Design Variable Concepts

19 Mar 09

Lab 7 Lecture Notes

Nomenclature

W	total weight $(= W_{\text{wing}} + W_{\text{fuse}} + W_{\text{pay}})$	b	wing span
S	reference area (wing area)	c	average wing chord
$A\!\!\!R$	wing aspect ratio	$T_{\max}$	maximum thrust
c_r	root wing chord	C_L	lift coefficient
c_t	tip wing chord	C_D	drag coefficient
λ	taper ratio $(= c_t/c_r)$	c_d	wing profile drag coefficient
I_o	wing root bending inertia	CDA_0	drag area of non-wing components
E	Young's modulus	κ_o	wing root bending curvature
M_o	wing root bending moment	δ	tip deflection

Design Space

Design Variables are numbers whose values can be freely varied by the designer to define a designed object. As a very simple example, consider a rectangular wing with a pre-defined airfoil. It can be defined by deciding on the values of the following two design variables:

$$\{\, b \,,\, c \,\} \tag{1}$$

Placing these variables along orthogonal axes defines a *design space*, or set of all possible design options. Each point in the design space corresponds to a chosen design, as illustrated in Figure 1.

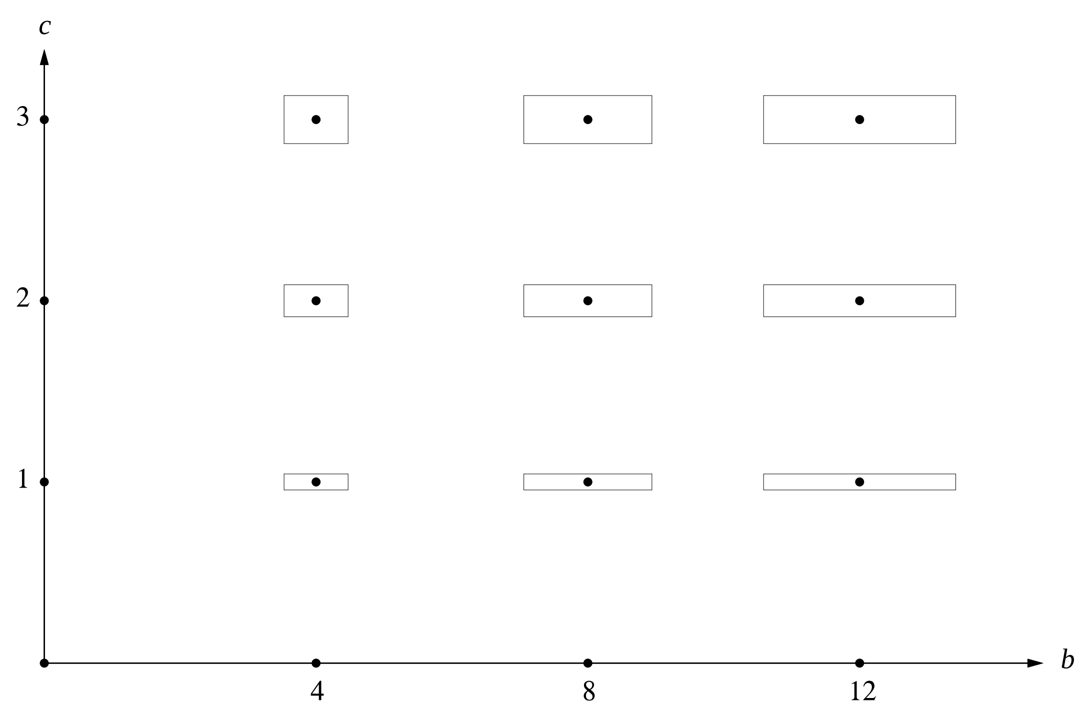

Figure 1: Two-variable design space of a rectangular wing.

