problem 1: Linearity, Time Invariance, and Causality [24 points]

Consider the following systems with a real input signal $x(t)$ and a real output signal $y(t)$.

System A:

$$y(t) = \frac{1}{3}\left[x(t-2) + x(t-1) + x(t)\right]$$

System B:

$$y(t) = \int_{-\infty}^{t} x(\tau)\cos(t-\tau)\,d\tau$$

System C:

$$y(t) = \begin{cases} x(t), & \text{if } x(t) \leq t_0 \\ A_0, & \text{if } x(t) > t_0 \end{cases}$$

System D:

$$y(t) = \lfloor x(t+5) \rfloor$$

where $\lfloor z \rfloor$ is defined as the largest integer smaller than or equal to z.

For each of the following questions give a detailed explanation justifying your answer.

(a) [6 points]

Is System A linear?	**Yes** or **No**	YES
Is System A time invariant?	**Yes** or **No**	YES
Is System A causal?	**Yes** or **No**	YES

Justification:

- $x_1(t) \longrightarrow y_1(t)$

$x_2(t) \longrightarrow y_2(t)$

$$a x_1(t) + b x_2(t) \longrightarrow y(t) = \frac{1}{3}\Big[a x_1(t-2) + b x_2(t-2) + a x_1(t-1) + b x_2(t-1) + a x_1(t) + b x_1(t)\Big]$$

$$= a y_1(t) + b y_2(t)$$

<u>Linear!</u>

- $x(t) \rightarrow y(t)$

$$x(t-T) \rightarrow \frac{1}{3}\left[x(t-T-2) + x(t-T-1) + x(t-T)\right]$$

$$= y(t-T)$$

<u>time invariant!</u>

- <u>Causal</u> since $y(t)$ depends only on the current and past two values of $x(t)$

(b) [6 points]

Is System B linear?	**Yes** or **No**	YES
Is System B time invariant?	**Yes** or **No**	YES
Is System B causal?	**Yes** or **No**	YES

Justification:

$x_1(t) \longrightarrow y_1(t)$

$x_2(t) \longrightarrow y_2(t)$

$$a x_1(t) + b x_2(t) \longrightarrow \int_{-\infty}^{t} \left[a\, x_1(\tau) + b\, x_2(\tau)\right] \cos(t-\tau)\, d\tau$$

$$= a\, y_1(t) + b\, y_2(t)$$

Linear !

$x(t) \to y(t)$

$$x(t-T) \longrightarrow \int_{-\infty}^{t} x(\tau - T) \cos(t-\tau)\, d\tau$$

let $\lambda = \tau - T$

$$= \int_{-\infty}^{t-T} x(\lambda) \cos(t - T - \lambda)\, d\lambda$$

$$= y(t-T)$$

time invariant !

causal since $y(t)$ depends on current and past values of $x(t)$.

(c) [6 points]

Is System C linear? Yes or No $\boxed{\text{No}}$

Is System C time invariant? Yes or No $\boxed{\text{YES}}$

Is System C causal? Yes or No $\boxed{\text{YES}}$

Justification:

$$x_1(t) = 2t_0 \quad \longrightarrow y_1(t) = A_0$$

$$x_2(t) = 4t_0 \quad \longrightarrow y_2(t) = A_0$$

$$\neq 2y_1(t)$$

not linear !

$$x(t) \longrightarrow y(t)$$

$$x(t-T) \longrightarrow \tilde{y}(t) = \begin{cases} x(t-T) & \text{if } x(t-T) \leq t_0 \\ A_0 & \text{if } x(t-T_0) > t_0 \end{cases}$$

$$= y(t-T)$$

time invariant !

Causal since $y(t)$ depends only on current value of $x(t)$.

(d) [6 points]

Is System D linear?	Yes or No	No
Is System D time invariant?	Yes or No	YES
Is System D causal?	Yes or No	No

Justification:

$x_1(t) = 1 \longrightarrow y_1(t) = 1$

$x_2(t) = 0.5\, x_1(t) \longrightarrow y_2(t) = 0$

$\neq 0.5\, y_1(t)$

not linear!

