Note Title 5/3/2009

Inverse Laplace Transform

In general, we need to find the inverse transform of a function that has the form

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}$$

the coefficients a_n's and b_m's are real constants

$F(s)$ is a proper rational function if $m > n$
an improper " " $m \leq n$

- We first consider proper rational function. For improper rational function, we use long hand division to turn it into proper rational function (we will come back to this point later).

- We will consider

A) Proper rational functions with distinct real roots

B) " " " " complex "

C) " " " repeated real roots

D) " " " " complex "

We will illustrate each of the four cases using specific example.

A) Proper rational functions with distinct real roots

- Consider determining the coefficients in a partial fraction expansion when the roots of $D(s)$ are real and distinct.

  To find a "K" associated with a term that arises because of a distinct root of $D(s)$

  - multiply both side of the identity by a factor equal to the denominator beneath the desired K.
  - evaluate both side of the identity at the root corresponding to the multiplying factor

- example:

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} = \frac{K_1}{s} + \frac{K_2}{s+8} + \frac{K_3}{s+6}$$

- To find K_1,

$$K_1 = \lim_{s \to 0} s\,F(s)$$

$$= \lim_{s \to 0} \frac{96(s+5)(s+12)}{(s+8)(s+6)}$$

$$= \frac{\overset{12}{\cancel{96}}(5)(\overset{2}{\cancel{12}})}{(\cancel{8})(\cancel{6})}$$

$$K_1 = 120$$

- To find K_2,

$$K_2 = \lim_{s \to -8} (s+8) F(s)$$

$$= \lim_{s \to -8} \frac{96(s+5)(s+12)}{s\ (s+6)}$$

$$= \frac{\overset{12}{\cancel{96}}(-3)(\overset{2}{\cancel{4}})}{-\cancel{8}(-\cancel{2})}$$

$$K_2 = -72$$

- To find K_3,

$$K_3 = \lim_{s \to -6} (s+6) F(s)$$

$$= \lim_{s \to -6} \frac{96(s+5)(s+12)}{s(s+8)}$$

$$= \frac{\overset{48}{\cancel{96}}(-1)(\cancel{6})}{-\cancel{6}(\cancel{2})}$$

$$\underline{K_3 = 48}$$

Therefore

$$F(s) = \frac{96(s+5)(s+12)}{s(s+8)(s+6)} = \frac{120}{s} - \frac{72}{s+8} + \frac{48}{s+6}$$

At this point it is a good idea to protect against algebraic errors.

Since the above equation must hold for all values of s, we can test at convenient points namely, $s = -5$ and $s = -12$

Letting $s = -5$ gives

$$0 = \frac{\overset{24}{\cancel{120}}}{-\cancel{5}} - \frac{\overset{24}{\cancel{72}}}{\cancel{3}} + \frac{48}{1}$$

$$= -24 - 24 + 48$$

$$= 0 \qquad \text{check at } s = -5$$

Letting $s = -12$ gives

$$0 = \frac{\cancel{120}^{10}}{-\cancel{12}} + \frac{\cancel{72}^{18}}{\cancel{4}} + \frac{\cancel{48}^{8}}{-\cancel{6}}$$

$$= -10 + 18 - 8$$

$$= 0 \qquad \text{check at } s = -12$$

Note: While checking at 2 points does not gaurantee that our solution at hand is correct, this is a good check.

Finally

$$\mathcal{L}^{-1}\{F(s)\} = \left(120 + 48e^{-6t} - 72e^{-8t}\right) u(t)$$

B) Proper rational functions with distinct complex roots

The only difference between finding the coefficients associated with distinct complex roots and finding those associated with distinct real roots is that, the algebra associated with "the distinct complex roots case" involves complex numbers.

example:

$$F(s) = \frac{100\,(s+3)}{(s+6)(s^2+6s+25)}$$

note that

$$(s^2+6s+25) = (s+3-j4)(s+3+j4)$$

$$F(s) = \frac{K_1}{(s+6)} + \frac{K_2}{(s+3-j4)} + \frac{K_3}{(s+3+j4)}$$

$$K_1 = \lim_{s \to -6} (s+6)\,F(s)$$

$$= \lim_{s \to -6} \frac{100\,(s+3)}{s^2+6s+25}$$

$$= \frac{100(-3)}{36 - 36 + 25}$$

$$K_1 = -12$$

$$K_2 = \lim_{s \to -3+j4} (s+3-j4)\, F(s)$$

$$= \lim_{s \to -3+j4} \frac{100(s+3)}{(s+6)(s+3+j4)}$$

$$= \frac{100\,(j4)}{(3+j4)\,(j8)}$$

$$K_2 = 6 - j8 = 10\, e^{-j\,53.13^\circ}$$

$$K_2 = \lim_{s \to -3-j4} (s+3+j4)\, F(s)$$

$$= \lim_{s \to -3-j4} \frac{100(s+3)}{(s+6)(s+3-j4)}$$

$$= \frac{100\,(-j4)}{(3-j4)(-j8)}$$

$$K_3 = 6+j8 = 10\,e^{+j\,53.13^\circ}$$

Therefore

$$F(s) = \frac{-12}{(s+6)} + \frac{10\,e^{-j53.13^\circ}}{(s+3-j4)} + \frac{10\,e^{+j53.13^\circ}}{(s+3+j4)}$$

Observations:

- complex roots are in complex conjugate pairs
- K_3 is the conjugate of K_2

  Thus for complex conjugate pairs, we only need to calculate only half of the coefficients.