Variable Set Choice

Frequently, an alternative variable set can be defined in terms of the starting variable set. For example, we can define the same design space using the variable set

$$\{\, AR\,,\, S \,\} \tag{2}$$

with the following relations translating between the two alternative variable sets:

$$\begin{aligned} AR &= b/c \qquad & b &= \sqrt{S \times AR} \\ S &= b\,c \qquad & c &= \sqrt{S/AR} \end{aligned} \tag{3}$$

Figure 2 shows the alternative $\{\, AR\,,\, S \,\}$ design space. Other variable sets are possible for

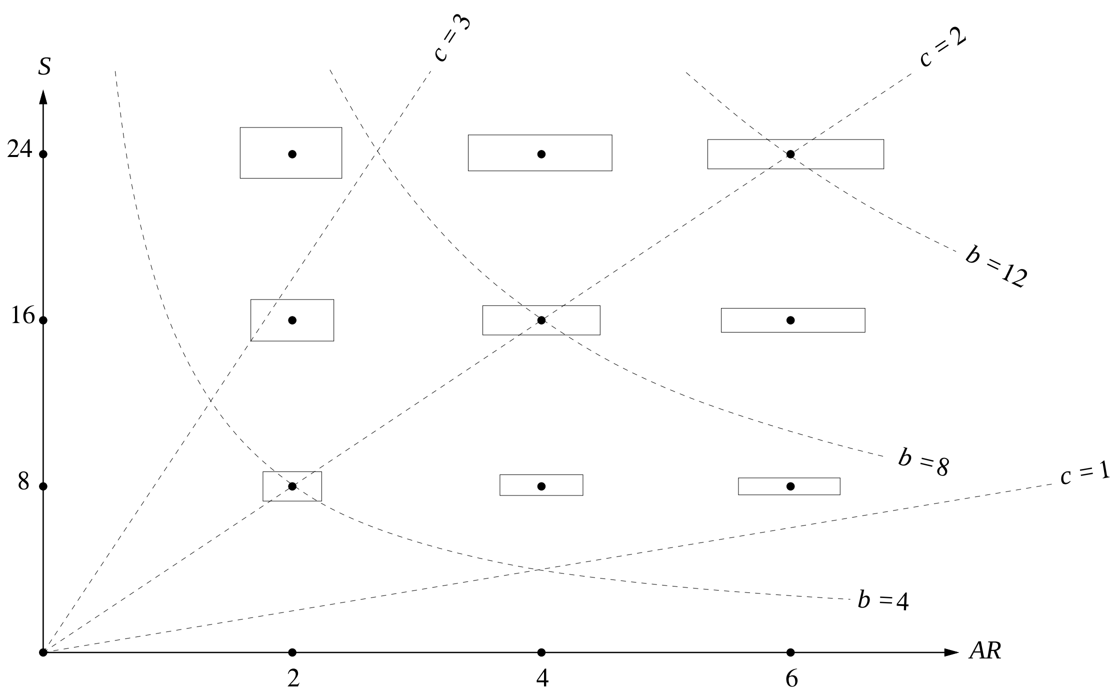

Figure 2: Alternative design variable set of a rectangular wing.

this case, such as $\{AR, b\}$, $\{S, b\}$, etc. The best variable set is usually the one which gives the simplest or clearest means to evaluate the *objective function*, or performance of the design, so that the best point in the design space can be selected.

Example Objective Function

As an example, we wish to determine a wing design which will maximize the payload weight of an electric aircraft, whose motor and propeller can generate at most some given maximum thrust $T_{\max}$. We begin with the level flight force-equilibrium relations,

$$W = L \tag{4}$$

$$T_{\max} = D = L\frac{C_D}{C_L} = W\frac{C_D}{C_L} \tag{5}$$

$$T_{\max} \;=\; (W_{\rm fuse} + W_{\rm wing} + W_{\rm pay}) \left(\frac{CDA_0/S}{C_L} \;+\; \frac{c_d}{C_L} \;+\; \frac{C_L}{\pi \, AR} \right) \tag{6}$$

Therefore, the objective function to be maximized is obtained by solving for the payload weight.

$$W_{\rm pay}(AR, S) \;=\; \frac{T_{\max}}{\dfrac{CDA_0/S}{C_L} + \dfrac{c_d}{C_L} + \dfrac{C_L}{\pi \, AR}} \;-\; W_{\rm fuse} - W_{\rm wing} \tag{7}$$

Since AR and S appear explicitly in this objective function definition, these are probably the best choices for the design variables.

Objective Function Contours

In practice, the dependence of $W_{\rm pay}$ on $\{AR, S\}$ is far more complex than what's explicitly visible in equation (7). For example, the wing weight $W_{\rm wing}$ will clearly depend on $\{AR, S\}$, as will c_d via the chord Reynolds number. Also, $T_{\max}$ will likely depend on the flight speed, which is influenced by wing loading and hence by S. Given quantitative models of all these effects, we can numerically determine the value of $W_{\rm pay}$ for every $\{AR, S\}$ combination. The results might be as shown in Figure 3, which shows the objective function as contours, or isolines. The point where the objective function has a maximum represents the optimum design.