$x(t) \longrightarrow y(t)$

$x(t-T) \longrightarrow \lfloor x(t-T+5) \rfloor = y(t-T)$

time invariant!

not causal since $y(t)$ depends on future value of $x(t)$.

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Problem 2: Convolution [26 points]

In this problem you will be asked to compute the discrete and continuous convolution. *You will receive partial credit only if you show your work.*

(a) [12 points] The discrete-time functions $x_1[n]$ and $x_2[n]$ are given in Figures 1 and 2, respectively.

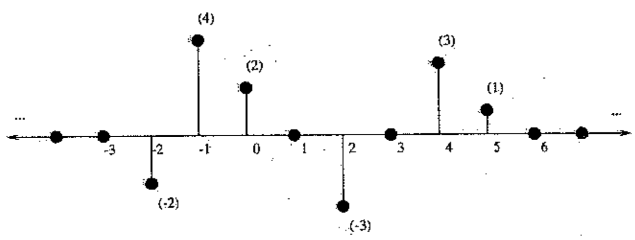

Figure 1: $x_1[n]$

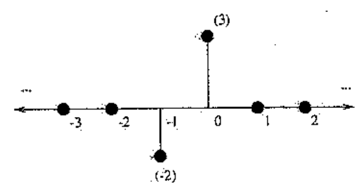

Figure 2: $x_2[n]$

Find the convolution $y[n] = x_1[n] * x_2[n]$.

$x_2[n] \longrightarrow \boxed{x_1[n]} \longrightarrow y[n]$

Provide a labeled plot of your final solution below:

	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$x_2[n]$					-2	4	2	0	-3	0	3	-1	
$3x_2[n]$					-6	12	6	0	-9	0	9	3	
$-2x[n+1]$				4	-8	-4	0	6	0	-6	-2		
$y[n]$				4	-4	8	6	6	-9	-6	7	3	

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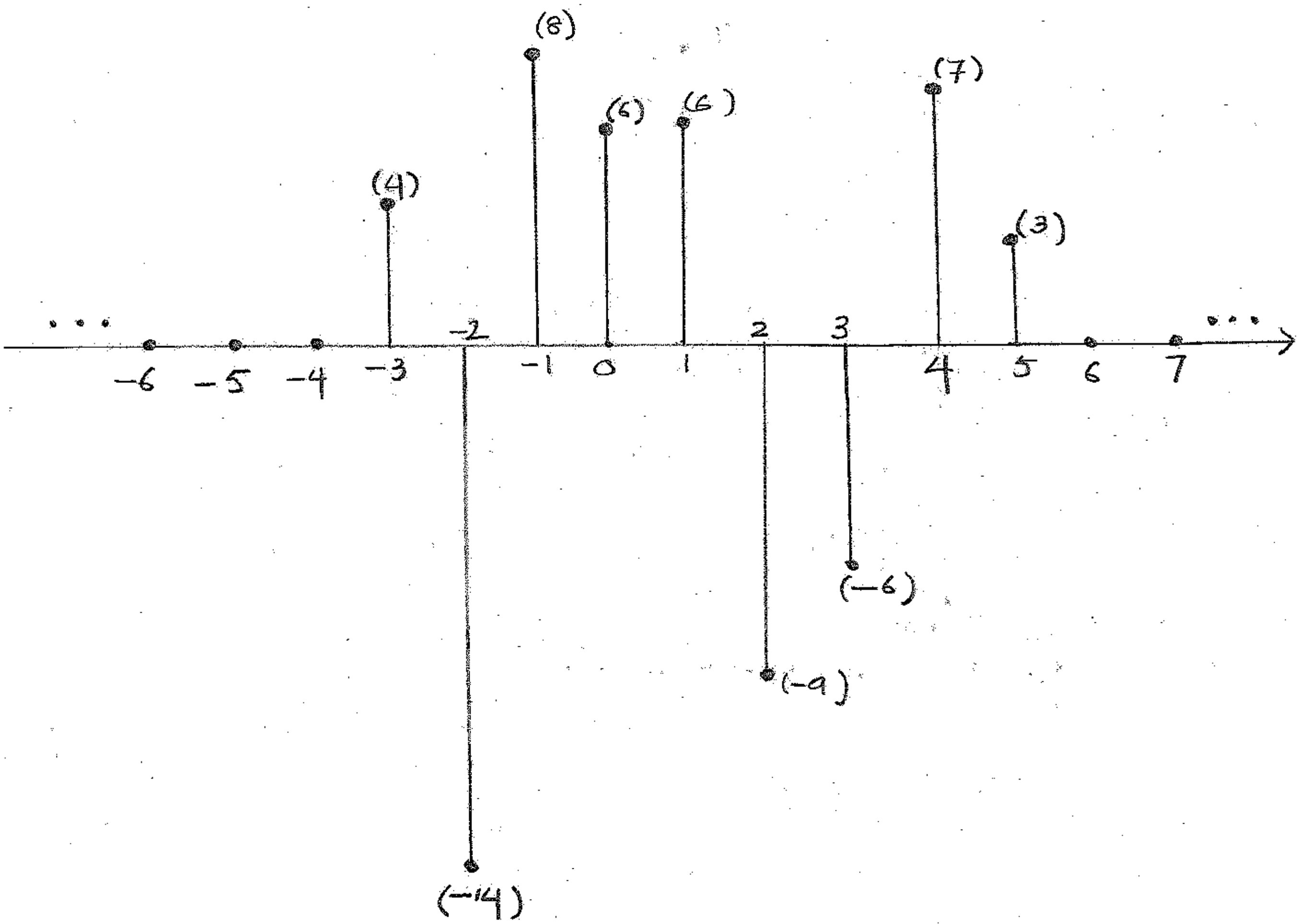