Check at $s = -3$

$$0 = \frac{-12}{3} + \frac{10\,e^{-j53.13^\circ}}{-j4} + \frac{10\,e^{+j53.13^\circ}}{+j4}$$

$$= -4 + 2.5\,e^{+j36.87} + 2.5\,e^{-j36.87^\circ}$$

$$= -4 + 2.0 + j1.5 + 2.0 - j1.5$$

$$= 0$$

check at $s = -3$

Finally

$$\mathcal{L}^{-1}\left\{F(s)\right\} =$$

$$\left[-12e^{-6t} + 10e^{-j53.13^\circ}e^{-(3-j4)t} + 10e^{+j53.13^\circ}e^{-(3+j4)t}\right]u(t)$$

Since, the terms involving imaginary components comes in complex conjugate pairs,

$$10e^{-j53.13^\circ}e^{-(3-j4)t} + 10e^{+j53.13^\circ}e^{-(3+j4)t}$$

$$= 10e^{-3t}\left[e^{j(4t-53.13^\circ)} + e^{-j(4t-53.13^\circ)}\right]$$

$$= 20e^{-3t}\cos(4t - 53.13^\circ)$$

Thus

$$\mathcal{L}^{-1}\{F(s)\} = \left[-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ)\right]u(t)$$

Summary:

Whenever $D(s)$ contains distinct complex roots, i.e., factors of the form

$$(s+\alpha-j\beta)(s+\alpha+j\beta)$$

then a pair of terms of the form

$$\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}$$

appears in the partial fraction expansion.

In polar form

$$K = |K|\, e^{+j\theta}$$

$$K^* = |K|\, e^{-j\theta}$$

$$\boxed{\mathcal{L}^{-1}\left\{ \frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta} \right\} = 2|K|\, e^{-\alpha t} \cos(\beta t + \theta)\, u(t)}$$

Note: Applying above Formula, it is important to note that

- K is defined as the coefficient associated with the denominator term $s+\alpha-j\beta$
- K^* is defined as the coefficient associated with the denominator term $s+\alpha+j\beta$

c) Proper rational functions with repeated real roots

To find the coefficents associated with the terms generated by a multiple root of multiplicity r, we use the approach illustrated by the following example.

$$F(s) = \frac{100(s+25)}{s(s+5)^3}$$

$$= \frac{K_1}{s} + \frac{K_2}{(s+5)^3} + \frac{K_3}{(s+5)^2} + \frac{K_4}{(s+5)}$$

$$K_1 = \lim_{s \to 0} s\,F(s)$$

$$= \lim_{s \to 0} \frac{100(s+25)}{(s+5)^3}$$

$$= \frac{100(\cancel{25})}{\cancel{125}_{\,5}}$$

$$K_1 = 20$$

$$K_2 = \lim_{s \to -5} (s+5)^3 F(s)$$

$$= \lim_{s \to -5} \frac{100(s+25)}{s}$$

$$= \frac{100(\cancel{20}^{\,4})}{-\cancel{5}}$$

$$K_2 = -400$$

$$K_3 = \lim_{s \to -5} \left[\frac{d}{ds} (s+5)^3 F(s) \right]$$

$$= \lim_{s \to -5} \left[\frac{d}{ds} \frac{100(s+25)}{s} \right]$$

$$= \lim_{s \to -5} \frac{d}{ds} \left(100 + \frac{2500}{s} \right)$$

$$= \lim_{s \to -5} -\frac{2500}{s^2}$$

$$K_3 = -100$$

$$K_4 = \lim_{s \to -5} \left[\frac{1}{2} \frac{d^2}{ds^2} (s+5)^3 F(s) \right]$$

$$= \lim_{s \to -5} \left[\frac{1}{2} \frac{d^2}{ds^2} \frac{100(s+25)}{s} \right]$$

$$= \lim_{s \to -5} \left[\frac{1}{2} \frac{d}{ds} \left(-\frac{2500}{s^2} \right) \right]$$

$$= \lim_{s \to -5} \left[\frac{1}{\cancel{2}} + \cancel{2} \frac{2500}{s^3} \right]$$