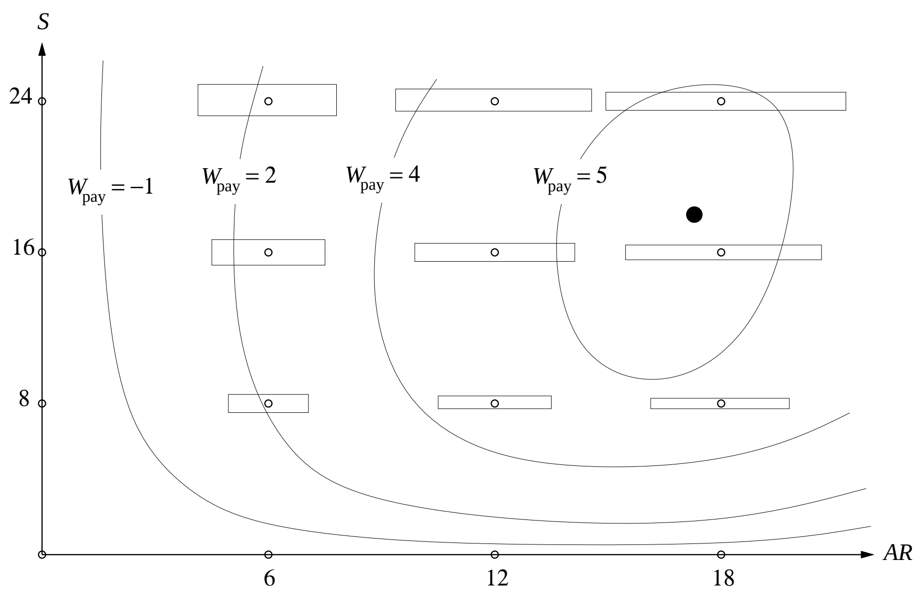

Figure 3: Objective function contours (isolines) in design space of a rectangular wing. Black dot shows the optimum-design maximum payload weight point.

Constraints

In almost any real design optimization problem, an objective function such as given by equation (7) does not capture all considerations which might go into selection of a design. Frequently one has to account for *constraints* which rule out certain regions of the design space. One typical constraint which appears in wing design is the structural requirement of adequate strength or stiffness.

To incorporate a stiffness constraint, we first must express the stiffness requirement in terms of the chosen design variables, or $\{AR, S\}$ in this case. We will require that the tip-deflection/span δ/b not exceed some reasonable upper limit, say

$$\delta/b \;\leq\; 0.05 \ldots 0.10 \tag{8}$$

in 1G flight. The tip deflection can be estimated using simple beam theory. Assuming the beam curvature to be roughly constant across the span and equal to its value κ_o at the root, the estimated tip deflection is given as follows.

$$\kappa_o \;=\; \frac{M_o}{EI_o} \tag{9}$$

$$\delta \;\simeq\; \frac{1}{2}\,\kappa_o \left(\frac{b}{2}\right)^2 \tag{10}$$

To use this, we still need to estimate the root bending moment M_o and root bending inertia I_o. A conservative estimate is that the net local loading/span is proportional to the chord. For a simple-taper wing with taper ratio $\lambda = c_t/c_r$, the root bending moment is then obtained by twice integrating this loading, giving

$$M_o \;=\; \frac{1}{12}\frac{1+2\lambda}{1+\lambda}\,(W_{\rm fuse} + W_{\rm pay})\,b \tag{11}$$

Assuming the wing is constructed out of a solid material the bending inertia of its root airfoil cross section is approximately

$$I_o \;\simeq\; 0.036\,c_r\,t_r(t_r^2 + h_r^2) \;=\; 0.036\,c_r^4\,\tau(\tau^2 + \varepsilon^2) \tag{12}$$

where t_r is the maximum root airfoil thickness, h_r is the maximum root airfoil camber height, $\tau = t/c$ is the airfoil thickness/chord ratio, and $\varepsilon = h/c$ is the airfoil camber/chord ratio. For a straight-taper wing of taper ratio λ, the root chord is related to the average chord by

$$c_r \;=\; c\,\frac{2}{1+\lambda} \tag{13}$$

Combining equations (10), (11), (12), and (13) gives

$$\frac{\delta}{b} \;=\; 0.018\,\frac{W_{\rm fuse} + W_{\rm pay}}{E\tau(\tau^2+\varepsilon^2)}(1+\lambda)^3(1+2\lambda)\frac{b^2}{c^4} \tag{14}$$