(b) [14 points] The continuous-time function $x_3(t)$ is given in Figure 3.

$x_3(t) = u(t) - 2u(t-\tau)$

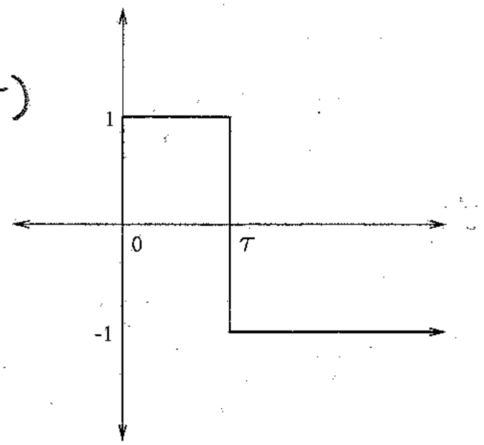
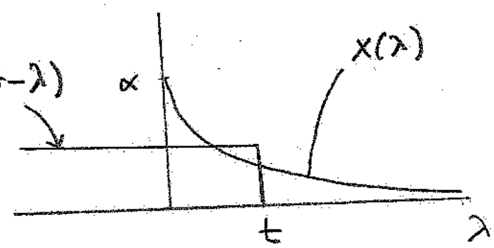


Figure 3: $x_3(t)$

Let $x_4(t) = \alpha e^{-\alpha t} u(t)$. Find the convolution $y(t) = x_3(t) * x_4(t)$.



$$y_a(t) = u(t) * x_4(t)$$

$$= \int_0^\infty u(t-\lambda)\, \alpha e^{-\alpha\lambda}\, d\lambda$$

$$= 0 \quad \text{if } t < 0$$

$$y_a(t) = \int_0^t \alpha e^{-\alpha\lambda}\, d\lambda \quad \text{if } t \geq 0$$