$$\underline{K_4 = -20}$$

Therefore

$$F(s) = \frac{100(s+25)}{s(s+5)^3}$$

$$= \frac{20}{s} + \frac{-400}{(s+5)^3} + \frac{-100}{(s+5)^2} + \frac{-20}{(s+5)}$$

check at $s = -25$

$$0 = \frac{20}{-25} + \frac{\cancel{400}}{(20)^{\cancel{3}}} - \frac{\cancel{100}}{\cancel{20^2}_{4}} + \frac{\overset{1}{\cancel{20}}}{\cancel{20}}$$

$$0 = -\frac{4}{5} + \frac{1}{20} - \frac{1}{4} + 1$$

$$= \frac{-16 + 1 - 5 + 20}{20}$$

$$= 0$$ check at $s = -25$

$$\mathcal{L}^{-1}\{F(s)\} =$$

$$\left[20 - 400t^2 e^{-5t} - 100te^{-5t} - 20e^{-5t}\right]u(t)$$

c) Proper rational functions with repeated complex roots

The only difference between finding the coefficents associated with repeated complex roots and finding those associated with repeated real roots is that, the algebra associated with "the repeated complex roots case" involves complex numbers.

example

$$F(s) = \frac{768}{(s^2+6s+25)^2}$$

$$= \frac{768}{(s+3-j4)^2(s+3+j4)^2}$$

$$= \frac{K_1}{(s+3-j4)^2} + \frac{K_2}{(s+3-j4)}$$

$$\frac{K_1^*}{(s+3+j4)^2} + \frac{K_2^*}{(s+3+j4)}$$

$$K_1 = \lim_{s \to -3+j4} (s+3-j4)^2 F(s)$$

$$= \lim_{s \to -3+j4} \frac{768}{(s+3+j4)^2}$$

$$= \frac{768}{(j8)^2}$$

$$\underline{K_1 = -12}$$

$$K_2 = \lim_{s \to -3+j4} \frac{d}{ds} (s+3-j4)^2 F(s)$$

$$= \lim_{s \to -3+j4} \frac{-2(768)}{(s+3+j4)^3}$$

$$= \frac{-2(768)}{(j8)^3}$$

$$\underline{K_2 = -j3} = 3e^{-j90^\circ}$$

$$\Rightarrow \quad K_1^* = -12$$

$$K_2^* = 3e^{+j90}$$

Therefore

$$F(s) = \frac{-12}{(s+3-j4)^2} + \frac{3e^{-j90^\circ}}{(s+3-j4)}$$

$$\frac{-12}{(s+3+j4)^2} + \frac{3e^{+j90^\circ}}{(s+3+j4)}$$

$$\mathcal{L}^{-1}\{F(s)\} = \Big[-24te^{-3t}\cos(4t).$$

$$+6e^{-3t}\cos(4t-90^\circ)\Big]u(t)$$

Summary:

If $F(s)$ has a real root α of multiplicity r in its denominator, the term in partial fraction expansion is of the form

$$\frac{K}{(s+\alpha)^r}$$

The inverse transform of this term is

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha)^r}\right\} = \frac{K\, t^{r-1} e^{-\alpha t}}{(r-1)!} \quad u(t)$$

If $F(s)$ has a complex root of $(\alpha+j\beta)$ of multiplicity r in its denominator, the term in partial fraction expansion is the conjugate pair

$$\frac{K}{(s+\alpha-j\beta)^r} + \frac{K^*}{(s+\alpha+j\beta)^r}$$

The inverse transform of this term is

$$\mathcal{L}^{-1}\left\{\frac{K}{(s+\alpha-j\beta)^r}+\frac{K^*}{(s+\alpha+j\beta)^r}\right\}$$

$$=\left[\frac{2|K|\,t^{r-1}}{(r-1)!}e^{-\alpha t}\cos(\beta t+\theta)\right]u(t)$$

where $K=|K|e^{j\theta}$

Partial Fraction Expansion for Improper Rational Functions

Improper rational functions pose no serious obstacle since they can be expanded into a polynomial plus a proper rational function.

example:

$$F(s) = \frac{s^4 + 13s^3 + 66s^2 + 200s + 300}{s^2 + 9s + 20}$$

Long hand division gives

$$= s^2 + 4s + 10 + \frac{30s + 100}{s^2 + 9s + 20}$$

expanding

$$\frac{30s + 100}{s^2 + 9s + 20} = \frac{30s + 100}{(s+4)(s+5)}$$

$$= \frac{-20}{(s+4)} + \frac{50}{(s+5)}$$

Therefore

$$F(s) = s^2 + 4s + 10 + \frac{-20}{(s+4)} + \frac{50}{(s+5)}$$

$$\mathcal{L}^{-1}\{F(s)\} = \left[\frac{d^2}{dt^2}\delta(t) + 4\frac{d}{dt}\delta(t) + 10\,\delta(t) - 20e^{-4t} + 50e^{-5t}\right]u(t)$$