Putting b and c in terms of our chosen design variables $\{AR, S\}$ as given by (3), the deflection/span ratio finally becomes

$$\frac{\delta}{b} \;=\; 0.018\,\frac{W_{\rm fuse} + W_{\rm pay}}{E\tau(\tau^2+\varepsilon^2)}(1+\lambda)^3(1+2\lambda)\frac{AR^3}{S} \tag{15}$$

In the design space, the isolines of δ/b are given by rearranging equation (15) into

$$S \;=\; \left[\frac{0.018}{(\delta/b)}\,\frac{W_{\rm fuse} + W_{\rm pay}}{E\tau(\tau^2+\varepsilon^2)}(1+\lambda)^3(1+2\lambda)\right] AR^3 \tag{16}$$

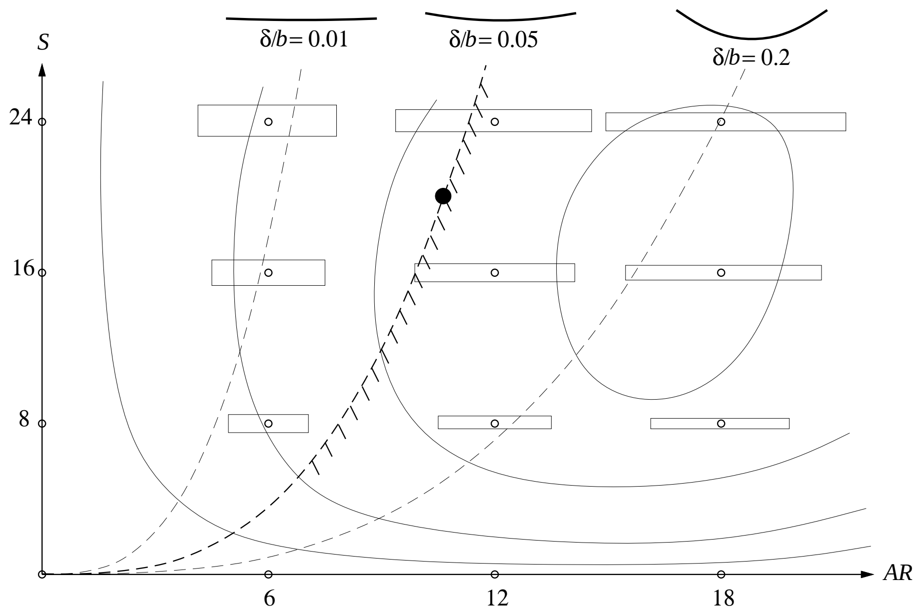

Figure 4: Wing deflection/span contours (dashed) superimposed on objective function contours (solid). The contour $\delta/b = 0.05$ is the chosen constraint boundary. Black dot shows the constrained maximum-payload weight point.

which is shown in Figure 4 for three values of δ/b. All points above the $\delta/b = 0.05$ isoline satisfy a chosen deflection constraint (8), and hence constitute the *feasible design space*. The new *constrained optimum* design is the point of maximum objective function which still lies in the feasible design space.

Additional Design Variables

Most practical design problems have vastly more than the two design variables $\{AR, S\}$ assumed in the examples above. A basic rule is that any adjustable quantity which is likely to have a strong effect on the constrained objective function should be considered as a design variable. One such candidate is the wing taper ratio $c_t/c_r = \lambda$, which clearly has a powerful effect on the tip deflection in relation (15). If λ is chosen as a new design variable, the design space is now three dimensional as shown in Figure 5.

$$\{\, AR \,,\; S \,,\; \lambda \} \tag{17}$$

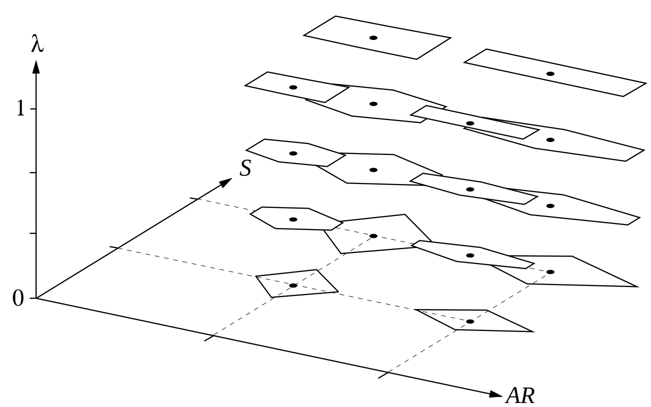

Figure 5: Three-variable design space. As before, each point represents a unique design.