$$= 1 - e^{-\alpha t}$$

$$\therefore \quad y_a(t) = \left[1 - e^{-\alpha t}\right] u(t)$$

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By Linearity,

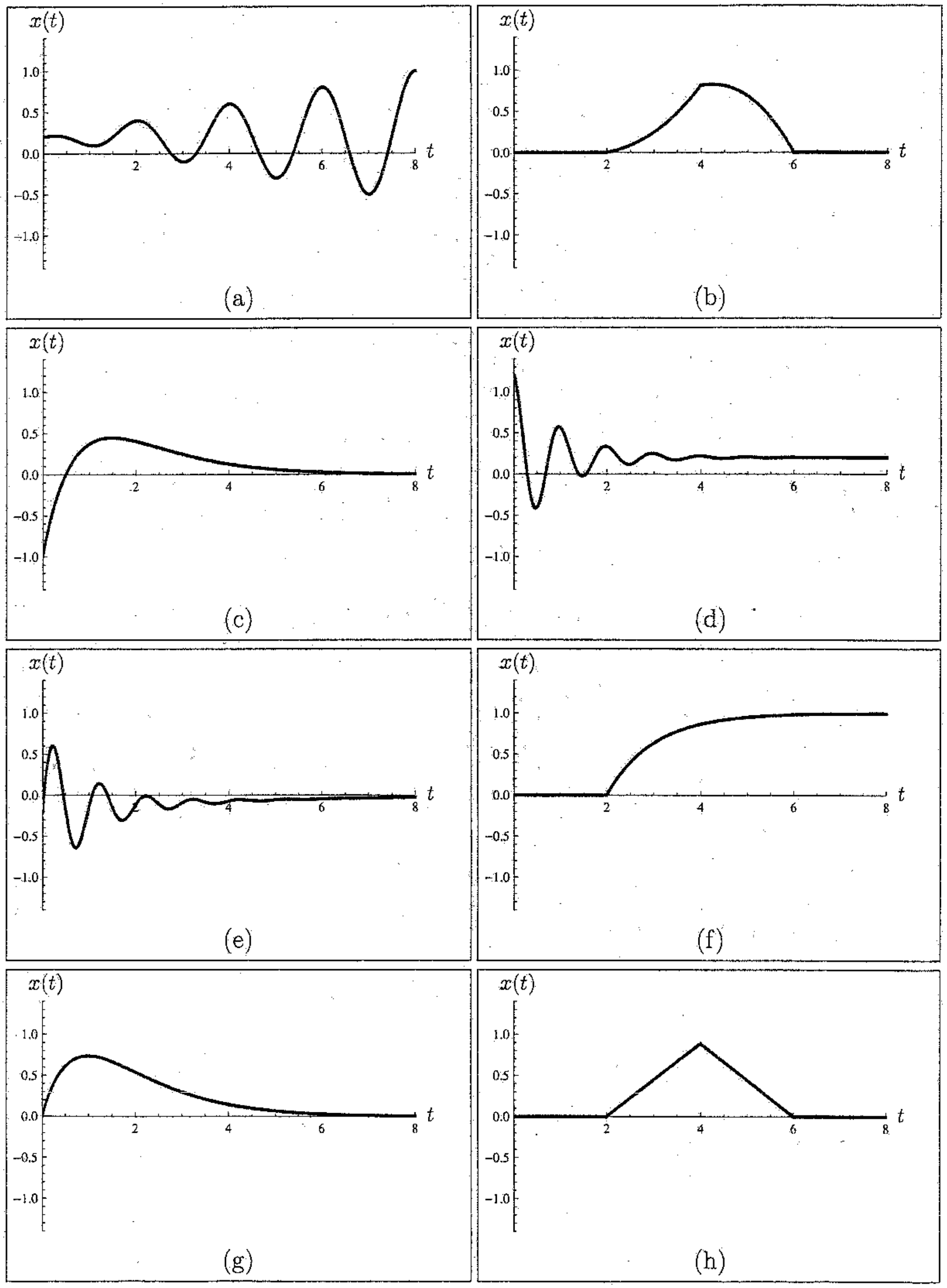
$$\boxed{y(t) = \left[1 - e^{-\alpha t}\right] u(t) - 2\left[1 - e^{-\alpha(t-T)}\right] u(t-T)}$$

Problem 3: Inverse Laplace Transforms [30 points]

For each of the unilateral Laplace transforms listed in the table below select the appropriate time signal $x(t)$ from the plots given on the next page.

You must provide a brief explanation for your answer, describing which transform properties you used to determine your answer. Please do so **on page 14**.