In reality, there would also be other variables such as the modulus E of the construction material, C_L and c_d via airfoil shape, etc. The design space would then be

$$\{ \; AR \, , \, S \, , \, \lambda \, , \, E \, , \, C_L \, , \, c_d \; \ldots \} \tag{18}$$

Design Space Slicing

Because the entire design space of many dimensions is impossible to visualize graphically, we typically attempt to get its character by *slicing* it with a plane defined by only two variables, by choosing unique values for all the others. For example, the 2D space in Figure 4 is the same as the 3D space in Figure 5 sliced along the $\lambda = 1$ plane. Two of the three possible slice orientations are also shown in Figure 6.

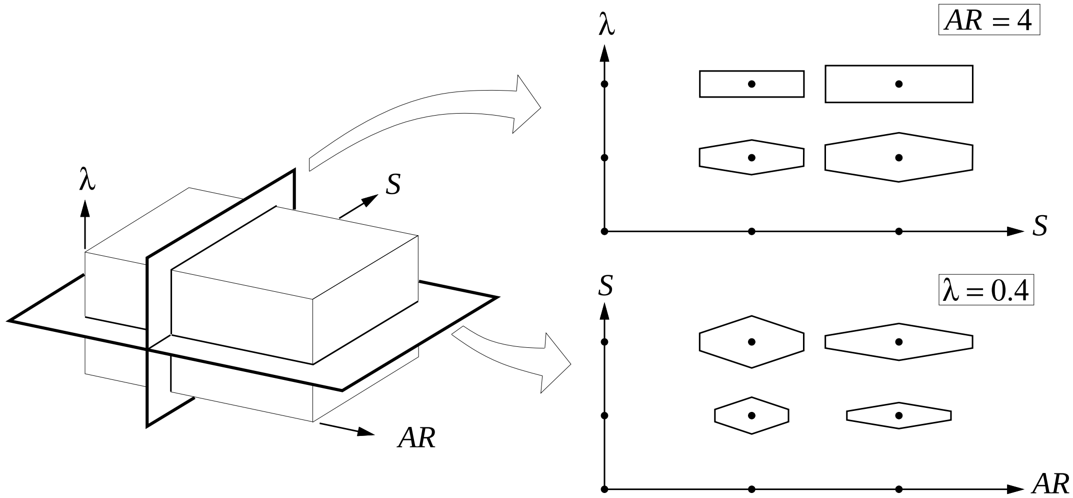

Figure 6: Two 2D slices through a 3D design space. The variable(s) which are not along the axes of a slice are held fixed.

Occasionally it is also useful to slice a design space using 1D lines rather than 2D planes. This allows plotting of a quantity of interest, such as W_{pay} in the current example, along each slice line. This is an alternative means of locating the optimum design, in lieu of the contour technique shown in Figure 3.

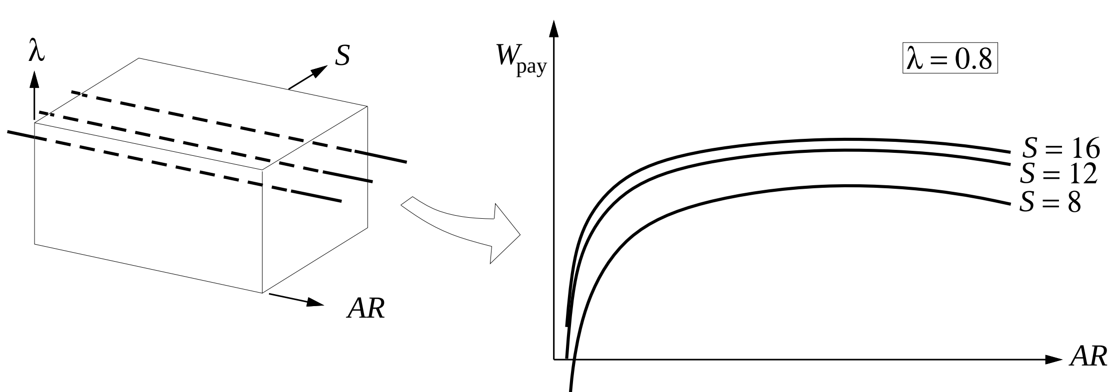

Figure 7: Three line slices through a 3D design space.