	Laplace Transform	Function
I	$\frac{1}{s}\frac{e^{-2s}}{s+1}$	(F)
II	$-\frac{d}{ds}\left(\frac{1}{10}\frac{s}{s^2+\pi^2}\right)+\frac{2}{10}\frac{1}{s}$	(a)
III	$\frac{2}{(s+1)^2}$	(g)
IV	$\left(\frac{1}{s}e^{-s}-\frac{1}{s}e^{-3s}\right)^2$	(h)



(a) (b) (c) (d) (e) (f) (g) (h)

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Justification for I:[7 points]

$$e^{-(t-2)}u(t-2) \longleftrightarrow \frac{e^{-2s}}{s+1}$$

$$\int_0^t e^{-(\lambda-2)}u(\lambda-2)\,d\lambda \longleftrightarrow \frac{1}{s}\,\frac{e^{-2s}}{s+1}$$

$$= \int_0^{t-2} e^{-y}\,dy = \left[1-e^{-(t-2)}\right]u(t-2)$$

Justification for II:[8 points]

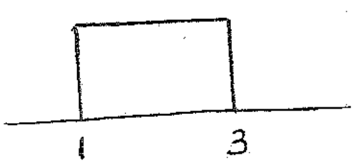
$$\frac{1}{10}\,t\cos \pi t + \frac{2}{10}\,u(t)$$

Justification for III:[7 points]

$$2te^{-t}$$

Justification for IV:[8 points]

$$u(t-1) - u(t-3) \longleftrightarrow \frac{1}{s}e^{-s} - \frac{1}{s}e^{-3s}$$



$$x(t) * x(t) \longleftrightarrow X(s)\;X(s)$$

Problem 4: Laplace Transform [20 points]

In this problem you will compute the Laplace transform.

You will be graded on the correctness of the method; if you obtain the correct answer without showing your work, you will not receive most of the credit for this problem.

(a) [10 points] Determine the Laplace transform of the following function:

$$y(t) = \cosh(\beta t)u(t)$$

[*Hint*: $\cosh(\beta t) = \frac{e^{\beta t}+e^{-\beta t}}{2}$]

$$\mathcal{L}\{y(t)\} = \int_0^\infty \frac{e^{\beta t} + e^{-\beta t}}{2} u(t) e^{-st} dt$$

$$= \frac{1}{2} \int_0^\infty e^{-(s-\beta)t} dt + \frac{1}{2} \int_0^\infty e^{-(s+\beta)t} dt$$

$$= \frac{1}{2} \frac{1}{s-\beta} + \frac{1}{2} \frac{1}{s+\beta}$$

$$\boxed{\mathcal{L}\{y(t)\} = \frac{s}{s^2 - \beta^2}}$$

(b) [10 points] Determine the Laplace transform of the following function:

$$y(t) = (t-b)^2 e^{a(t-b)} \cosh(\beta(t-b))u(t-b)$$

$$\mathcal{L}\{y(t)\} = e^{-sb} \, \mathcal{L}\{t^2 e^{at} \cosh(\beta t)\, u(t)\}$$

$$= e^{-sb} \frac{d^2}{ds^2} \mathcal{L}\left\{e^{+at}\left[\frac{e^{\beta t} + e^{-\beta t}}{2}\right]u(t)\right\}$$

$$= \frac{1}{2} e^{-sb} \frac{d^2}{ds^2}\left[\frac{1}{(s-a-\beta)} + \frac{1}{(s-a+\beta)}\right]$$

$$= \frac{1}{2} e^{-sb}\left[\frac{2}{(s-a-\beta)^3} + \frac{2}{(s-a+\beta)^3}\right]$$

$$\boxed{\mathcal{L}\{y(t)\} = e^{-sb}\left[\frac{1}{(s-a-\beta)^3} + \frac{1}{(s-a+\beta)^3}\right]}$$

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An Abbreviated List of Laplace Transform Pairs

Type	$f(t)\ (t > 0-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

An Abbreviated List of Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
nth derivative (time)	$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{df^2(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a),\ a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at),\ a > 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$